ECE216: Signals and Systems

Winter 2024

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Complete Course Notes

This Version: December 20, 2023

About these slides

These slides are prepared and maintained by Prof. Simpson-Porco. They are based on material from many sources, including EE102 notes from Stanford, ECE301 notes from Purdue, ELE301 from Princeton, MATH334 from Queen's, textbooks by Oppenheim & Willsky, Lathi, Kwakernaak & Sivana, and Levan, and previous course notes written by Prof. S. Draper and Prof. M. Wonham.

Please report any typos you find in the slides to Prof. Simpson-Porco at `jwsimpson@ece.utoronto.ca`.

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1. Introduction

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- what are signals & systems?
- motivating engineering examples

Course mechanics

- course website: Quercus
- course information sheet and course website are the authoritative administrative references
- lectures: multiple sections, multiple instructors
- these are the course notes. There is no official textbook. Extensive supplementary reading from standard textbooks is listed
- **evaluation:** labs, midterm(s), final

Core prerequisites

- ECE 212: Circuit Analysis
  - transient/steady-state response of a circuit
  - Laplace transform analysis of circuits
  - phasor analysis of circuits

- MAT290: Advanced Engineering Mathematics
  - manipulating complex numbers
  - solution of constant-coefficient linear ODEs, homogeneous and particular solutions
  - Laplace transforms

We are assuming that you are comfortable with the mechanics of the above material, i.e., that you can perform the basic computations. Please refer to your previous notes as needed.

About these course notes

- the instructor for your section will let you know the precise manner in which these notes will be used, if at all

- the notes <u>are not</u> an exhaustive textbook

  - they give a succinct (but reasonably complete) presentation

  - supplementary reading is listed at the end of each chapter of these notes; see for additional exposition and extra problems

  - blank space at end of each chapter for your personal notes and additional examples from lecture

  - all `MATLAB` code used in notes is on `Quercus`! **Enjoy!**

Major topics & outline

- fundamentals of continuous-time (CT) and discrete-time (DT) signals
- analysis of signals in the ***frequency-domain***
  - *Fourier series* analysis of periodic signals
  - *Fourier transform* analysis of signals
- fundamentals of ***CT systems***: LTI systems, causality, convolution
- analysis of CT systems in the ***frequency domain***
  - *Laplace transform* analysis of CT systems
  - *Fourier transform* analysis of CT systems
- fundamentals of ***DT systems***: LTI systems, causality, convolution
- example applications: signal processing, communications, control

Why study signals and systems?

- it is a ***foundational*** subject; methods used across all engineering disciplines, and within many non-engineering disciplines
- it is a ***unifying*** subject; synthesis of ideas from engineering and math
- it can be broadly ***applied*** in wireless/digital communications, audio/image/video processing, feedback control, automotive and aerospace, medical imaging and diagnostics, energy systems, economics and finance, spectroscopy, crystallography ...

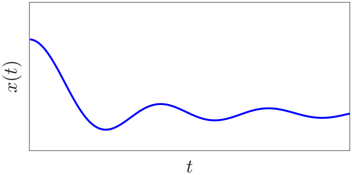
> If you let it, this course will open your eyes to the interplay between mathematics and engineering, and the common math foundations of many disparate engineering topics

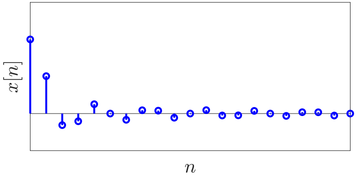
Strategies for success

- if you need to, review complex numbers, ODEs, and Laplace transforms; we are assuming you know this material
- attend lectures
- be competent in performing the calculations, but make sure you understand the *ideas*
- attempt all homework problems
- take advantage of tutorials and instructor office hours

What is a signal?

- a ***signal*** is any phenomenon carrying quantitative information
- could describe a physical quantity, or a variable within an engineering algorithm; most often, we think of signals as changing over ***time***





- **note:** the "signal" is the ***entire*** plot, i.e., the whole function
- $x(t)$ or $x[n]$ is the ***value*** of the signal at the time instant t or n

Examples of signals

- AM or FM radio signals
- audio, video, images
- ethernet or wireless internet signals
- pressure, temperature, concentration, volume
- position, velocity, force
- voltage, current, charge, flux
- quarterly revenue at a company
- the daily price of a stock
- hourly CAD/USD exchange rate

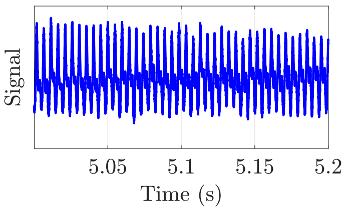
Key questions about signals

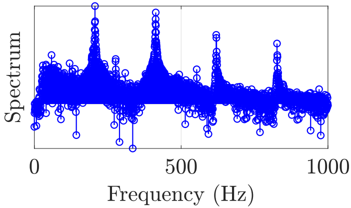
(i) what are some important ***types*** and ***properties*** of signals?

(ii) can we analyze a complicated signal by ***decomposing*** it into simpler "building block" signals?

(iii) conversely, can we combine "building block" signals and combine them to ***synthesize*** more complicated signals?

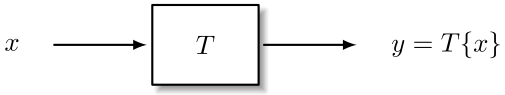
(iv) what is the "***spectrum***" or "***frequency-domain***", and how does it help us understand, analyze, and design signals?





What is a system?

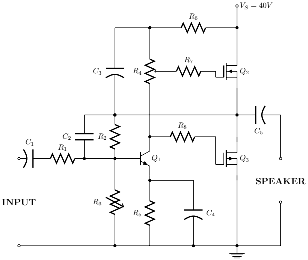
- a ***system*** is some transformation — call it T — that turns one signal x (the *input*) into another signal $T\{x\}$ (the *output*)

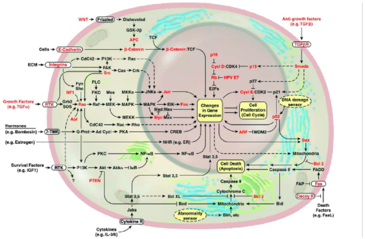


- the *value* of the output signal at time t is $y(t) = T\{x\}(t)$
- the operation T could be
  - very simple (e.g., multiplication by a constant)
  - very complex (e.g., described by a long piece of computer code)
- important intermediate case: ***linear time-invariant systems***

Examples of systems

- electrical circuits
- biological circuits
- your cellphone
- robotics systems
- automotive/aerospace systems
- digital signal processors
- machine vision systems
- the economy
- the stock market

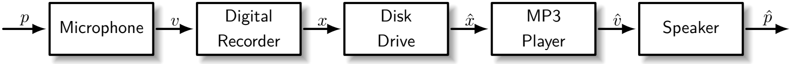




Example: wireless audio communications

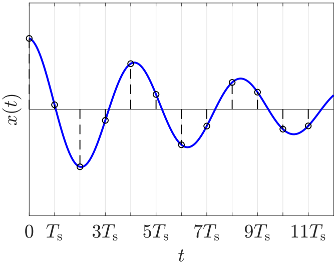
- $p, \hat{p}$ are pressure signals, $v, \hat{v}$ are voltages, and $x, \hat{x}$ are EM waves
- if this communication system works well, then we should have $\hat{p} \approx p$
- the overall system consists of a combination of ***subsystems***; we typically build a model for each subsystem (i.e., block) in the diagram
- from an engineering perspective, not all blocks are the same
  - some blocks we get to design (e.g., the radio receiver)
  - some blocks are determined by nature (e.g., the atmosphere)

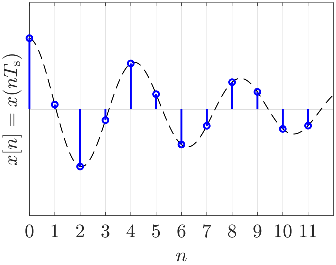
Example: recording and storing audio



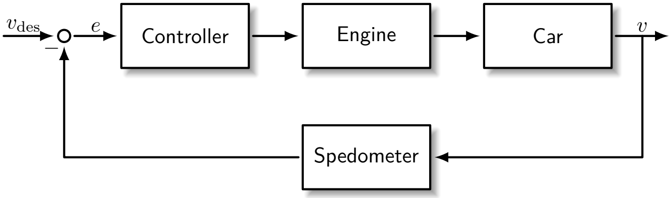
- $p, \hat{p}$ are pressure signals, $v, \hat{v}$ are voltages, and $x, \hat{x}$ are bit sequences

> The process of sampling converts a continuous-time (analog) signal into a discrete-time (digital) signal





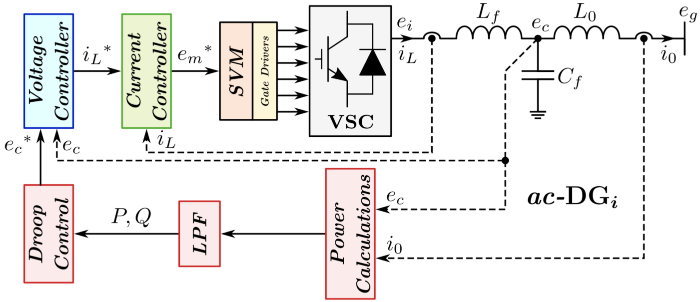
Example: feedback control system



- a ***cruise-control*** system attempts to maintain a car at a desired velocity v_{des} despite changes in road conditions and wind
- to do this, the cruse controller measures the velocity v, computes the velocity error $e = v_{des} - v$, and the ***controller*** decides how to adjust the gas/brake in order to reduce the error
- you will be able to study control systems in detail in ECE311

Example: control of power converters

- a DC/AC power converter with associated control loops



- feedback control is used extensively in renewable energy applications
- typically many interconnected subsystems, multiple feedback loops

Key questions about systems

(i) what are the most important ***models*** of systems?

(ii) what do standard ***properties*** of systems mean, including

- linearity
- causality
- time-invariance
- stability

and how are these properties characterized?

(iii) why are ***linear time-invariant (LTI) systems*** so important?

(iv) how can we analyze, understand, and design LTI systems?

Mathematics notation

- if X and Y are sets, then $f : X \to Y$ means that f is a *function* which assigns to each $x \in X$ a value $f(x) \in Y$
- X is the *domain* of f: the set of all allowable arguments x we can give f. The set Y is the *codomain* of f, the set where f takes values
- in this course the sets X, Y are usually one of
  - (i) the set of all ***integers*** $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
  - (ii) the set of all ***real*** numbers $\mathbb{R}$
  - (iii) the set of all ***complex*** numbers $\mathbb{C}$
- we sometimes consider ***Cartesian products*** of these sets. For example, $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is the set of all ordered triples (x, y, z) of real numbers $x, y, z \in \mathbb{R}$
- for two sets A and B, $A \subset B$ means that A is a subset of B

Supplementary reading

- **O-W:** A. V. Oppenheim and S. Willsky, *Signals and Systems*, 2nd Ed.
- **BB:** B. Boulet, *Fundamentals of Signals and Systems*.
- **BPL:** B. P. Lathi, *Signal Processing and Linear Systems*.
- **K-S:** H. Kwakernaak and R. Sivan, *Modern Signals and Systems*.
- **EL-PV:** E. Lee and P. Varaiya, *Structure and Interpretation of Signals and Systems*, 2nd Ed.
- **ADL:** A. D. Lewis, *The Mathematical Theory of Signals and Systems*.

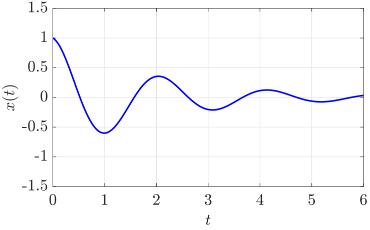
Topic	**O-W**	**BB**	**BPL**	**K-S**	**EL-PV**	**ADL**
Big Picture / Motivation	1.0, 2.0	1	1	1.1–1.3	1.1–1.3	V4 1.1, 2.1

2. Fundamentals of Continuous and Discrete-Time Signals

- continuous-time (CT) and discrete-time (DT) signals
- manipulating signals
- periodic signals
- support and finite-duration signals
- even and odd signals
- action, energy, and amplitude of signals
- special CT and DT signals
- sinusoidal signals, complex numbers
- CT complex exponential signals
- DT complex exponential signals

Continuous-time (CT) signals

Definition 2.1. A continuous-time signal is a function of one (or more) independent variables which range over the real numbers $\mathbb{R}$.



