MAT290F - ADVANCED ENGINEERING MATHEMATICS
QUIZ 2 Solution Outline, Section TUT01-04

1. Find all solutions to the equation:

$$e^z = 3\sinh z\,.$$

$$\begin{aligned} \mathrm{e^z} &= \frac{3}{2}(\mathrm{e^z} - \mathrm{e^{-z}}) \\ (\mathrm{e^z})^2 &= 3\,. \end{aligned}$$

Hence

$$\begin{aligned} 2z &= \ln(3) + i(2\pi n), \qquad n = 0, \pm1, \pm2, \ldots \\ z &= \ln(\sqrt{3}) + i(\pi n), \qquad n = 0, \pm1, \pm2, \ldots\,. \end{aligned}$$

2. If Γ is the circle $|z| = 1$, evaluate

$$\int_\Gamma |z+1||dz|\,.$$

The curve can be parametrized as

$$z(t) = e^{it} \qquad \text{where} \qquad 0 \le t \le 2\pi\,.$$

$$\int_\Gamma |z+1||dz| = \int_0^{2\pi} |e^{it}+1||ie^{it}|dt = \int_0^{2\pi} |e^{it}+1|dt$$

Using Euler formula:

$$|e^{it}+1| = |\cos t + 1 + i\sin t| = ((\cos t+1)^2 + (\sin t)^2)^{1/2} = |2(\cos t+1)|^{1/2} = 2|\cos(\frac{t}{2})|\,.$$

Therefore:

$$\int_0^{2\pi} 2|\cos(\frac{t}{2})|dt = 8\int_0^{\frac{\pi}{2}} \cos u du = 8\,.$$

3. If a is a real positive number and Γ is a curve from point a to point $a + ia$, prove that:

$$\left| \int_\Gamma e^{-z^2} dz \right| \leq a.$$

$$\Gamma(t) = a + ati \qquad 0 \leq t \leq 1.$$

$$|e^{-z^2}| = |e^{-a^2+a^2t^2-2a^2ti}| = |e^{-a^2+a^2t^2}||e^{2a^2ti}| = |e^{-a^2+a^2t^2}| = \left| \frac{1}{e^{a^2(1-t^2)}} \right|$$

where:

$$1 \leq e^{a^2(1-t^2)} \leq e^{a^2} \qquad 0 \leq t \leq 1$$

and

$$\frac{1}{e^{a^2}} \leq \frac{1}{e^{a^2(1-t^2)}} \leq 1 \Longrightarrow |e^{-z^2}| \leq 1$$

By the Bound on Complex Integral theorem,

$$\left| \int_\Gamma \mathrm{e}^{-\mathrm{z}^2} \mathrm{dz} \right| \leq a\,.$$

4. Suppose

$$z^w = exp(wLog(z))$$

where $Log(z)$ is the principle value of $log(z)$.

(a) Prove that for nonzero z

$$z^{w+w_1} = z^w z^{w_1}$$

is always true.

(b) Prove that for nonzero z and nonzero z_1

$$(zz_1)^w = z^w {z_1}^w$$

is not always true. Note: do not use a counterexample.

(a)

$$z^{w+w_1} = exp((w + w_1)Log(z)) = exp(wLog(z) + w_1Log(z))$$

Invoking Theorem 17.22.3:

$$= exp(wLog(z))exp(w_1Log(z)) = z^w z^{w_1} \text{[2 marks]}$$

(b)

$$(zz_1)^w = exp(wLog(zz_1))$$

and

$$z^w {z_1}^w = exp(wLog(z))exp(wLog(z_1)) = exp(w(Log(z) + Log(z_1)))$$

Therfore we must show that $Log(zz_1) = Log(z) + Log(z_1)$ is not always true. If $z = |z|e^{i(\alpha+2n\pi)}$ and $z_1 = |z_1|e^{i(\beta+2n\pi)}$

$$Log(z) + Log(z_1) = \ln|z| + \ln|z_1| + i(\alpha + \beta)$$

while,

$$Log(zz_1) = \ln|zz_1| + i(\alpha + \beta) = \ln|z| + \ln|z_1| + i(\alpha + \beta) \qquad \text{if} \qquad \alpha + \beta < 2\pi$$

$$Log(zz_1) = \ln|z| + \ln|z_1| + i(\alpha + \beta - 2\pi n) \qquad \text{if} \qquad \alpha + \beta > 2\pi n$$

Therefore the equality is not always true. [3 marks]