MAT290F - ADVANCED ENGINEERING MATHEMATICS
QUIZ 2 Solution Outline, Section TUT05-06

1. (a) Find and sketch the region of convergence of the series:

$$\sum_{n=0}^{\infty} e^{-nz^2}$$

   (b) Find the sum of the series.

(a) The n^{th} term in the series is

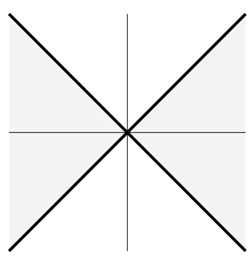
$$c_n = e^{-nz^2}.$$

Applying the ratio test, we get

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{e^{-(n+1)z^2}}{e^{-nz^2}}\right| \Rightarrow \left|e^{-z^2}\right|.$$

$$\left|e^{-z^2}\right| = \left|e^{-(x^2-y^2+2ixy)}\right| = e^{y^2-x^2} < 1 \Rightarrow \quad y^2 - x^2 < 0$$

So the region of convergence is $-x < y < x$. **[3 marks]**



[1 mark]

(b) Using the geometric series formula the sum of the series can be calculated as follows.

$$\sum_{n=0}^{\infty} e^{-nz^2} = \frac{1}{1-e^{-z^2}}$$

[1 mark]

2. Prove that $tanh^{-1}(z) = \frac{1}{2} \log \frac{1+z}{1-z}$.

Let

$$w = tanh^{-1}(z) \,.$$

Then

$$\begin{aligned} z = tanh(w) &= \frac{sinh(w)}{cosh(w)} \\ &= \frac{e^w - e^{-w}}{e^w + e^{-w}} \\ &= \frac{e^{2w} - 1}{e^{2w} + 1} \,. \end{aligned}$$

[2 marks]

Then

$$\begin{gathered} ze^{2w} + z = e^{2w} - 1 \\ (1 + z) = (1 - z)e^{2w} \\ e^{2w} = \tfrac{1+z}{1-z} \\ 2w = \log \tfrac{1+z}{1-z} \,. \end{gathered}$$

Finally,

$$w = tanh^{-1}(z) = \frac{1}{2} \; \log \frac{1 + z}{1 - z}$$

[3 marks]

3. Find all values of $(1+i\sqrt{3})^{2-3i}$. **[5 marks]**

$$\begin{aligned} w &= (1+i\sqrt{3})^{2-3i} \\ \log(w) &= (2-3i)\log(1+i\sqrt{3}) = (2-3i)\log[2e^{i(\frac{\pi}{3}+2n\pi)}]\,, \end{aligned}$$

where $n = 0, \pm 1, \pm 2, \ldots$. Then

$$\begin{aligned} \log(w) &= (2-3i)[\ln(2) + i(\frac{\pi}{3}+2n\pi)] \\ &= [2\ln(2) + 3(\frac{\pi}{3}+2n\pi)] + i[2(\frac{\pi}{3}+2n\pi) - 3\ln(2)]\,. \end{aligned}$$

Then

$$\begin{aligned} w &= \mathrm{e}^{2\ln(2)+\pi+6\mathrm{n}\pi}\mathrm{e}^{\mathrm{i}(\frac{2\pi}{3}-3\ln(2))} \\ &= 4\mathrm{e}^{\pi+6\mathrm{n}\pi}\mathrm{e}^{\mathrm{i}(\frac{2\pi}{3}-3\ln(2))} \\ &= 4\mathrm{e}^{\pi+6\mathrm{n}\pi}\cos[\frac{2\pi}{3}-3\ln(2)] + \mathrm{i}\sin[\frac{2\pi}{3}-3\ln(2)]. \end{aligned}$$

where $n = 0, \pm 1, \pm 2, \ldots$. **[5 marks]**

4. Let Γ be any piecewise smooth curve from 0 to $1+2i$. Evaluate the integral

$$\int_\Gamma zcos(z)dz.$$

Note: your final answer should be of the form $a + ib$ where a and b are functions of real numbers. **[5 marks]**

The function $zcos(z)$ is continuous everywhere in the complex plane; hence we can use the Antiderivative theorem. We have

$$\begin{aligned}
\int_\Gamma zcos(z)dz &= zsin(z)|_0^{1+2i} - \int_\Gamma sin(z)dz \\
&= (1+2i)sin(1+2i) + cos(z)|_0^{1+2i} \\
&= (1+2i)sin(1+2i) + cos(1+2i) - cos(0) \\
&= (1+2i)[sin(1)cosh(2) + icos(1)sinh(2)] \\
&\quad + \ cos(1)cosh(2) - isin(1)sinh(2) - 1 \\
&= [sin(1)cosh(2) - 2cos(1)sinh(2) + cos(1)cosh(2) - 1] \\
&\quad + \ i[2sin(1)cosh(2) + cos(1)sinh(2) - sin(1)sinh(2)] \,.
\end{aligned}$$