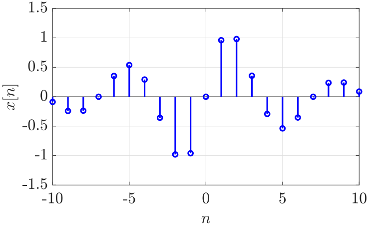
Examples

- 1D real: $x : \mathbb{R} \to \mathbb{R}$
- 1D complex: $x : \mathbb{R} \to \mathbb{C}$
- 2D: a topographical map $x : \mathbb{R}^2 \to \mathbb{R}$
- 3D: video $x : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$

- x is the signal, we write $x(t)$ for the signal's value at time $t \in \mathbb{R}$
- the signal is usually defined ***for all*** arguments $t \in \mathbb{R}$

Discrete-time (DT) signals

Definition 2.2. A discrete-time signal is a function of one (or more) independent variables which range over the integers $\mathbb{Z}$.



Examples

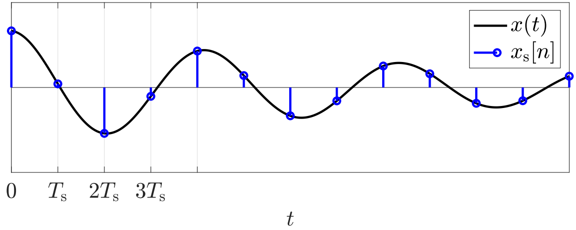
- 1D real: $x : \mathbb{Z} \to \mathbb{R}$
- 1D complex: $x : \mathbb{Z} \to \mathbb{C}$
- Bit stream: $x : \mathbb{Z} \to \{0, 1\}$
- Grayscale image: $x : \mathbb{Z}^2 \to [0, 1]$

- x is the signal, we write $x[n]$ for the signal's value at time $n \in \mathbb{Z}$
- the signal is defined ***only*** at integer times n, not "in-between".

Sampling: from CT signals to DT signals

- some signals are naturally modelled in discrete-time (e.g., text)
- DT signals also arise via regular ***sampling*** of a CT signal x
- this produces a sampled DT signal x_s defined as

$$x_s[n] = x(t)\Big|_{t=nT_s} = x(nT_s), \qquad T_s = \text{sampling period.}$$

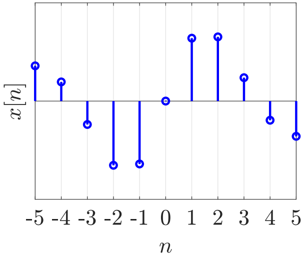


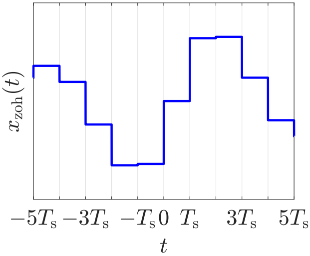
- **Examples:** audio (44,100 samples/sec), video (30 samples/sec), digital camera (5,000,000 RGB samples per image).

Interpolation: from DT signals to CT signals

- given a DT signal x and a sampling period T_s, we can construct a corresponding CT signal x_zoh via a ***zero-order hold***

$$x_\mathrm{zoh}(t) = x[n] \quad \text{for } nT_\mathrm{s} \leq t < (n+1)T_\mathrm{s}$$





- we ***hold*** the value $x[n]$ until we reach sampling time $n+1$, and then we change to the new value $x[n+1]$ (other possibilities?)

Pointwise operations on CT and DT signals

- ***pointwise addition:*** if we have two signals f, g we can build a new signal $h = f + g$ which takes the values

$$h(t) = f(t) + g(t), \qquad h[n] = f[n] + g[n]$$

- ***pointwise scaling:*** we can scale a signal f by any constant α to build a new signal $h = \alpha f$ which takes the values

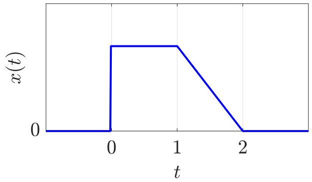
$$h(t) = \alpha f(t), \qquad h[n] = \alpha f[n]$$

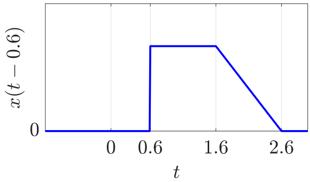
- ***pointwise multiplication:*** if we have two signals f, g, we can build a new signal $h = f \cdot g$ which takes the values

$$h(t) = f(t)g(t), \qquad h[n] = f[n]g[n]$$

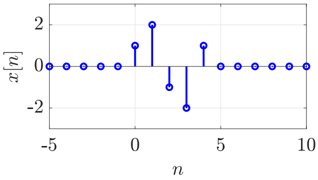
More operations: time-shifting a signal

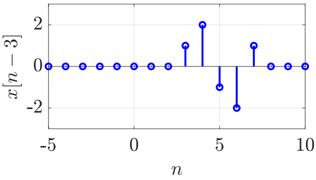
- we can ***time-shift*** a CT signal x by $t_0 \in \mathbb{R}$ to obtain $x(t - t_0)$





- we can time-shift a DT signal x by $n_0 \in \mathbb{Z}$ to obtain $x[n - n_0]$

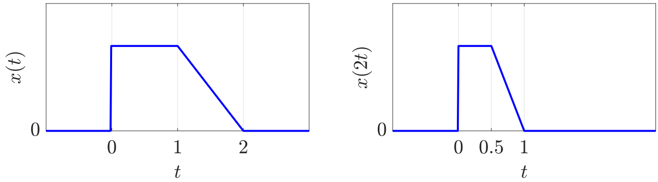




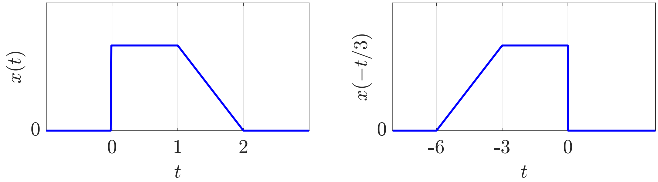
- $t_0, n_0 > 0$ means we ***delay*** in time; negative means we ***advance***

More operations: time-scaling a CT signal

- we can ***time-scale*** a CT signal x by $\alpha \in \mathbb{R}$ to obtain $x(\alpha t)$

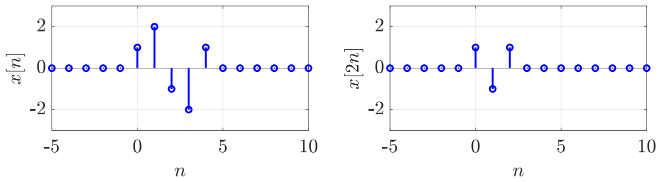


- $|\alpha| > 1$ compresses the time axis, $|\alpha| < 1$ expands time
- if $\alpha < 0$, time effectively ***flips*** (runs backward)



More operations: time-scaling a DT signal

- you can ***time-scale*** a DT signal x by any $\alpha \in \mathbb{Z}$, yielding $x[\alpha n]$

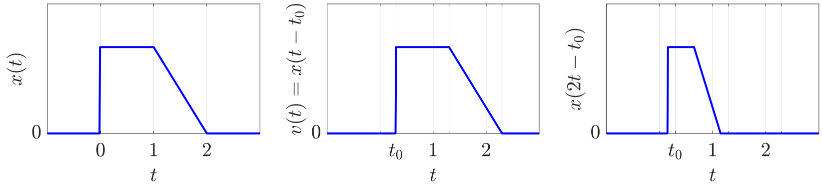


- this operation ***sub-samples*** the signal x; in the above example with $\alpha = 2$, we keep only every *other* sample
- α must be an ***integer***! The expression $x[n/3]$ makes no sense (why?)

More operations: combine shifting and scaling

- you can time-shift and time-scale $y(t) = x(\alpha t - \beta)$ in two steps
  - (i) time shift x to obtain $v(t) = x(t - \beta)$
  - (ii) time scale v to obtain $y(t) = v(\alpha t)$

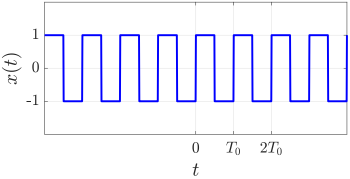
$$\text{Check:} \qquad v(\alpha t) = x(t - \beta)|_{t=\alpha t} = x(\alpha t - \beta) = y(t).$$



- **note:** the above is *not* the same as first time-scaling by α, then time-shifting by β; you should try this for yourself and see.

Periodic CT signals

Definition 2.3. A CT signal x is ***periodic*** if $x(t) = x(t+T)$ for some value $T > 0$ and all $t \in \mathbb{R}$. The smallest such value of T, denoted by T_0, is called the ***fundamental*** period of x.

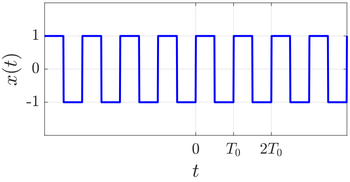


Examples

- $\sin(t)$ has fundamental period $T_0 = 2\pi$
- $\sin(2\pi t/\tau)$ has fundamental period $T_0 = \tau$
- square waves, triangle waves, ...

- the period is not uniquely defined, but the *fundamental* period is
- ***edge case***: is a *constant* CT signal is periodic? Fund. period?

Periodic CT signals



- the fundamental period T_0 tells us ***how long*** a periodic signal takes to repeat itself

- the inverse $f_0 = 1/T_0$ tells us ***how often*** or ***how fast*** the signal repeats, and is called the ***fundamental frequency***.

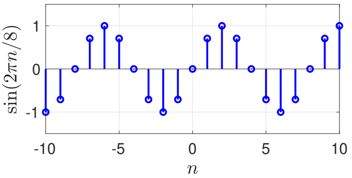
- the units of f_0 are cycles/second, called Hertz (Hz)

- we often instead work with the ***fundamental angular frequency*** ω_0

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}, \quad \text{units} = \text{radians/second}$$

Periodic DT signals

Definition 2.4. A DT signal x is ***periodic*** if $x[n] = x[n+N]$ for some integer $N \geq 1$ and all $n \in \mathbb{Z}$. The smallest such value of N, denoted by N_0, is called the ***fundamental*** period of x.



Examples

- $\sin(2\pi n/8)$ has $N_0 = 8$
- $\cos(2\pi n)$ has $N_0 = 1$

- we often use the ***frequency*** and ***angular frequency***

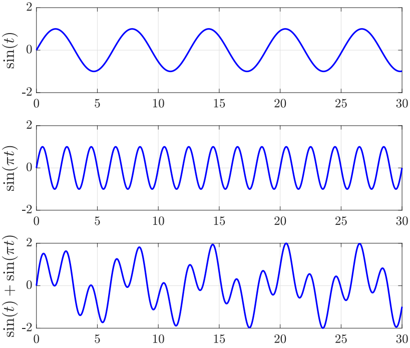
$$f_0 = \frac{1}{N_0} \text{ [cycles/sample]} \qquad \omega_0 = 2\pi f_0 = \frac{2\pi}{N_0} \text{ [radians/sample]}$$

- is a *constant* DT signal is periodic? Fund. period?

Periodicity of sums of signals

Is the ***sum*** of two periodic signals always periodic? No.

- **counter-example:** consider $\sin(t)$ and $\sin(\pi t)$



- $\sin(t)$ has $T_0 = \frac{2\pi}{1} = 2\pi$
- $\sin(\pi t)$ has $T_0 = \frac{2\pi}{\pi} = 2$

Even if we wait ***forever***, the two periods will never "line up" with each other, because $\frac{2\pi}{2} = \pi$ is irrational . . .

Periodicity of sums of signals

- the two periods will eventually line up with one another if one period is a *rational multiple* of the other ...

Theorem 2.1. Let x_1, x_2 be periodic CT signals with fundamental periods T_1 and T_2. If T_1/T_2 is rational, i.e., if $T_1/T_2 = k/l$ for some positive integers k, l, then $x_1 + x_2$ is periodic with period $T = lT_1 = kT_2$.

Proof: Let $z(t) = x_1(t) + x_2(t)$. We calculate for any $t \in \mathbb{R}$ that

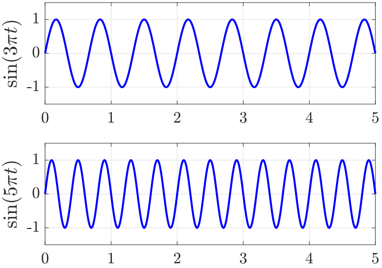
$$\begin{aligned} z(t + lT_1) &= x_1(t + lT_1) + x_2(t + lT_1) \\ &= x_1(t) + x_2(t + kT_2) && \text{(periodicity of } x_1\text{)} \\ &= x_1(t) + x_2(t) && \text{(periodicity of } x_2\text{)} \\ &= z(t) \end{aligned}$$

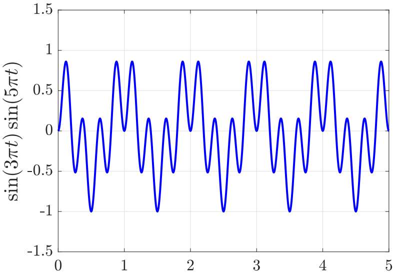
so z is periodic with period lT_1. Similar calculation shows that $z(t + kT_2) = z(t)$.

