MAT290F - ADVANCED ENGINEERING MATHEMATICS
MIDTERM TEST SOLUTION OUTLINE

1. (a) [5 marks] Suppose $f(t)$ is a Laplace transformable periodic function, of period T. Derive the following expression for its Laplace transform:

$$F(s) = \frac{1}{1-\mathrm{e}^{-sT}} \int_0^T \mathrm{e}^{-st} f(t)\ dt.$$

(b) [5 marks] Let $f(t)$ be the function

$$f(t) = \begin{cases} 1, & n \leq t < n+1, \qquad n \text{ even positive integer} \\ 0, & n \leq t < n+1, \qquad n \text{ odd positive integer} \end{cases}$$

Is f Laplace transformable? If it is not, explain why. If it is, explain why and find its Laplace transform.

(a) This is homework problem 3.1.24. Start with

$$F(s) = \int_0^\infty \mathrm{e}^{-st} f(t)dt = \sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} \mathrm{e}^{-st} f(t)dt$$

Then change variables in the integral, using periodicity of f.

(b) Function f is piecewise continuous and of exponential order. Hence f is Laplace transformable.

(c) We have $T = 2$, so

$$\begin{aligned} F(s) &= \frac{1}{1-\mathrm{e}^{-2s}} \int_0^1 \mathrm{e}^{-st} dt \\ &= \frac{1-\mathrm{e}^{-s}}{s(1-\mathrm{e}^{-2s})}. \end{aligned}$$

2. [10 marks] Solve the following equation for $f(t)$.

$$f(t) = te^t - 2e^t \int_0^t e^{-\tau} f(\tau) d\tau \,.$$

$$f(t) = te^t - 2e^t * f(t) \,.$$

$$F(s) = \frac{1}{(s-1)^2} - \frac{2}{(s-1)} F(s) \,.$$

Solving for $F(s)$, we have

$$F(s) = \frac{1}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} \,.$$

$$A = F(s)(s-1)|_{s=1} = \frac{1}{2} \,.$$

$$B = F(s)(s+1)|_{s=-1} = -\frac{1}{2} \,.$$

$$f(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} \,.$$

3. [15 marks]

(a) Let $f(t)$ be the function

$$f(t) = \begin{cases} 1 & \text{if} \quad 0 \le t < 1\,, \\ -1 & \text{if} \quad 1 \le t < 3\,, \\ 1 & \text{if} \quad 3 \le t < 4\,, \\ 0 & \text{if} \quad t \ge 4\,. \end{cases}$$

Calculate $F(s)$.

(b) Let $g(t)$ be the function

$$g(t) = \begin{cases} t & \text{if} \quad 0 \le t < 1\,, \\ 1-(t-1) & \text{if} \quad 1 \le t < 3\,, \\ t-4 & \text{if} \quad 3 \le t < 4\,, \\ 0 & \text{if} \quad t \ge 4\,. \end{cases}$$

Calculate $G(s)$.

$$\begin{aligned} f(t) &= [H(t) - H(t-1)] - [H(t-1) - H(t-3)] + [H(t-3) - H(t-4)] \\ &= 1 - 2H(t-1) + 2H(t-3) - H(t-4)\,. \end{aligned}$$

$$F(s) = \frac{1}{s} - 2\frac{\mathrm{e}^{-s}}{s} + 2\frac{\mathrm{e}^{-3s}}{s} - \frac{\mathrm{e}^{-4s}}{s}\,.$$

$$G(s) = \frac{1}{s}F(s) = \frac{1}{s^2} - 2\frac{\mathrm{e}^{-s}}{s^2} + 2\frac{\mathrm{e}^{-3s}}{s^2} - \frac{\mathrm{e}^{-4s}}{s^2}\,.$$

(c) Using Laplace transforms, solve the initial value problem

$$\dot{y} + y = g(t)\,, \qquad y(0) = 0\,,$$

where g is given in part (b).

$$sY(s) - y(0) + Y(s) = G(s)\,.$$

$$(s+1)Y(s) = \frac{1}{s^2}\left[1 - 2\mathrm{e}^{-s} + 2\mathrm{e}^{-3s} - \mathrm{e}^{-4s}\right]\,.$$

$$Y(s) = \frac{1}{s^2(s+1)}\left[1 - 2\mathrm{e}^{-s} + 2\mathrm{e}^{-3s} - \mathrm{e}^{-4s}\right]\,.$$

$$\frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s+1)}\,.$$

Solution of partial fraction expansion: $A = -1$, $B = 1$, $C = 1$.

