MAT290F - ADVANCED ENGINEERING MATHEMATICS
MIDTERM TEST, October 24, 2002, 6:10pm - 8:10pm

FAMILY NAME ______________________________

GIVEN NAME(S) ______________________________

STUDENT NUMBER ______________________________

Instructions
NO AIDS, NO CALCULATOR
Write your solutions clearly in the spaces provided below the problem statements.
Use the back sides of the pages for rough work.

You may use the following table of Laplace transforms:

$f(t)$	t^n	e^{at}	$\sin \omega t$	$\cos \omega t$
$F(s)$	$\frac{n!}{s^{n+1}}$	$\frac{1}{s-a}$	$\frac{\omega}{s^2+\omega^2}$	$\frac{s}{s^2+\omega^2}$

Problem	Mark
1	/5
2	/8
3	/7
4	/12
5	/10
6	/8
Total	/50

1. (a) [2 marks] Given the complex number z draw a picture in the complex plane $\mathbb{C}$ showing $\overline{z}$, $-z+1$, $\frac{1}{z}$, and z^2.

   (b) [3 marks] Using Euler's formula, prove that

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)\,.$$

2. (a) [4 marks] Let S denote the following set of complex numbers in $\mathbb{C}$:

$$S = \{z : z = e^{a+bi},\ a \le -1, \frac{\pi}{4} \le b \le \frac{7\pi}{4}\}.$$

What are its interior points? What are its boundary points? Is S open? Is S closed? (Write your answer in set notation $\{\}$. Proofs not needed.)

(b) [4 marks] Repeat the above questions for the set

$$S = \{z : z = e^{a+bi},\ \frac{\pi}{4} \le b \le \frac{7\pi}{4}, a = -1, -2, -3, \dots\}.$$

3. (a) [3 marks] Write in logic format the definition that z is not an interior point of the set S. Your definition should begin with either $\forall$ or $\exists$.

   (b) [3 marks] Let S be a set in $\mathbb{C}$ that is not $\emptyset$ and not $\mathbb{C}$ itself. If a point z is not an interior point of S and it is not an interior point of $\mathbb{C} - S$, the complement of S, can you determine what kind of point z is? If yes, use the definition of part (a) to give a formal logic proof. (Do not use limit points).

4. (a) [2 marks] Write the $\varepsilon - \delta$ definition of the statement

$$\lim_{z \to z_0} f(z) = w_0,$$

where the function f is defined at all points of an open disk centred at z_0.

(b) [4 marks] Using the preceding $\varepsilon - \delta$ definition, prove that

$$\lim_{z \to 0} \frac{z^2}{\overline{z}} = 0.$$

(c) [2 marks] Write the $\varepsilon - \delta$ definition of the statement

$$\lim_{z \to z_0} f(z) \neq w_0,$$

where the function f is defined in an open disk centred at z_0.

(d) [4 marks] Using this $\varepsilon - \delta$ definition, prove that

$$\lim_{z \to 0} \frac{\operatorname{Re}(z)}{\overline{z}} \neq 0.$$

5. (a) [2 marks] State the Cauchy-Riemann equations.

   (b) [2 marks] State a necessary condition for a function $f(z)$ to be differentiable at $z = z_0$.

   (c) [2 marks] State sufficient conditions for a function $f(z)$ to be differentiable at $z = z_0$.

(d) [4 marks] Consider the function

$$f(z) = \mathrm{Re}(z)\mathrm{e}^{\overline{z}}\,.$$

Using the Cauchy-Riemann equations alone, make all the conclusions that you can about differentiability of $f(z)$ at $z_0 = \frac{\pi}{2}i$.

6. [8 marks] Using the Laplace transform method, solve the initial-value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = \mathrm{e}^{-t}\cos t$$

$$y'(0) = 0, \quad y(0) = 0.$$