Periodicity of products of signals

- the same idea works for products of signals!

Theorem 2.2. Let x_1, x_2 be periodic CT signals with fundamental periods T_1 and T_2. If T_1/T_2 is rational, i.e., if $T_1/T_2 = k/l$ for some positive integers k, l, then $x_1 \odot x_2$ is periodic with period $T = lT_1 = kT_2$.





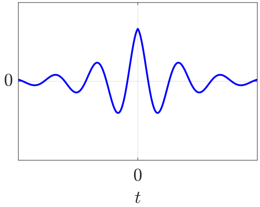
Signal support

Definition 2.5. The ***support*** $\mathrm{supp}(x)$ of a signal x is

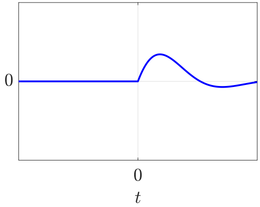
- **CT case:** the smallest closed set containing $\{t \in \mathbb{R} \mid x(t) \neq 0\}$
- **DT case:** the set $\{n \in \mathbb{Z} \mid x[n] \neq 0\}$.

Basic Idea: The support tells you where the signal is non-zero.

$\mathrm{supp}(x) = \mathbb{R}$



$\mathrm{supp}(x) = \{t \in \mathbb{R} \mid t \geq 0\}$



Right-sided signals

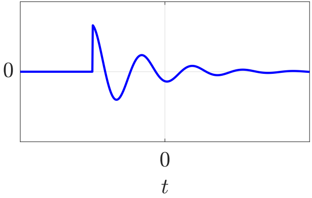
- in many applications signals "begin" at some time, then continue

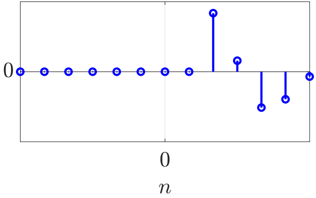
Definition 2.6. A CT signal x is

- ***right-sided from time T*** if $x(t) = 0$ for all $t < T$
- ***right-sided*** if it is right-sided from time T, for some T.

A DT signal x is

- ***right-sided from time N*** if $x[n] = 0$ for all $n < N$
- ***right-sided*** if it is right-sided from time N, for some N.



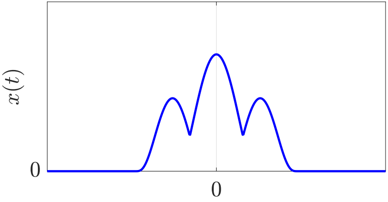


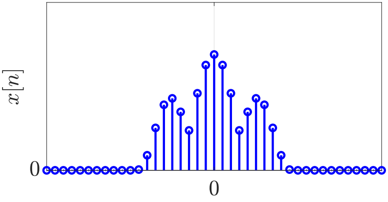
Finite-duration signals

- while our signals are defined for all $t \in \mathbb{R}$ or all $n \in \mathbb{Z}$, in practice, we often measure a signal over a *finite duration* ...

Definition 2.7. A signal x is of ***finite duration*** if it equals zero outside some bounded time interval. Otherwise, the signal is of ***infinite duration***.

- put differently, $\mathrm{supp}(x)$ is contained in a bounded interval



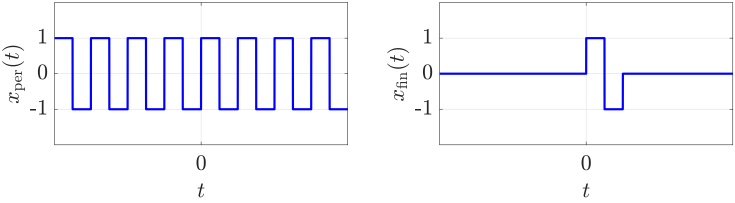


Finite-duration signals $\iff$ periodic signals

Periodic signals $\Longrightarrow$ finite-duration signals (CT)

- let x_{per} be a periodic signal with fund. period T_0. Then we can define a finite-duration signal

$$x_{\text{fin}}(t) = \begin{cases} x_{\text{per}}(t) & \text{if } 0 \leq t < T_0 \\ 0 & \text{otherwise} \end{cases}$$



Nothing fancy; we just cut the periodic signal off after one period.

Periodic signals $\Longleftarrow$ finite-duration signals (CT)

- now let x_{fin} be a finite-duration signal with $\text{supp}(x_{\text{fin}}) \subseteq (0, T_0]$ for some $T_0 > 0$, and define

$$x_{\text{per}}(t) = \sum_{k=-\infty}^{\infty} x_{\text{fin}}(t - kT_0)$$

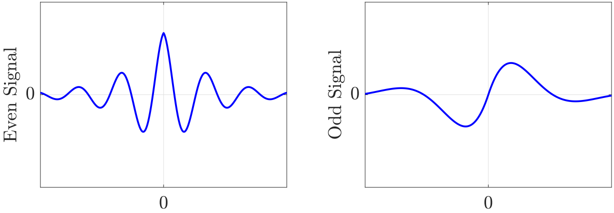
- ***claim:*** x_{per} is periodic with period T_0.
- to see this, we can calculate that

$$\begin{aligned} x_{\text{per}}(t + T_0) &= \textstyle\sum_{k=-\infty}^{\infty} x_{\text{fin}}(t + T_0 - kT_0) \\ &= \textstyle\sum_{k=-\infty}^{\infty} x_{\text{fin}}(t - \underbrace{(k-1)}_{m} T_0) \\ &= \textstyle\sum_{m=-\infty}^{\infty} x_{\text{fin}}(t - mT_0) \\ &= x_{\text{per}}(t)! \end{aligned}$$

Even and odd signals

Definition 2.8. A CT signal x is ***even*** if $x(t) = x(-t)$ for all $t \in \mathbb{R}$, and is ***odd*** if $x(t) = -x(-t)$ for all $t \in \mathbb{R}$.

- the corresponding definition for DT signals is obvious



- we can decompose ***any*** signal x as $x = x_{\mathrm{even}} + x_{\mathrm{odd}}$, where

$$x_{\mathrm{even}}(t) = \frac{x(t) + x(-t)}{2}, \quad x_{\mathrm{odd}}(t) = \frac{x(t) - x(-t)}{2}$$

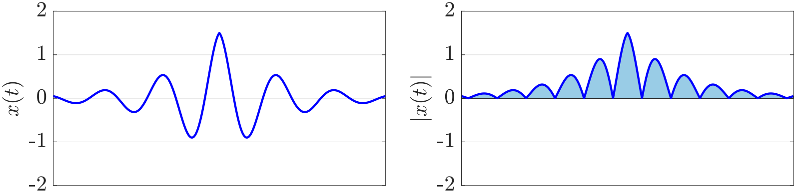
The size of a signal: action, energy, amplitude

- **question:** how "big" is a given signal? There are many ***completely distinct*** ways to answer this question, all useful in different contexts.

Definition 2.9. The ***action*** $\|x\|_1$ of a signal x is defined as

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)| \, \mathrm{d}t, \qquad \|x\|_1 = \sum_{n=-\infty}^{\infty} |x[n]|.$$

If the action is finite, then we write $x \in \mathsf{L}_1$ (CT case) or $x \in \ell_1$ (DT case).



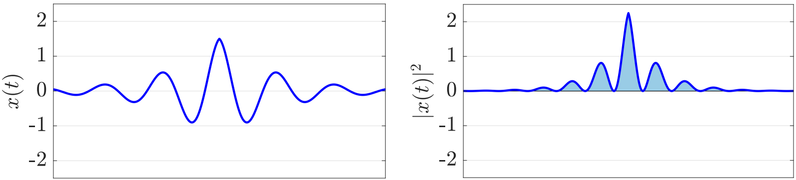
The size of a signal: action, energy, amplitude

Definition 2.10. The ***energy*** $\|x\|_2^2$ of a signal x is defined as

$$\|x\|_2^2 = \int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t, \qquad \|x\|_2^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2.$$

If the energy is finite, then we write $x \in \mathsf{L}_2$ (CT case) or $x \in \ell_2$ (DT case).

- **intuition:** if x represents electrical current, then $|x(t)|^2$ is proportional to power, and we integrate to obtain energy



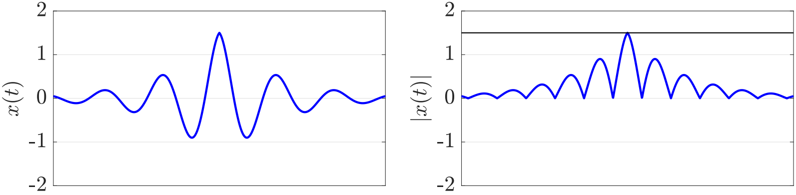
The size of a signal: action, energy, amplitude

Definition 2.11. The ***amplitude*** $\|x\|_\infty$ of a signal x is defined as

$$\|x\|_\infty = \max_{t\in\mathbb{R}} |x(t)| \qquad\qquad \|x\|_\infty = \max_{n\in\mathbb{Z}} |x[n]|.$$

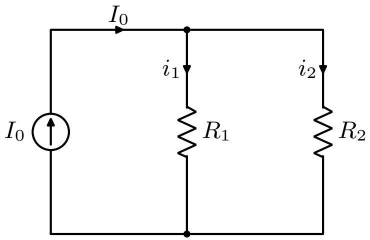
If the amplitude is finite, then we write $x \in \mathsf{L}_\infty$ (CT) or $x \in \ell_\infty$ (DT).

- finite amplitude signals are also called ***bounded*** signals



- **note:** we should write "sup" instead of max, but we won't worry about it.

Example: action, energy, amplitude in a circuit



The current i_1 through R_1 is

$$i_1(t) = \frac{R_2}{R_1 + R_2} I_0(t)$$

Suppose that $I_0(t) = e^{-2t}$ for $t \geq 0$ and $I_0(t) = 0$ for $t < 0$

$$\|i_1\|_1 = \tfrac{R_2}{R_1+R_2} \int_0^\infty e^{-2t}\,\mathrm{d}t = \tfrac{1}{2}\tfrac{R_2}{R_1+R_2}$$

$$\|i_1\|_2^2 = \left(\tfrac{R_2}{R_1+R_2}\right)^2 \int_0^\infty e^{-4t}\,\mathrm{d}t = \tfrac{1}{4}\left(\tfrac{R_2}{R_1+R_2}\right)^2$$

$$\|i_1\|_\infty = \max_{t \geq 0} \tfrac{R_2}{R_1+R_2} e^{-2t} = \tfrac{R_2}{R_1+R_2}$$

What's with the "L" notation?

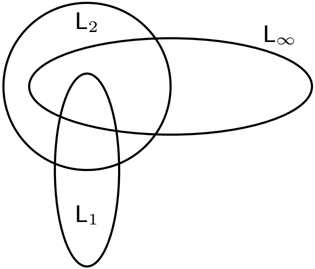
- the sets $\mathsf{L}_1, \mathsf{L}_2, \mathsf{L}_\infty$ are actually ***vector spaces*** of ***signals***

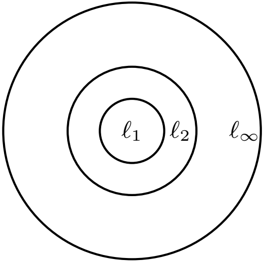
$$\mathsf{L}_1 = \{f : \mathbb{R} \to \mathbb{C} \mid \|f\|_1 \text{ is finite}\}, \qquad \mathsf{L}_2 = \cdots, \qquad \text{etc.}$$

- you can find a bit more on the vector space perspective in the appendix, and in the supplementary textbooks
- this provides a powerful and far-reaching viewpoint for signal analysis, but is a bit beyond our overall scope in ECE216
- ***FYI:*** the "L" stands for ***Lebesgue***, the French mathematician who formalized these ideas

Relationships between action, energy, amplitude

> Do these sets of signals "overlap"? Yes, but only partially, and the relationships between them are a bit complicated.





