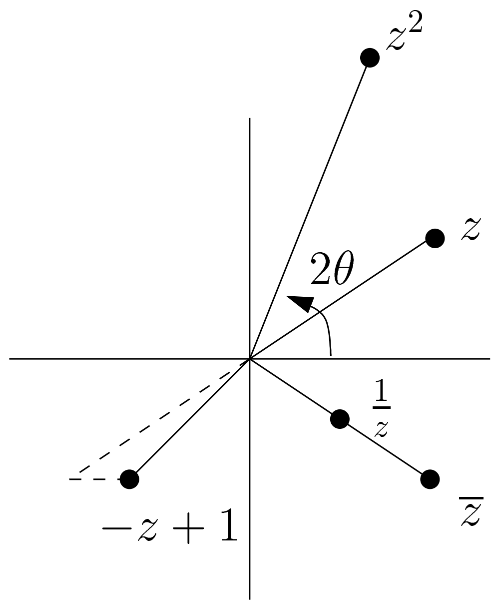
MAT290F - ADVANCED ENGINEERING MATHEMATICS MIDTERM TEST SOLUTION OUTLINE

1. (a) [2 marks] Given the complex number z draw a picture in the complex plane $\mathbb{C}$ showing $\overline{z}$, $-z+1$, $\frac{1}{z}$, and z^2.

   (b) [3 marks] Using Euler's formula, prove that

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)\,.$$

(a) Using polar coordinates let $z = re^{i\theta}$. Then $z^2 = r^2e^{i2\theta}$ and $\frac{1}{z} = \frac{1}{r}e^{-i\theta}$.



(b) By Euler's formula

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta\,. \end{aligned}$$

Summing and rearranging we have

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right).$$

Hence

$$\begin{aligned} \cos^2(\theta) &= \frac{1}{4}\left(e^{i2\theta} + 2 + e^{-i2\theta}\right) \\ &= \frac{1}{2}\cos 2\theta + \frac{1}{2}\,. \end{aligned}$$

2. (a) [4 marks] Let S denote the following set of complex numbers in $\mathbb{C}$:

$$S = \{z : z = \mathrm{e}^{a+bi},\ a \leq -1, \frac{\pi}{4} \leq b \leq \frac{7\pi}{4}\}.$$

What are its interior points? What are its boundary points? Is S open? Is S closed? (Write your answer in set notation $\{\}$. Proofs not needed.)

(b) [4 marks] Repeat the above questions for the set

$$S = \{z : z = \mathrm{e}^{a+bi}, \frac{\pi}{4} \leq b \leq \frac{7\pi}{4}, a = -1, -2, -3, \dots\}.$$

(a)

$$S^\circ = \{z : z = \mathrm{e}^{a+bi}, a < -1, \frac{\pi}{4} < b < \frac{7\pi}{4}\}.$$

$$\partial S = \{z : z = \mathrm{e}^{a+bi},\ a < -1, b = \frac{\pi}{4}, \frac{7\pi}{4}\} \cup \{0\} \cup \{z : z = \mathrm{e}^{-1+bi}, \frac{\pi}{4} \leq b \leq \frac{7\pi}{4}\}$$

S is not open because $S \neq S^\circ$. S is not closed because $\partial S \not\subset S$.

(b)

$$S^\circ = \emptyset.$$

$$\partial S = S \cup \{0\}.$$

S is not open. S is not closed.

3. (a) [3 marks] Write in logic format the definition that z is not an interior point of the set S. Your definition should begin with either $\forall$ or $\exists$.

   (b) [4 marks] Let S be a set in $\mathbb{C}$ that is not $\emptyset$ and not $\mathbb{C}$ itself. If a point z is not an interior point of S and it is not an interior point of $\mathbb{C} - S$, the complement of S, can you determine what kind of point z is? If yes, use the definition of part (a) to give a formal logic proof. (Do not use limit points).

(a) z is not an interior point if

$$(\forall \varepsilon > 0)(\exists w \text{ with } |w - z| < \varepsilon) w \notin S.$$

(b) If z is not an interior point of S then

$$(\forall \varepsilon > 0)(\exists w_1 \text{ with } |w_1 - z| < \varepsilon) w_1 \notin S.$$

If z is not an interior point of $\mathbb{C} - S$ then

$$(\forall \varepsilon > 0)(\exists w_2 \text{ with } |w_2 - z| < \varepsilon) w_2 \notin \mathbb{C} - S \implies w_2 \in S.$$

Hence, w_2 is a boundary point of S (as well as of $\mathbb{C} - S$).