$$Y(s) = \left[\frac{-1}{s} + \frac{1}{s^2} + \frac{1}{(s+1)}\right]\left[1 - 2\mathrm{e}^{-s} + 2\mathrm{e}^{-3s} - \mathrm{e}^{-4s}\right]\,.$$

$$\begin{aligned} y(t) &= \left(-1 + t + \mathrm{e}^{-t}\right) H(t) - 2\left(-1 + (t-1) + \mathrm{e}^{-(t-1)}\right) H(t-1) \\ &+ 2\left(-1 + (t-3) + \mathrm{e}^{-(t-3)}\right) H(t-3) - \left(-1 + (t-4) + \mathrm{e}^{-(t-4)}\right) H(t-4)\,. \end{aligned}$$

4. [10 marks] Let S denote the following set of complex numbers in $\mathbb{C}$:

$$S = \left\{ z : |z+1| < |z-i| \text{ and } |z-1+i| > \sqrt{2} \right\} .$$

Sketch the set S. What are its interior points? What are its boundary points? Is S open? Is S closed? (Write your answer in set notation $\{\}$. Proofs not needed.)

$$\begin{aligned} S^\circ &= S \\ \partial S &= \{ \ z = x+iy \ : \ y = -x, x \leq 0 \ \} \bigcup \\ &\left\{ \ z \ : \ z = 1 - i + \sqrt{2}\mathrm{e}^{i\theta}, \ \theta \in [\frac{3\pi}{4}, \frac{7\pi}{4}] \ \right\} \bigcup \\ &\{ z = x+iy \ : \ y = -x, x \geq 2 \ \} . \end{aligned}$$

S is open since $S = S^\circ$. S is not closed since $\partial S \not\subset S$.

5. [10 marks] Prove that the sequence

$$z_n = \frac{i}{2n} - \frac{1}{2n^4} - 3i$$

converges.

The sequence converges to $3i$. To prove it we must show:

$$(\forall \epsilon > 0)(\exists N \geq 1, n \geq N)\ |z_n - 3i| < \epsilon\,.$$

Let $\epsilon > 0$ be arbitrary. Pick N to be a positive integer greater than $\frac{1}{\epsilon}$. Let n be an integer greater than N. Then,

$$\begin{aligned} |z_n - (-3i)| &= \left|\frac{i}{2n} - \frac{1}{2n^4}\right| \\ &\leq \frac{1}{2n} + \frac{1}{2n^4} \\ &\leq \frac{1}{2n} + \frac{1}{2n} \\ &= \frac{1}{n} \\ &\leq \frac{1}{N} \\ &\leq \epsilon\,. \end{aligned}$$

as required. QED.

6. [10 marks]

(a) Using the $\varepsilon - \delta$ definition, prove that $f(z) = \overline{z} + z$ is continuous at an arbitrary point $a \in \mathbb{C}$.

We must show

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall z, |z-a| < \delta)\ |f(z) - f(a)| < \varepsilon\,.$$

Let $\varepsilon > 0$ be arbitrary. Consider

$$\begin{aligned} |\overline{z} + z - \overline{a} - a| &= |(\overline{z} - \overline{a}) + (z - a)| \\ &\leq |\overline{z} - \overline{a}| + |z - a| \\ &= |\overline{z - a}| + |z - a| \\ &\leq 2\delta\,. \end{aligned}$$

Let $\delta = \frac{\epsilon}{2}$. Then the result follows. QED.

Let $f(z) = z^2 \cdot \overline{z}$.

(b) Is f a bounded function in the set

$$S = \{-2003 < \text{Re}(z) < 2003, -2003 \le \text{Im}(z) \le 2003\}\,?$$

If yes, give a proof. If no, explain.

(c) Does the limit

$$\lim_{z \to 0} \frac{f(z)}{z}$$

exist? If yes, evaluate the limit. If no, explain.

(d) Is f differentiable at $z = 0$? If yes, evaluate the derivative. If no, explain.

(e) Is f differentiable at $z = 3 + 3i$? If yes, evaluate the derivative. If no, explain.

- If $z \in S$ then $|x| < 2003$, $|y| < 2003$. Therefore $|z| < \sqrt{2} \cdot 2003$ and $|f(z)| < (\sqrt{2} \cdot 2003)^3 = 2\sqrt{2} \cdot 2003^3$. Hence, f is bounded in S.
- The limit exists and is 0.
- Check Cauchy-Riemann equations. We have

  $$u = x^3 + xy^2, \qquad v = yx^2 + y^3\,.$$

  Then

  $$\begin{aligned} u_x &= 3x^2 + y^2 \\ u_y &= 2xy \\ v_x &= 2xy \\ v_y &= x^2 + 3y^2\,. \end{aligned}$$

  If $z = 0$, then $u_x = v_y$ and $u_y = -v_x$. Also, u_x, u_y, v_x, v_y are continuous. By the sufficiency theorem, $f(z)$ is differentiable at $z = 0$.
- At $z = 3 + 3i$, $u_x = v_y$, $u_y \neq -v_x$, so $f(z)$ is not differentiable at $z = 3 + 3i$.