- curious minds can find more information in the handout on Quercus

Action, energy, amplitude for periodic signals

- our definitions so far are for *non-periodic* signals
- if x is periodic, simply restrict the sums and integrals to one period

$$\|x\|_1 = \int_0^{T_0} |x(t)|\,\mathrm{d}t \quad \|x\|_2^2 = \int_0^{T_0} |x(t)|^2\,\mathrm{d}t \quad \|x\|_\infty = \max_{t\in\mathbb{R}} |x(t)|$$

$$\|x\|_1 = \sum_{n=0}^{N_0-1} |x[n]| \quad \|x\|_2^2 = \sum_{n=0}^{N_0-1} |x[n]|^2 \quad \|x\|_\infty = \max_{n\in\mathbb{Z}} |x[n]|$$

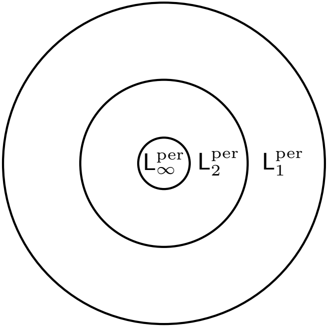
- to distinguish the periodic and non-periodic cases, we use the notation

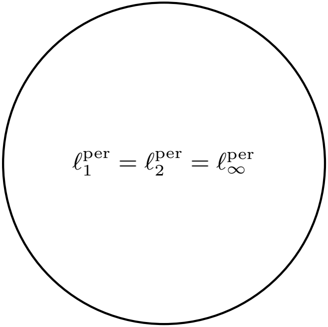
$$\mathsf{L}_1^{\mathrm{per}},\ \mathsf{L}_2^{\mathrm{per}},\ \mathsf{L}_\infty^{\mathrm{per}} \qquad \ell_1^{\mathrm{per}},\ \ell_2^{\mathrm{per}},\ \ell_\infty^{\mathrm{per}}$$

for periodic signals with finite action, energy, and amplitude

Action, energy, amplitude for periodic signals

The inclusion relationships between the sets of signals is a bit simpler for periodic signals.

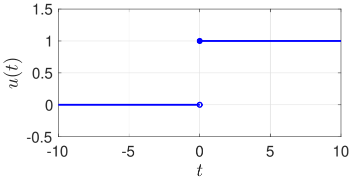




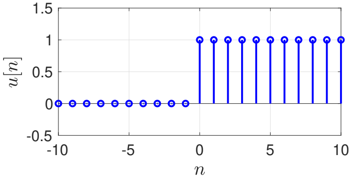
Special signals: the CT and DT unit steps

- the CT ***unit step*** $u(t)$ and DT ***unit step*** $u[n]$

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$



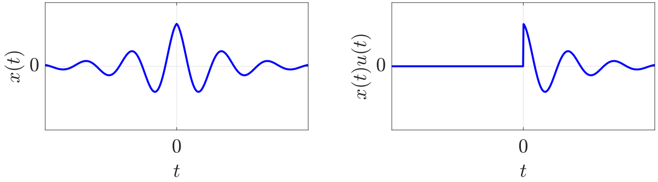
$$u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$



- also known as the ***Heaviside*** function

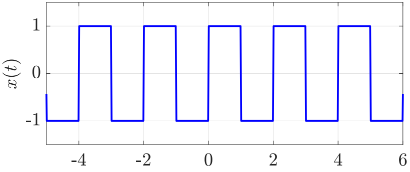
Special signals: the CT and DT unit steps

- multiplying by the step creates a signal that is right-sided



- the unit step is useful for *building* more complex signals, e.g.,

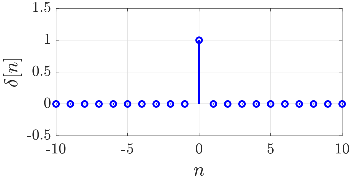
$$x(t) = \sum_{k=-\infty}^{\infty} (-1)^k (u(t-k) - u(t-1-k))$$



Special signals: the DT unit impulse

- the DT ***unit impulse*** $\delta[n]$ is the signal

$$\delta[n] = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases}$$



- shifted signal $\delta[n-k]$ places the impulse at time $k \in \mathbb{Z}$
- useful for building new signals, e.g.,

$$x[n] = \delta[n] + 2\delta[n-1] + 10\delta[n-3] + \cdots$$

- ***exercise:*** convince yourself that the following relationships hold:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k], \qquad \delta[n] = u[n] - u[n-1].$$

Special signals: the DT unit impulse

- for any DT signal x, we have the so-called ***sifting formula***:

$$\text{for any time } n_0 \in \mathbb{Z}: \quad \sum_{n=-\infty}^{\infty} x[n]\delta[n-n_0] = x[n_0].$$

- the proof is by direct calculation:

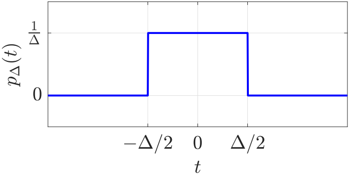
$$\begin{aligned}\sum_{n=-\infty}^{\infty} \delta[n-n_0]x[n] &= \cdots + (0)x[n_0-1] + (1)x[n_0] + (0)x[n_0+1] + \cdots \\ &= x[n_0]\end{aligned}$$

Multiplying any DT signal by a DT impulse and summing over all time "picks out" the value of the signal at the location of the impulse.

Special signals: the CT unit pulse

- the CT ***symmetric unit pulse of duration*** $\boldsymbol{\Delta > 0}$ is the signal

$$p_\Delta(t) = \begin{cases} \frac{1}{\Delta} & \text{if } -\frac{\Delta}{2} \leq t < \frac{\Delta}{2} \\ 0 & \text{otherwise} \end{cases}$$



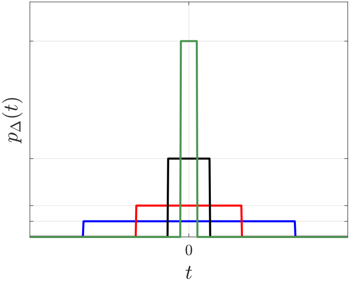
- **note:** the ***area*** under the unit pulse is always equal to one, since

$$\int_{-\infty}^{\infty} p_\Delta(t)\,\mathrm{d}t = \int_{-\Delta/2}^{\Delta/2} \frac{1}{\Delta}\,\mathrm{d}t = 1$$

Fun to think about: what would happen if we made this pulse *shorter and shorter*?

Special signals: the CT unit impulse

- the CT ***unit impulse*** $\boldsymbol{\delta(t)}$ is defined as $\delta(t) = \lim_{\Delta \to 0} p_\Delta(t)$



Main Idea: $\delta(t)$ is an ***idealized*** pulse at $t = 0$ which is *very* fast and *very* large in size.

- **Note:** $\delta(t)$ has unit area, since

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} \lim_{\Delta \to 0} p_\Delta(t)\,\mathrm{d}t = \lim_{\Delta \to 0} \int_{-\infty}^{\infty} p_\Delta(t)\,\mathrm{d}t = 1$$

- shifting by t_0, i.e., $\delta(t - t_0)$, places the impulse at time t_0

Sifting property of the CT unit impulse

Fact: $x(t)\delta(t) = x(0)\delta(t)$ for any signal x that is continuous at $t = 0$.

Proof: By definition, we have

$$x(t)\delta(t) = \lim_{\Delta \to 0} p_\Delta(t)x(t) = \begin{cases} \lim_{\Delta \to 0} \frac{1}{\Delta}x(t) & \text{if } -\frac{\Delta}{2} \leq t \leq \frac{\Delta}{2} \\ 0 & \text{otherwise.} \end{cases}$$

As $\Delta \to 0$, $x(t)$ over $-\frac{\Delta}{2} \leq t \leq \frac{\Delta}{2}$ will get closer and closer to $x(0)$, so

$$x(t)\delta(t) = x(0) \lim_{\Delta \to 0} p_\Delta(t) = x(0)\delta(t).$$

- **extension:** we have $x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$, which leads to

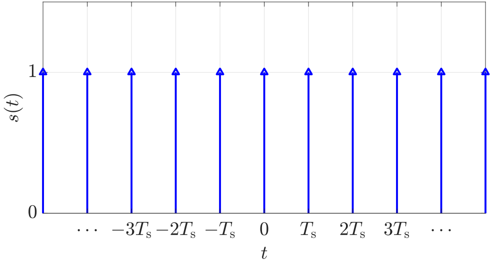
"Sifting formula": for any time $t_0 \in \mathbb{R}$: $\displaystyle\int_{-\infty}^{\infty} x(t)\delta(t - t_0)\,\mathrm{d}t = x(t_0).$

Plotting the CT unit impulse

- we can't plot the CT impulse like a normal signal
- nonetheless, it's ***useful*** to plot impulses by drawing vertical arrows
- for example, for a constant $T_\mathrm{s} > 0$, we can plot

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_\mathrm{s})$$

as



Rules for working with the CT impulse

- **question:** if x is a continuous CT signal, should

$$\int_0^\infty x(t)\delta(t)\,\mathrm{d}t \quad \text{equal } 0? \quad \text{equal } x(0)?$$

- **answer:** The expression is ambiguous; we must avoid it!

- to fix this, we let 0^- and 0^+ be values *infinitesimally* to the left and to the right of 0, i.e., $0^- < 0 < 0^+$. Then, formally, we write

$$\int_{0^-}^\infty x(t)\delta(t)\,\mathrm{d}t = x(0^-) = \lim_{t\uparrow 0} x(t) \qquad \int_{-\infty}^{0^+} x(t)\delta(t)\,\mathrm{d}t = x(0^+) = \lim_{t\downarrow 0} x(t)$$

- we will consider these formulas valid even if $x(0)$ has a jump at $t = 0$, as long as the appropriate left or right limit exists.

Final comments on the CT unit impulse

- the CT impulse is not a "normal" signal, like the ones you are used to from studying calculus; it is known as a *generalized signal*

- the sifting property can be thought of as the defining property of the CT impulse; we understand $\delta(t)$ *via how it acts under an integral*, rather than by the *values* it takes at any particular $t \in \mathbb{R}$

- we (sadly) can't do some rather elementary things with $\delta(t)$. For instance, the expression $\delta(t)^2$ is meaningless, as is asking "What is the value of $\delta(t)$ at $t = 0$?"

- remarkably, ***we can still do calculus*** with $\delta(t)$, and in fact

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau, \qquad \delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}u(t).$$

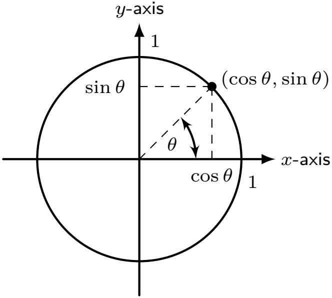
Special signals: sinusoidal signals

- **recall:** some basic trigonometry on the ***unit circle***

$$\cos\theta = \frac{\text{adj.}}{\text{hyp.}} = \text{adj.} = x \text{ coord.}$$

$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} = \text{opp.} = y \text{ coord.}$$

$$(x, y) = (\cos\theta, \sin\theta)$$



- ***sinusoidal signals*** are obtained by making θ a linear function of time

$$\theta(t) = \omega_0 t, \qquad x(t) = \cos(\omega_0 t), \qquad y(t) = \sin(\omega_0 t)$$

Special signals: sinusoidal signals

$$x - \text{coordinate}: \quad x(t) = \cos(\theta(t)) = \cos(\omega_0 t) = \cos(2\pi f_0 t)$$
$$y - \text{coordinate}: \quad y(t) = \sin(\theta(t)) = \sin(\omega_0 t) = \sin(2\pi f_0 t)$$

- larger ω_0 leads to faster revolution around the circle
- both signals are periodic with fundamental period $T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f_0}$

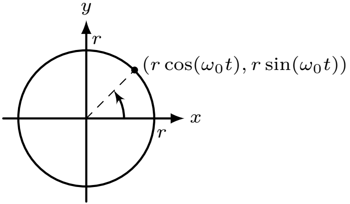
$$\sin(\omega_0(t + T_0)) = \sin(\omega_0 t + \omega_0 T_0) = \sin(\omega_0 t + 2\pi) = \sin(\omega_0 t).$$

- if the circle instead has radius r, we simply change things to

$$x(t) = r\cos(\omega_0 t), \qquad y(t) = r\sin(\omega_0 t)$$

Special signals: sinusoidal signals

$$x(t) = r\cos(\omega_0 t)$$
$$y(t) = r\sin(\omega_0 t)$$



- at $t = 0$, $(x, y) = (r, 0)$ $\Rightarrow$ ***initial angle*** with the x-axis is zero.
- for an initial angle equal to ϕ, we can add a ϕ as a ***phase shift***

$$x(t) = r\cos(\omega_0 t + \phi), \qquad y(t) = r\sin(\omega_0 t + \phi)$$

- **note:** if $\phi < 0$, this ***delays*** the sinusoid by $\tau = -\phi/\omega_0$ seconds
- therefore, *phase shifting* is just another word for ***time shifting*** of sinusoidal signals

Complex numbers

- a ***complex number*** $z \in \mathbb{C}$ is typically written as $z = x + \mathbf{j}y$ where
  - $x = \mathrm{Re}\{z\} \in \mathbb{R}$ is the ***real part*** of z
  - $y = \mathrm{Im}\{z\} \in \mathbb{R}$ is the ***imaginary part*** of z
  - $\mathbf{j} = \sqrt{-1}$, i.e., $\mathbf{j}^2 = -1$
- this is called the ***Cartesian*** representation of a complex number, because we think of this as defining a point (x, y) in the plane.
- the distance from the point (x, y) to the origin is the ***magnitude***

$$r \triangleq |z| = \sqrt{\mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2} = \sqrt{x^2 + y^2}$$