4. (a) [2 marks] Write the $\varepsilon - \delta$ definition of the statement

$$\lim_{z \to z_0} f(z) = w_0,$$

where the function f is defined at all points of an open disk centred at z_0.

(b) [4 marks] Using the preceding $\varepsilon - \delta$ definition, prove that

$$\lim_{z \to 0} \frac{z^2}{\overline{z}} = 0.$$

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i.

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall z \text{ with } |z - z_0| < \delta)\ |f(z) - w_0| < \varepsilon$$

ii. Let $\varepsilon > 0$ be arbitrary. We have

$$\left| \frac{z^2}{\overline{z}} \right| = |z|$$

So take $\delta > 0$ such that $\delta = \varepsilon$.

(c) [2 marks] Write the $\varepsilon - \delta$ definition of the statement

$$\lim_{z \to z_0} f(z) \neq w_0,$$

where the function f is defined in an open disk centred at z_0.

(d) [4 marks] Using this $\varepsilon - \delta$ definition, prove that

$$\lim_{z \to 0} \frac{\operatorname{Re}(z)}{\overline{z}} \neq 0.$$

(a)

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists z \text{ with } |z - z_0| < \delta)\ |f(z) - w_0| \geq \varepsilon\,.$$

(b) Let $\delta > 0$ be arbitrary. Let $\varepsilon = \frac{1}{2}$ and let $z = \frac{\delta}{2} + i0$. Then

$$|f(z) - w| = 1 \geq \varepsilon\,.$$

5. (a) [2 marks] State the Cauchy-Riemann equations.

   (b) [2 marks] State a necessary condition for a function $f(z)$ to be differentiable at $z = z_0$.

   (c) [2 marks] State sufficient conditions for a function $f(z)$ to be differentiable at $z = z_0$.

(a) Let $u_x = \frac{\partial u}{\partial x}$, etc. Then the CR equations are

$$u_x = v_y, \quad u_y = -v_x.$$

(b) One necessary condition is that $f(z)$ be continuous at $z = z_0$. Another is that the CR equations hold at $z = z_0$.

(c) The CR equations hold at z_0 and the four partials u_x, u_y, v_x, v_y are continuous at $z = z_0$.

(d) [4 marks] Consider the function

$$f(z) = \mathrm{Re}(z)\mathrm{e}^{\overline{\mathrm{z}}}\,.$$

Using the Cauchy-Riemann equations alone, make all the conclusions that you can about differentiability of $f(z)$ at $z_0 = \frac{\pi}{2}i$.

$$f(z) = x\mathrm{e}^{\mathrm{x}}(\cos\mathrm{y} - \mathrm{i}\sin\mathrm{y})\,.$$

$$u(x,y) = x\mathrm{e}^{\mathrm{x}}\cos\mathrm{y}\,, \qquad \mathrm{v(x,y)} = -\mathrm{xe^{x}}\sin\mathrm{y}\,.$$

$$\begin{aligned} u_x &= x\mathrm{e}^{\mathrm{x}}\cos\mathrm{y} + \mathrm{e}^{\mathrm{x}}\cos\mathrm{y} \\ u_y &= -x\mathrm{e}^{\mathrm{x}}\sin\mathrm{y} \\ v_x &= -x\mathrm{e}^{\mathrm{x}}\cos\mathrm{y} + \mathrm{e}^{\mathrm{x}}\cos\mathrm{y} \\ v_y &= -x\mathrm{e}^{\mathrm{x}}\cos\mathrm{y} \end{aligned}$$

At $z_0 = \frac{\pi}{2}i$, $u_x = 0$, $u_y = 0$, $v_x = -1$ and $v_y = 0$. Since CR do not hold anywhere, the function is not differentiable anywhere.

6. [8 marks] Using the Laplace transform method, solve the initial-value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = \mathrm{e}^{-t}\cos t$$

$$y'(0) = 0, \quad y(0) = 0.$$

$$\mathrm{e}^{-t}\cos t \longrightarrow \frac{s+1}{(s+1)^2+1}$$

Taking LT in the equation gives

$$s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + Y(s) = \frac{(s+1)}{(s+1)^2+1}$$

Thus

$$(s^2 + 2s + 1)Y(s) = \frac{(s+1)}{(s+1)^2+1}$$

$$Y(s) = \frac{1}{(s+1)[(s+1)^2+1]}$$

$$Y(s) = \frac{1}{(s+1)} - \frac{(s+1)}{(s+1)^2+1}.$$

Thus

$$y(t) = \mathrm{e}^{-t} - \mathrm{e}^{-t}\cos t, \qquad t \geq 0.$$