- the angle (in radians) made with the positive x-axis is the ***phase***

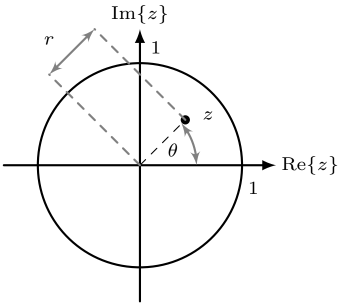
$$\theta \triangleq \angle z = \begin{cases} \arctan \frac{\mathrm{Im}\{z\}}{\mathrm{Re}\{z\}} & \text{if } \mathrm{Re}\{z\} \geq 0 \\ \arctan \frac{\mathrm{Im}\{z\}}{\mathrm{Re}\{z\}} + \pi & \text{if } \mathrm{Re}\{z\} < 0 \end{cases}$$

Complex numbers: polar representation

- magnitude and phase give us the ***polar*** representation of z:

$$z = x + \mathbf{j}y = r\cos(\theta) + \mathbf{j}r\sin(\theta) = r(\cos\theta + \mathbf{j}\sin\theta).$$

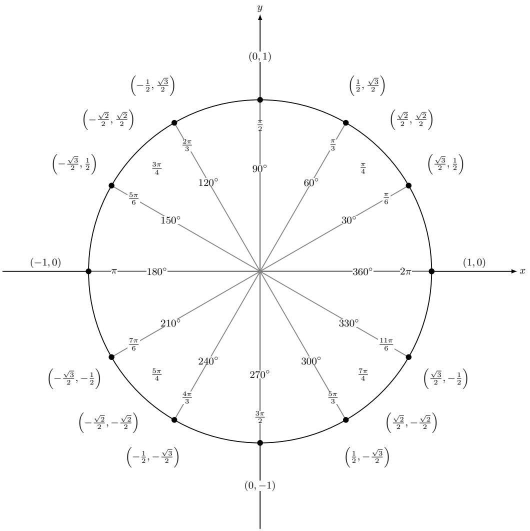
- ***Euler's Relation:*** $\cos(\theta) + \mathbf{j}\sin(\theta) = e^{\mathbf{j}\theta}$
- therefore, we can write any z as $\boldsymbol{z = re^{\mathbf{j}\theta}}$ (*polar representation*)



Some useful special values:

$$e^{\mathbf{j}0} = 1$$
$$e^{\mathbf{j}\frac{\pi}{2}} = \mathbf{j}$$
$$e^{\mathbf{j}\pi} = e^{-\mathbf{j}\pi} = -1$$
$$e^{\mathbf{j}\frac{3\pi}{2}} = e^{-\mathbf{j}\frac{\pi}{2}} = -\mathbf{j}$$

Common angles on the unit circle



Complex numbers: Cartesian vs. polar representation

- the Cartesian representation makes ***addition*** easy. If we have $z_1 = x_1 + \mathbf{j}y_1$ and $z_2 = x_2 + \mathbf{j}y_2$ then

$$z_1 + z_2 = (x_1 + x_2) + \mathbf{j}(y_1 + y_2)$$

"the real and imaginary parts add"

- the polar representation makes ***multiplication*** easy. If we have $z_1 = r_1 e^{\mathbf{j}\theta_1}$ and $z_2 = r_2 e^{\mathbf{j}\theta_2}$, then

$$z_1 z_2 = r_1 r_2 e^{\mathbf{j}\theta_1} e^{\mathbf{j}\theta_2} = r_1 r_2 e^{\mathbf{j}(\theta_1 + \theta_2)}$$

"the magnitudes multiply and the phases add"

Complex numbers: Cartesian vs. polar representation

- like multiplication, the polar representation makes ***division*** easy. If we have $z_1 = r_1 e^{\mathbf{j}\theta_1}$ and $z_2 = r_2 e^{\mathbf{j}\theta_2}$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{e^{\mathbf{j}\theta_1}}{e^{\mathbf{j}\theta_2}} = \frac{r_1}{r_2}e^{\mathbf{j}\theta_1}e^{-\mathbf{j}\theta_2} = \frac{r_1}{r_2}e^{\mathbf{j}(\theta_1-\theta_2)}$$

- therefore we have that

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}, \qquad \angle\frac{z_1}{z_2} = \theta_1 - \theta_2 = \angle z_1 - \angle z_2$$

"the magnitudes divide and the phases subtract"

Complex numbers: the complex conjugate

- the ***complex conjugate*** z^* of $z = x + \mathbf{j}y$ is $z^* = x - \mathbf{j}y$
- useful for expressing the magnitude:

$$\begin{aligned} zz^* &= (x + \mathbf{j}y)(x - \mathbf{j}y) \\ &= x^2 + \mathbf{j}(xy - yx) - \mathbf{j}^2 y^2 \\ &= x^2 + y^2 \\ &= |z|^2 \end{aligned}$$

- in the polar representation $z = re^{\mathbf{j}\theta} = r\cos(\theta) + \mathbf{j}r\sin(\theta)$, we have

$$z^* = r\cos(\theta) - r\mathbf{j}\sin(\theta) = r\cos(-\theta) + r\mathbf{j}\sin(-\theta) = re^{-\mathbf{j}\theta}$$

since $\cos$ is even and $\sin$ is odd. Therefore (again)

$$zz^* = re^{\mathbf{j}\theta}re^{-\mathbf{j}\theta} = r^2 e^{\mathbf{j}(\theta-\theta)} = r^2 = |z|^2.$$

Complex numbers: the complex conjugate

- **note:** we can write z and z^* as

$$z = \mathrm{Re}\{z\} + \mathbf{j}\mathrm{Im}\{z\}, \qquad z^* = \mathrm{Re}\{z\} - \mathbf{j}\mathrm{Im}\{z\}$$

- adding and subtracting the two equations leads to

$$\mathrm{Re}\{z\} = \frac{z + z^*}{2}, \qquad \mathrm{Im}\{z\} = \frac{z - z^*}{2\mathbf{j}}$$

- thus, we can express the real and imaginary parts of z using z and z^*

CT complex exponential signals

- to generate a signal, we can take our complex number $re^{\mathbf{j}\theta}$ and now let $\theta(t) = \omega_0 t$ be a linear function of time with angular freq. ω_0

- this defines a ***complex exponential signal*** $z : \mathbb{R} \to \mathbb{C}$ taking values

$$z(t) = e^{\mathbf{j}\omega_0 t} = \cos(\omega_0 t) + \mathbf{j}\sin(\omega_0 t)$$

  with real and imaginary parts

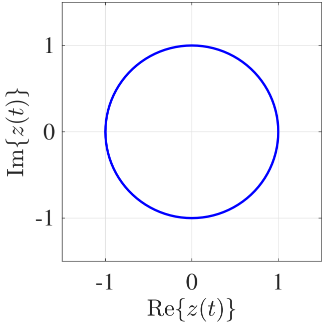
$$\mathrm{Re}\{z(t)\} = \cos(\omega_0 t)$$
$$\mathrm{Im}\{z(t)\} = \sin(\omega_0 t)$$

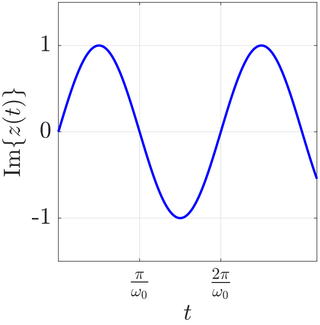
- **note:** we can equivalently write the real and imaginary parts as

$$\mathrm{Re}\{z(t)\} = \frac{e^{\mathbf{j}\omega_0 t} + e^{-\mathbf{j}\omega_0 t}}{2}, \qquad \mathrm{Im}\{z(t)\} = \frac{e^{\mathbf{j}\omega_0 t} - e^{-\mathbf{j}\omega_0 t}}{2\mathbf{j}}.$$

CT complex exponential signals

$$z(t) = e^{\mathbf{j}\omega_0 t} = \cos(\omega_0 t) + \mathbf{j}\sin(\omega_0 t), \qquad T_0 = \frac{2\pi}{\omega_0}$$





Why do we care about complex exponential signals?

1. they are fundamental building blocks for more general signals
2. they are mathematically easy to manipulate; differentiation and integration again yield complex exp. signals
3. they simplify many formulas vs. using $\sin$ and $\cos$
4. the "frequency spectrum" of such a signal is very simple; all energy is concentrated at frequency ω_0

Key relationships

$$e^{\mathbf{j}\omega_0 t} = \cos(\omega_0 t) + \mathbf{j}\sin(\omega_0 t)$$

$$\cos(\omega_0 t) = \frac{e^{\mathbf{j}\omega_0 t} + e^{-\mathbf{j}\omega_0 t}}{2} \qquad \sin(\omega_0 t) = \frac{e^{\mathbf{j}\omega_0 t} - e^{-\mathbf{j}\omega_0 t}}{2\mathbf{j}}.$$

DT complex exponential signals

- DT complex exponential signals are *superficially* similar to their CT cousins, but now the time index takes only *integer values*
- ***this produces very important differences with the CT case***
- **recall:** our definition of the CT complex exponential signal:

$$e^{\mathbf{j}\omega t} = \cos(\omega t) + \mathbf{j}\sin(\omega t)$$

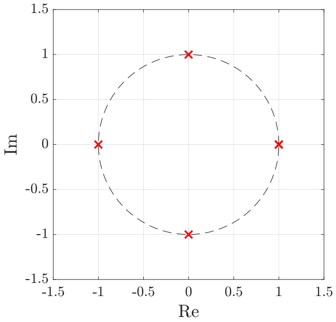
- let's now ***sample*** this signal with sampling period $T_{\mathrm{s}} = 1$

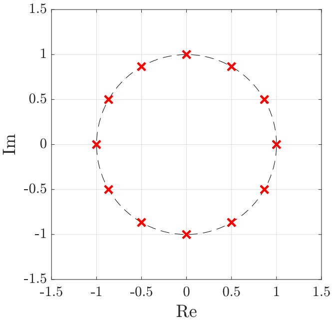
$$x[n] = e^{\mathbf{j}\omega(n\cdot 1)} = \cos(\omega n) + \mathbf{j}\sin(\omega n)$$

- while $x(t) = e^{\mathbf{j}\omega t}$ ***rotates smoothly*** around the unit circle, $x[n] = e^{\mathbf{j}\omega n}$ ***jumps*** from point to point around the circle.

Complex exponential DT signals $x[n] = e^{\mathbf{j}\omega n}$

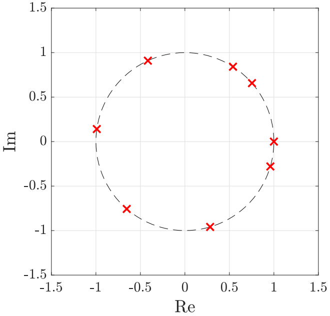
- **example:** suppose that $\omega = \pi/2$. Then $x[n] = e^{\mathbf{j}\frac{\pi}{2}n}$, which has the values $\{1, \mathbf{j}, -1, -\mathbf{j}, 1\}$ for $n \in \{0, 1, 2, 3, 4\}$. We are hopping in steps of $90°$ around the circle, the signal is periodic, and the fundamental period is $N_0 = 4$.
- **example:** suppose that $\omega = 5\pi/6$. The values of this for $n \in \{0, 1, 2, \ldots\}$ are plotted below. When $n = 12$, we get $e^{\mathbf{j}10\pi} = 1$, so we come back to the initial point. The signal is periodic with fundamental period $N_0 = 12$.





Complex exponential DT signals $x[n] = e^{\mathbf{j}\omega n}$

- **example:** suppose that $\omega = 1$. Then $x[n] = e^{\mathbf{j}n}$, so we are hopping by one radian each time we increase n. However, if we hop by one radian, we will ***never*** again hop back to an integer multiple of 2π, because π is irrational. Therefore ***this complex exponential is not periodic***.



$x[n] = e^{\mathbf{j}\omega n}$ can be periodic or aperiodic depending on ω!

Time-periodicity of complex exponential DT signals

So ... for what choices of ω is $x[n] = e^{\mathbf{j}\omega n}$ periodic?

- periodicity with period $N \in \mathbb{Z}_{\geq 1}$ requires that

$$e^{\mathbf{j}\omega n} = e^{\mathbf{j}\omega(n+N)} \qquad \text{for all } n \in \mathbb{Z}.$$

- dividing both sides by $e^{\mathbf{j}\omega n} \neq 0$ we find that $1 = e^{\mathbf{j}\omega N}$ or equivalently

$$e^{\mathbf{j}2\pi k} = e^{\mathbf{j}\omega N} \qquad \text{for any integer } k.$$

- it follows that we must have

$$2\pi k = \omega N \quad \Longleftrightarrow \quad \omega = k\frac{2\pi}{N}.$$

Conclusion: $e^{\mathbf{j}\omega n}$ is periodic with period N if the frequency ω is an ***integer multiple*** of $2\pi/N$

Frequency-periodicity of DT complex exponentials $e^{\mathbf{j}\omega n}$

- DT complex exponentials have *another* ***completely different*** periodicity property
- for any time n, note that

$$e^{\mathbf{j}(\omega+2\pi)n} = e^{\mathbf{j}\omega n} e^{\mathbf{j}2\pi n} = e^{\mathbf{j}\omega n}.$$

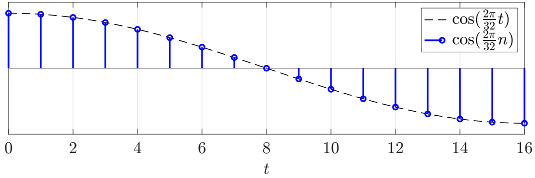
conclusion: $e^{\mathbf{j}\omega n}$ is always a ***periodic function of*** ω, with period 2π. We say that DT complex exponentials are "periodic in frequency".

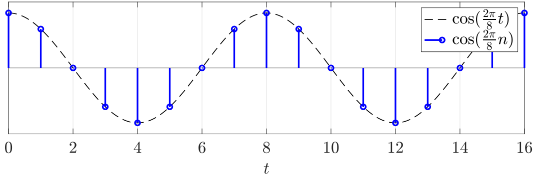
- **example:** $e^{\mathbf{j}\frac{5}{6}\pi n}$ and $e^{\mathbf{j}\frac{17}{6}\pi n} = e^{\mathbf{j}(\frac{5}{6}\pi+2\pi)n}$ are the *exact same* signal

A "larger" frequency does not *necessarily* mean faster oscillations!

Frequency-periodicity of DT complex exponentials $e^{\mathbf{j}\omega n}$

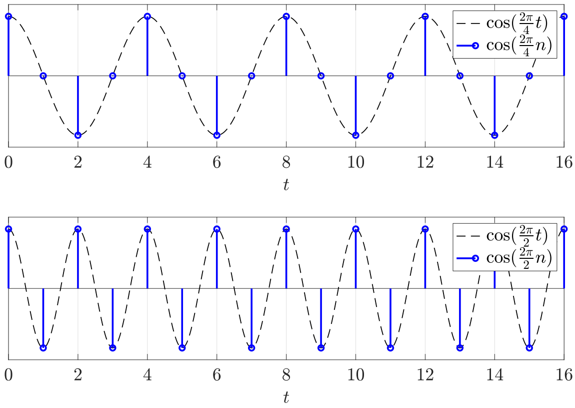
- let's plot out $\cos(\omega n) = \mathrm{Re}\{e^{\mathbf{j}\omega n}\}$ as ω increases from 0 to 2π





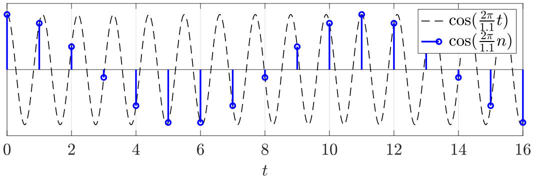
When ω is between 0 and π, increasing ω increases the rate of oscillation of the signal $e^{\mathbf{j}\omega n} = \cos(\omega n) + \mathbf{j}\sin(\omega n)$

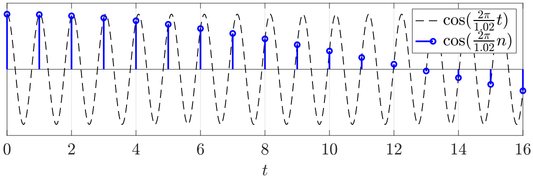
Frequency-periodicity of DT complex exponentials $e^{\mathbf{j}\omega n}$



When $\omega = \pi$, we obtain $e^{\mathbf{j}\pi n} = (-1)^n$, so the signal jumps back and forth between $+1$ and -1; this is the ***fastest*** that a signal can oscillate in discrete-time.

Frequency-periodicity of DT complex exponentials $e^{\mathbf{j}\omega n}$



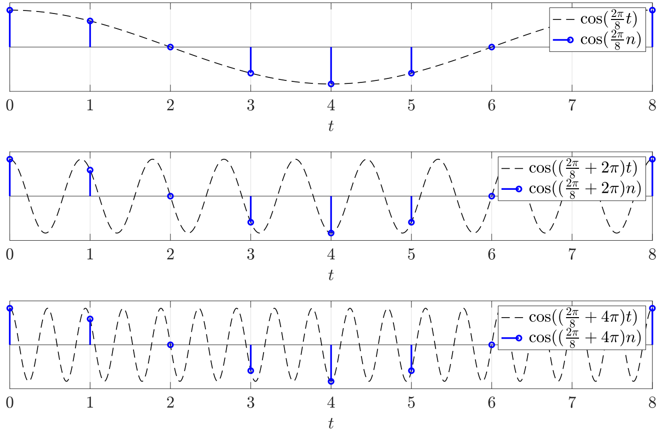


As ω increases beyond π, the oscillations become slower and slower as the frequency approaches $\omega = 2\pi$, since $e^{\mathbf{j}(2\pi)n} = 1$ (no oscillation at all).

Conclusion: a DT complex exponential will oscillate slowly when ω is near an ***even*** multiple of π, and quickly when ω is near an ***odd*** multiple of π.

Key consequence of frequency-periodicity: aliasing

Different continuous-time signals, when sampled, can produce the ***exact same*** discrete-time signal; this phenomenon is called aliasing.



Family of distinct DT complex exponential signals

- **Summary:**
  (i) $e^{\mathbf{j}\omega n}$ is periodic in n with period N iff $\omega = k\frac{2\pi}{N}$ for some $k \in \mathbb{Z}$
  (ii) $e^{\mathbf{j}\omega n}$ is always periodic in ω with period 2π

Theorem 2.3. Let $N_0 \in \mathbb{Z}_{\geq 1}$ be a desired period. Then there are exactly N_0 distinct DT complex exponential signals of period N_0, given by

$$\phi_k[n] = e^{\mathbf{j}k\omega_0 n}, \qquad k \in \{0, 1, \ldots, N_0 - 1\}$$

where $\omega_0 = 2\pi/N_0$.

Therefore, when working with DT exponentials of period N_0, there is only a finite set of "building blocks"

Proof of Theorem 2.3

First, we know from point (i) that if a DT complex exponential $e^{\mathbf{j}\nu n}$ has period N_0, we must have that $\nu N_0 = 2\pi k$ for some integer $k \in \mathbb{Z}$. Therefore

$$\nu = 2\pi \frac{k}{N_0} = \frac{2\pi}{N_0} k = \omega_0 k$$

We conclude that

$$\phi_k[n] = e^{\mathbf{j}\omega_0 k n}, \qquad k \in \mathbb{Z}.$$

are the only DT complex exponentials which are periodic with period N_0. From point (ii), we know that $e^{\mathbf{j}\omega_0 k n}$ is 2π periodic in the argument $\omega_0 k$, so these signals $\{\phi_k\}_{k\in\mathbb{Z}}$ are not all distinct. We have that $\phi_0[n] = 1$ and $\phi_{N_0}[n] = e^{\mathbf{j}\omega_0 N_0 n} = e^{\mathbf{j}2\pi n} = 1$. So $\phi_0, \phi_1, \ldots, \phi_{N_0-1}$ are distinct, and the rest can be discarded. •

Relevant MATLAB commands

- define and plot signals

```
%Define a "CT" signal
T_max = 6; h=0.001;
t = -T_max:h:T_max;
x = cos(3*t).*exp(-0.2*t) + ...
    heaviside(t+5)-heaviside(t-1);

%Sample the signal with period T_s
T_s = 0.5;
t_s = t(1:T_s/h:end);
x_s = x(1:T_s/h:end);

%Plot both signals
plot(t,x); hold on; stem(t_s,x_s); hold off;
```

Supplementary reading

- **O-W:** A. V. Oppenheim and S. Willsky, *Signals and Systems*, 2nd Ed.
- **BB:** B. Boulet, *Fundamentals of Signals and Systems*.
- **BPL:** B. P. Lathi, *Signal Processing and Linear Systems*.
- **K-S:** H. Kwakernaak and R. Sivan, *Modern Signals and Systems*.
- **EL-PV:** E. Lee and P. Varaiya, *Structure and Interpretation of Signals and Systems*, 2nd Ed.
- **ADL:** A. D. Lewis, *The Mathematical Theory of Signals and Systems*.

Topic	**O-W**	**BB**	**BPL**	**K-S**	**EL-PV**	**ADL**
Signal basics	1.1–1.4	1	1.1–, 8.1–	2.1–2.3	1.1, 2.2, 7.4	V4 1.1
Impulse signal	1.4, 2.5	1	1.4	2.5, App. C	9.1	V4 3.1
Signal size	1.1	1	1.1	2.4		V4 1.2, 1.3
Complex numbers	1.3	App B.	B.1	App. A	App. B	V2 3.1
Complex exp.	1.3	1	B.4	2.2	App. B	V4 1.2, 1.3

Personal Notes

Personal Notes

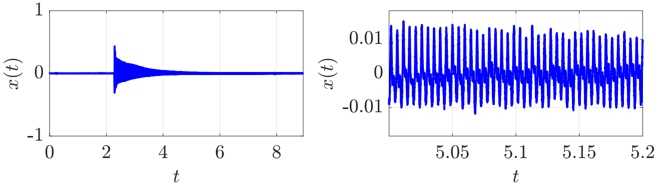
Personal Notes

3. The Fourier Series

- approximating and representing signals
- the cosine Fourier series
- the sine Fourier series
- the continuous-time Fourier series (CTFS)
- CTFS coefficients and signal manipulation
- existence of the CTFS coefficients
- convergence of the CTFS and Gibbs phenomenon
- the discrete-time Fourier series (DTFS)
- application: analysis of an audio signal

Approximating and representing signals

- suppose that we want to analyze (i.e., study, understand, interpret, . . .) the following finite-duration signal x



While x might be complicated, maybe we can express x — either approximately or exactly — as a ***weighted sum*** of many simple *basis* (building block) signals $\{\phi_0, \phi_1, \phi_2, \phi_3, \ldots\}$

Doing so may make analyzing x easier; this is the idea of ***Fourier analysis***

Approximating and representing signals

- given some building block signals $\{\phi_0, \phi_1, \ldots\}$, we can construct a new signal $\hat{x}$ via a ***linear combination***

$$\hat{x} = \alpha_0\phi_0 + \alpha_1\phi_1 + \alpha_2\phi_2 + \cdots = \sum_k \alpha_k\phi_k$$

where $\alpha_k \in \mathbb{C}$ are ***constant*** weighting coefficients

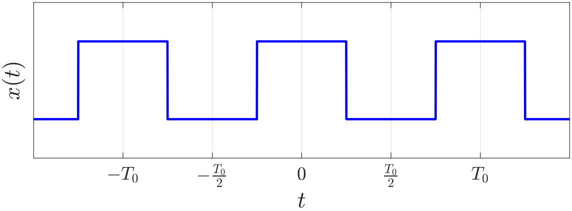
Signal approximation problem: Given a signal x and a set of basis signals $\{\phi_0, \phi_1, \ldots\}$, find the choice of coefficients $\{\alpha_0, \alpha_1, \ldots\}$ such that $\hat{x} = \sum_k \alpha_k\phi_k$ is the best approximation of x.

- if we can find $\{\alpha_0, \alpha_1, \ldots\}$ such that $\hat{x} = x$, then we say that $\hat{x} = x = \sum_k \alpha_k\phi_k$ is a ***representation*** of x in the basis $\{\phi_0, \phi_1, \ldots\}$.

Introductory example: an even periodic signal

- we start with a simplified case, then generalize. Assume that

x is ***real-valued***, ***even***, and ***periodic*** with fundamental period T_0



$$f_0 = \frac{1}{T_0}$$
$$\omega_0 = \frac{2\pi}{T_0}$$

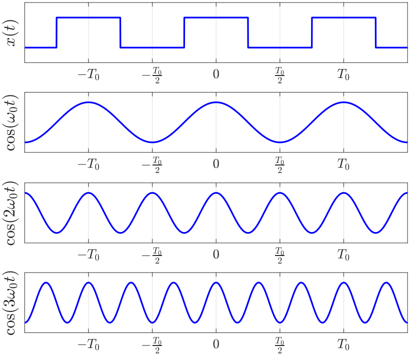
- since x is real, even, and T_0-periodic, it makes sense for our building blocks to *also* be real, even, and T_0-periodic! Let's use

$$\phi_0(t) = \frac{1}{2}, \quad \phi_1(t) = \cos(\omega_0 t), \quad \phi_2(t) = \cos(2\omega_0 t), \quad \ldots$$

Introductory example: an even periodic signal

- we therefore try to approximate x as

$$\hat{x}(t) = \frac{a_0}{2} + \sum_{\ell=1}^{\infty} a_\ell \cos(\ell\omega_0 t), \qquad a_0, a_1, \ldots \text{ are real constants}$$



note: all the cosines of different frequencies

$$\omega_0, 2\omega_0, 3\omega_0, \ldots$$

fit ***perfectly*** within our fundamental period T_0 of x

Introductory example: an even periodic signal

- naively, let's just set $\hat{x}(t) = x(t)$, and try to solve for $\{a_0, a_1, \ldots\}$:

$$x(t) = \frac{a_0}{2} + \sum_{\ell=1}^{\infty} a_\ell \cos(\ell\omega_0 t)$$

- the trick to solving is to use ***integration***

- to solve for a_0, just integrate both sides over one period:

$$\frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\,\mathrm{d}t = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{a_0}{2}\,\mathrm{d}t + \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{\ell=1}^{\infty} a_\ell \cos(\ell\omega_0 t)\,\mathrm{d}t$$

$$= a_0 + \frac{2}{T_0}\sum_{\ell=1}^{\infty} a_\ell \underbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(\ell\omega_0 t)\,\mathrm{d}t}_{=0}$$

- therefore, $a_0 = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\,\mathrm{d}t$

Introductory example: an even periodic signal

- to solve for a_k for $k \geq 1$, first multiply both sides by $\cos(k\omega_0 t)$

$$\cos(k\omega_0 t)x(t) = \cos(k\omega_0 t)\frac{a_0}{2} + \cos(k\omega_0 t)\sum_{\ell=1}^{\infty} a_\ell \cos(\ell\omega_0 t)$$

- next, integrate both sides like before

$$\frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(k\omega_0 t)x(t)\,\mathrm{d}t = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(k\omega_0 t)\left[\frac{a_0}{2} + \sum_{\ell=1}^{\infty} a_\ell \cos(\ell\omega_0 t)\right]\mathrm{d}t$$

$$= 0 + \sum_{\ell=1}^{\infty} a_\ell \underbrace{\frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(k\omega_0 t)\cos(\ell\omega_0 t)\,\mathrm{d}t}_{=?}$$

- let's evaluate this integral separately

Introductory example: an even periodic signal

- **recall:** $2\cos(A)\cos(B) = \cos(A-B) + \cos(A+B)$

$$\frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(k\omega_0 t)\cos(\ell\omega_0 t) = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} [\cos((k-\ell)\omega_0 t) + \cos((k+\ell)\omega_0 t)]\,\mathrm{d}t$$

- if $k \neq \ell$, then both integrals evaluate to ***zero*** over one cycle
- if $k = \ell$, then

$$\frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(k\omega_0 t)\cos(\ell\omega_0 t) = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} [1 + \cos((k+\ell)\omega_0 t)]\,\mathrm{d}t = 1 + 0 = 1$$

"Orthogonality":

$$\frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(k\omega_0 t)\cos(\ell\omega_0 t)\,\mathrm{d}t = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases}$$

Introductory example: an even periodic signal

- completing our calculation then, we find that

$$\frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\,\mathrm{d}t = \sum_{\ell=1}^{\infty} a_\ell \underbrace{\frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(k\omega_0 t)\cos(\ell\omega_0 t)\,\mathrm{d}t}_{=1 \text{ if and only if } \ell=k} = a_k$$

Theorem 3.1. If x is a real-valued, even, and periodic CT signal with fund. period T_0, then we can represent x via the ***cosine Fourier series***

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t), \qquad \omega_0 = \frac{2\pi}{T_0}$$

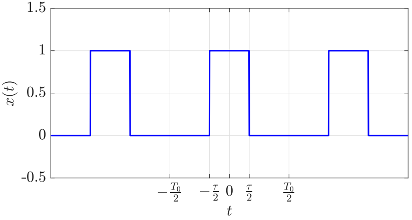
$$a_k = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos(k\omega_0 t)\,\mathrm{d}t, \qquad k \in \{0, 1, 2, \ldots\}.$$

- **note:** The integrals can be taken over ***any interval of length*** T_0

Example: a square wave

For $T_0 > 0$ and $0 < \tau \leq T_0$ consider

$$x_{\text{fin}}(t) = \begin{cases} 1 & \text{if } -\frac{\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$$



- we T_0-*periodize* x_{fin} to generate the signal x in the figure above
- to compute the cosine Fourier series, we first compute a_0

$$a_0 = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\,\mathrm{d}t = \frac{2}{T_0} \int_{-\tau/2}^{\tau/2} (1)\,\mathrm{d}t = \frac{2\tau}{T_0}$$

Example: a square wave

- for $k \geq 1$ we compute that

$$
\begin{aligned}
a_k = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \cos(k\omega_0 t)\, \mathrm{d}t &= \frac{2}{T_0} \int_{-\tau/2}^{\tau/2} \cos(k\omega_0 t)\, \mathrm{d}t \\
&= \frac{2}{T_0 k \omega_0} \left[\sin(k\omega_0\tau/2) - \sin(-k\omega_0\tau/2)\right] \\
&= \frac{2}{\pi k} \sin(k\omega_0\tau/2)
\end{aligned}
$$

  where we used that $\omega_0 T_0 = 2\pi$

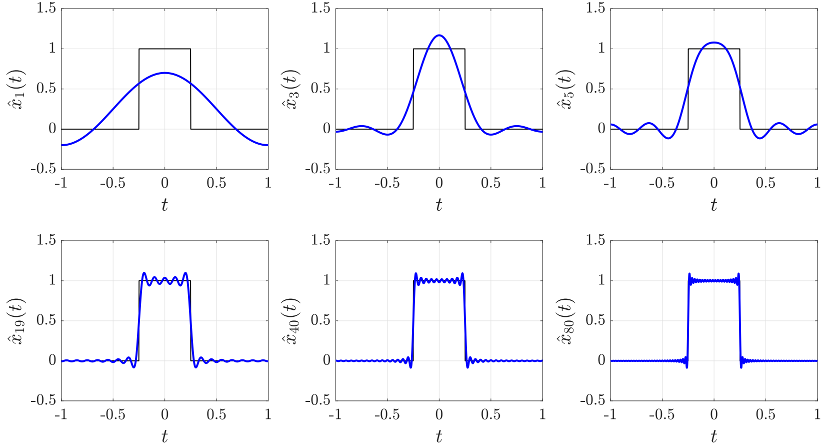
- therefore, we find that x can be ***represented*** as

$$x(t) = \frac{\tau}{T_0} + \sum_{k=1}^{\infty} \frac{2\sin(k\omega_0\tau/2)}{\pi k} \cos(k\omega_0 t)$$

- if we keep only K terms in the sum, we instead get the ***approximation***

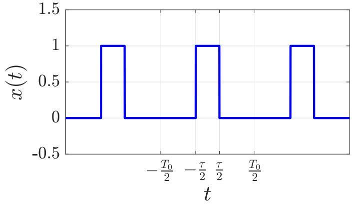
$$\hat{x}_K(t) = \frac{\tau}{T_0} + \sum_{k=1}^{K} \frac{2\sin(k\omega_0\tau/2)}{\pi k} \cos(k\omega_0 t)$$

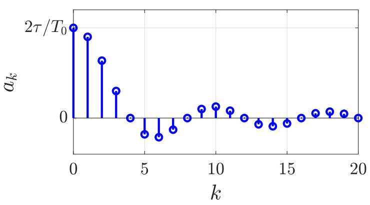
Example: a square wave – approximation quality



More terms = closer approximation of original signal

Example: a square wave – the frequency domain





- the coefficient $a_k = \frac{2}{\pi k}\sin(k\omega_0\tau/2)$ multiplies $\cos(k\omega_0 t)$
- a_k tells us how much the ***harmonic*** $k\omega_0$ contributes to the overall signal $x(t)$
- we can think of $\{a_k\}_{k=0}^{\infty}$ as equivalently representing $x(t)$ in the ***frequency domain*** – the domain consisting of multiples of the fundamental frequency. We call $\{a_k\}_{k=0}^{\infty}$ the ***spectrum*** of x.

Fourier series for odd periodic signals

What if the signal x of interest is periodic and ***odd***?

- as you might guess, we could instead express x as

$$x(t) = \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t), \qquad b_1, b_2, \ldots \text{ are constants}$$

  and follow the exact same procedure!

Theorem 3.2. If x is a real, odd, and periodic CT signal with fund. period T_0, then we can represent x via the ***sine Fourier series***

$$x(t) = \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t), \qquad b_k = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \sin(k\omega_0 t)\, \mathrm{d}t.$$

- **note:** The integral can be taken over ***any interval of length*** T_0

Fourier series for periodic signals

What if the signal x of interest is periodic, but ***not*** even or odd?

- we can just combine the two methods, and write

$$x(t) = \underbrace{\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t)}_{\text{even part of } x} + \underbrace{\sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)}_{\text{odd part of } x}$$

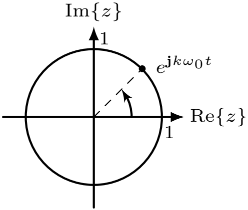
- this is called a ***sine/cosine Fourier series*** of x
- it turns out that working with these formulas is a pain . . . instead, using ***complex exponential signals*** will
  - (i) make our notation shorter and easier to read
  - (ii) allow us to work easily with *complex-valued* periodic signals

The complex exponential Fourier series

starting point: remember that

$$e^{\mathbf{j}k\omega_0 t} = \cos(k\omega_0 t) + \mathbf{j}\sin(k\omega_0 t)$$

is periodic with period T_0



- we now take $\phi_k(t) = e^{\mathbf{j}k\omega_0 t}$ as our building blocks, and try to represent x as a weighted sum of these signals

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\mathbf{j}k\omega_0 t}, \qquad \alpha_k \in \mathbb{C}.$$

- **note:** the sum runs from $-\infty$ to ∞, so both $e^{\mathbf{j}k\omega_0 t}$ and $e^{-\mathbf{j}k\omega_0 t}$ are part of the sum; this will allow us to pair up $e^{\mathbf{j}k\omega_0 t}$ and $e^{-\mathbf{j}k\omega_0 t}$ to form sine or cosine

Continuous-time Fourier series (CTFS)

Theorem 3.3 (CTFS). Let x be a periodic CT signal with fundamental period T_0 and angular frequency $\omega_0 = 2\pi/T_0$. Then

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\mathbf{j}k\omega_0 t}$$

is called the ***continuous-time Fourier series (CTFS)*** of the signal x, where the ***Fourier series coefficients*** α_k are given by

$$\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-\mathbf{j}k\omega_0 t}\,\mathrm{d}t.$$

- **note:** The integral can be taken over ***any interval of length*** T_0

Derivation of the CTFS via approximation

- we will take an alternative path to arrive at the formula for α_k
- simultaneously, we will find an interpretation of what it means if we only consider a ***finite*** number of terms in the Fourier series expansion
- for some positive integer K, consider the ***order K approximation***

$$\hat{x}_K(t) = \sum_{k=-K}^{K} \alpha_k e^{\mathbf{j}k\omega_0 t} \qquad \text{(note: } \hat{x} = \lim_{K\to\infty} \hat{x}_K\text{)}$$

- how can we quantify how close the approximation $\hat{x}_K$ is to the original signal x? Let's look at the ***energy of the error***

$$J(\alpha_{-K},\ldots,\alpha_K) = \frac{1}{T_0}\int_0^{T_0} |x(t) - \hat{x}_K(t)|^2 \,\mathrm{d}t, \qquad \text{(mean-square error)}$$

Our goal: find the choice of constants $\{\alpha_k\}_{k=-K}^{K}$ which ***minimizes*** J.

Derivation of the CTFS via approximation

Theorem 3.4. The selection of coefficients $\{\alpha_k\}_{k=-K}^{K}$ which minimizes the mean-squared error J is

$$\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-\mathbf{j}k\omega_0 t}\,\mathrm{d}t$$

(which are exactly the CTFS coefficients!).

- if we choose the coefficients as above, then the finite approximation

$$\hat{x}_K(t) = \sum_{k=-K}^{K} \alpha_k e^{\mathbf{j}k\omega_0 t}$$

is the ***best**** ***possible*** approximation to x that one can build using $2K+1$ complex exponential signals

---

*Where "best" means "the error has the smallest possible energy"

Orthogonality of complex exponentials

- for the cosine Fourier series, we used an "orthogonality result"; something similar holds for exponentials. For any $m, \ell \in \mathbb{Z}$:

"Orthogonality": $$\frac{1}{T_0}\int_0^{T_0} e^{\mathbf{j}m\omega_0 t}e^{-\mathbf{j}\ell\omega_0 t}\,\mathrm{d}t = \begin{cases} 1 & \text{if } m = \ell \\ 0 & \text{if } m \neq \ell \end{cases}$$

The calculation for $m = \ell$ gives $\frac{1}{T_0}\int_0^{T_0}(1)\,\mathrm{d}t = 1$. If $m \neq \ell$, then we have

$$\begin{aligned} \frac{1}{T_0}\int_0^{T_0} e^{\mathbf{j}(m-\ell)\omega_0 t}\,\mathrm{d}t &= \frac{1}{T_0}\frac{1}{\mathbf{j}(m-\ell)\omega_0}\left[e^{\mathbf{j}(m-\ell)\omega_0 t}\right]_{t=0}^{T_0} \\ &= \frac{1}{T_0}\frac{1}{\mathbf{j}(m-\ell)\omega_0}\left[e^{\mathbf{j}(m-\ell)\omega_0 T_0} - 1\right] \\ &= \frac{1}{T_0}\frac{1}{\mathbf{j}(m-\ell)\omega_0}\left[e^{\mathbf{j}(m-\ell)2\pi} - 1\right] \\ &= 0 \end{aligned}$$

where we used that $\omega_0 = 2\pi/T_0$ and that $e^{\mathbf{j}2\pi n} = 1$ for all integers n.

Proof of Theorem 3.4

The function of interest is

$$J = \frac{1}{T_0}\int_0^{T_0} |x(t) - \hat{x}_K(t)|^2 \,\mathrm{d}t = \frac{1}{T_0}\int_0^{T_0} (x(t) - \hat{x}_K(t))^*(x(t) - \hat{x}_K(t))\,\mathrm{d}t$$

where we used that $|z|^2 = z^* z$. Expanding out, we have

$$J = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2 - x(t)^*\hat{x}_K(t) - \hat{x}_K(t)^* x(t) + |\hat{x}_K(t)|^2 \,\mathrm{d}t$$

The last term can be written as

$$\begin{aligned} |\hat{x}_K(t)|^2 &= \left(\sum\nolimits_{\ell=-K}^{K} \alpha_\ell e^{\mathbf{j}\omega_0 \ell t}\right)^* \left(\sum\nolimits_{m=-K}^{K} \alpha_m e^{\mathbf{j}\omega_0 m t}\right) \\ &= \sum_{\ell=-K}^{K} \sum_{m=-K}^{K} \alpha_\ell^* \alpha_m e^{-\mathbf{j}\ell\omega_0 t} e^{\mathbf{j}m\omega_0 t}. \end{aligned}$$

Optimal selection of coefficients

Substituting into J, we can write things out as

$$J = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2 \,\mathrm{d}t - \frac{1}{T_0}\int_0^{T_0} x(t)^* \left[\sum_{m=-K}^{K} \alpha_m e^{\mathbf{j}\omega_0 m t}\,\mathrm{d}t\right]$$

$$- \frac{1}{T_0}\int_0^{T_0} \left[\sum_{m=-K}^{K} \alpha_m^* e^{-\mathbf{j}\omega_0 m t}\right] x(t)\,\mathrm{d}t + \frac{1}{T_0}\int_0^{T_0} \sum_{\ell=-K}^{K}\sum_{m=-K}^{K} \alpha_\ell^* \alpha_m e^{-\mathbf{j}\ell\omega_0 t} e^{\mathbf{j}m\omega_0 t}\,\mathrm{d}t$$

If we define $\beta_m = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-\mathbf{j}\omega_0 m t}\,\mathrm{d}t$, then we can more simply write this as

$$J = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2 \,\mathrm{d}t - \sum_{m=-K}^{K} (\alpha_m \beta_m^* + \alpha_m^* \beta_m) + \sum_{\ell=-K}^{K}\sum_{m=-K}^{K} \alpha_\ell^* \alpha_m \frac{1}{T_0}\int_0^{T_0} e^{\mathbf{j}\omega_0 (m-\ell) t}\,\mathrm{d}t$$

Optimal selection of coefficients

The third term in J therefore simplifies to

$$\sum_{\ell=-K}^{K}\sum_{m=-K}^{K}\alpha_\ell^*\alpha_m\frac{1}{T_0}\int_0^{T_0}e^{\mathbf{j}\omega_0(m-\ell)t}\,\mathrm{d}t=\sum_{m=-K}^{K}\alpha_m^*\alpha_m=\sum_{m=-K}^{K}|\alpha_m|^2$$

We therefore have that

$$\begin{aligned}J&=\frac{1}{T_0}\int_0^{T_0}|x(t)|^2\,\mathrm{d}t-\sum_{m=-K}^{K}(\alpha_m\beta_m^*+\alpha_m^*\beta_m)+\sum_{m=-K}^{K}|\alpha_m|^2\\&=\frac{1}{T_0}\int_0^{T_0}|x(t)|^2\,\mathrm{d}t+\sum_{m=-K}^{K}(-\alpha_m\beta_m^*-\alpha_m^*\beta_m+|\alpha_m|^2)\end{aligned}$$

Optimal selection of coefficients

Our expression is now

$$J = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2 \, \mathrm{d}t + \sum_{m=-K}^{K} \left(-\alpha_m \beta_m^* - \alpha_m^* \beta_m + |\alpha_m|^2\right)$$

If we add and subtract $|\beta_m|^2$ inside the sum, we can complete the square:

$$\begin{aligned} J &= \frac{1}{T_0}\int_0^{T_0} |x(t)|^2 \, \mathrm{d}t + \sum_{m=-K}^{K} \left(|\beta_m|^2 - \alpha_m \beta_m^* - \alpha_m^* \beta_m + |\alpha_m|^2\right) - \sum_{m=-K}^{K} |\beta_m|^2 \\ &= \frac{1}{T_0}\int_0^{T_0} |x(t)|^2 \, \mathrm{d}t + \sum_{m=-K}^{K} (\beta_m - \alpha_m)^*(\beta_m - \alpha_m) - \sum_{m=-K}^{K} |\beta_m|^2 \end{aligned}$$

The first and third terms do not depend at all on α! Therefore, the best thing we can do to minimize J is to make the middle term zero. We therefore find that $\alpha_k = \beta_k$, which completes the proof. •

Summary of CTFS results

order K approximation of x: $\hat{x}_K(t) = \sum_{k=-K}^{K} \alpha_k e^{\mathbf{j}k\omega_0 t}$

coefficients: $\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-\mathbf{j}k\omega_0 t}\,\mathrm{d}t, \qquad \omega_0 = \frac{2\pi}{T_0}.$

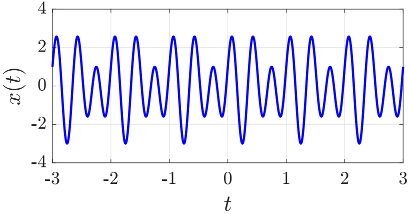
Comments:

- roughly speaking, the magnitude $|\alpha_k|$ of α_k tells us how strongly the frequency harmonic $k\omega_0$ appears in the overall signal x
- the $k = 0$ term in $\hat{x}_K(t)$ is constant; this is called the "dc" term
- you can do the integral over ***any*** interval of length T_0 and obtain the same result, e.g., from $-T_0/2$ to $T_0/2$ instead of 0 to T_0

Example: sum of harmonic signals

$$x(t) = \cos(4\pi t) + 2\sin(6\pi t)$$

- $\cos(4\pi t) = \cos(2 \cdot 2\pi t)$ has period $1/2$
- $\sin(6\pi t) = \sin(3 \cdot 2\pi t)$ has period $1/3$
- fundamental period T_0 is the ***least common multiple*** of the two periods, which is 1; this does indeed match up with the plot. So $\omega_0 = 2\pi/(1) = 2\pi$.



- using Euler's relation, we can rewrite $x(t)$ as

$$x(t) = \frac{e^{\mathbf{j}4\pi t} + e^{-\mathbf{j}4\pi t}}{2} + 2\frac{e^{\mathbf{j}6\pi t} - e^{-\mathbf{j}6\pi t}}{2\mathbf{j}}$$

Example: sum of harmonic signals

Recall our useful formula

$$\frac{1}{T_0}\int_0^{T_0} e^{\mathbf{j}\omega_0(k-\ell)t}\,\mathrm{d}t = \begin{cases} 1 & \text{if } k=\ell \\ 0 & \text{otherwise} \end{cases}$$

With $T_0 = 1$ and $\omega_0 = 2\pi$, we now compute α_k to be

$$\begin{aligned}
\alpha_k &= \frac{1}{1}\int_0^1 \left(\frac{e^{\mathbf{j}4\pi t}+e^{-\mathbf{j}4\pi t}}{2} + 2\frac{e^{\mathbf{j}6\pi t}-e^{-\mathbf{j}6\pi t}}{2\mathbf{j}}\right) e^{-\mathbf{j}2\pi k t}\,\mathrm{d}t \\
&= \int_0^1 \frac{1}{2}\left[e^{\mathbf{j}2\pi(2-k)t}+e^{\mathbf{j}2\pi(-2-k)t}\right] + \frac{1}{\mathbf{j}}\left[e^{\mathbf{j}2\pi(3-k)t}-e^{\mathbf{j}2\pi(-3-k)t}\right]\mathrm{d}t \\
&= \begin{cases} \frac{1}{2} & \text{if } k=\pm 2 \\ \frac{1}{\mathbf{j}} = -\mathbf{j} & \text{if } k=3 \\ \frac{-1}{\mathbf{j}} = +\mathbf{j} & \text{if } k=-3 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Therefore, $\alpha_2 = \alpha_{-2} = \dfrac{1}{2}, \quad \alpha_3 = -\mathbf{j}, \quad \alpha_{-3} = +\mathbf{j}$.

Example: sum of harmonic signals

- let's write out $\hat{x}_K(t) = \sum_{k=-K}^{K} \alpha_k e^{\mathbf{j}k\omega_0 t}$
- for $K = 1$ we have $\hat{x}_1(t) = 0 + 0e^{\mathbf{j}2\pi t} + 0e^{-\mathbf{j}2\pi t} = 0$
- for $K = 2$ we have

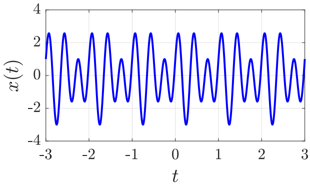
$$\hat{x}_2(t) = \frac{1}{2}e^{\mathbf{j}4\pi t} + \frac{1}{2}e^{-\mathbf{j}4\pi t} = \cos(4\pi t)$$

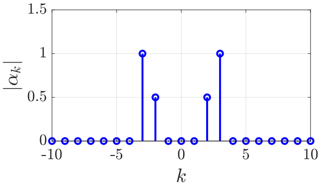
  so our approximation captures the ***lowest frequency component*** of the signal.

- for $K \geq 3$ we have

$$\begin{aligned}\hat{x}_3(t) &= \frac{1}{2}e^{\mathbf{j}4\pi t} + \frac{1}{2}e^{-\mathbf{j}4\pi t} + \frac{1}{\mathbf{j}}e^{\mathbf{j}6\pi t} - \frac{1}{\mathbf{j}}e^{-\mathbf{j}6\pi t} \\ &= \cos(2\pi t) + 2\frac{1}{2\mathbf{j}}(e^{\mathbf{j}6\pi t} - e^{-\mathbf{j}6\pi t}) = \cos(4\pi t) + 2\sin(6\pi t)\end{aligned}$$

  so our approximation captures ***both frequencies*** present in the signal x

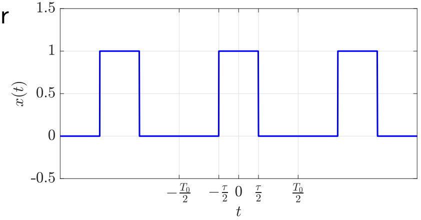




Example: a square wave (revisited)

For $T_0 > 0$ and $0 < \tau \leq T_0$ consider

$$x(t) = \begin{cases} 1 & \text{if } -\frac{\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0 & \text{if } \frac{\tau}{2} < |t| \leq \frac{T_0}{2} \end{cases}$$



- we copy this pattern every T_0 seconds to make things periodic
- first we compute the dc coefficient α_0

$$\alpha_0 = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\,\mathrm{d}t = \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} \mathrm{d}t = \frac{\tau}{T_0}$$

Note that the dc coefficient α_0 is just the ***average*** value of $x(t)$ over one period.

Example: a square wave (revisited)

For $k \neq 0$, recall that $\omega_0 T_0 = 2\pi$ and now compute that

$$\begin{aligned}\alpha_k &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-\mathbf{j}k\omega_0 t} = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} e^{-\mathbf{j}k\omega_0 t} \\ &= \frac{1}{T_0}\frac{1}{-\mathbf{j}k\omega_0}\left[e^{-\mathbf{j}k\omega_0 t}\right]_{t=-\tau/2}^{t=\tau/2} = \frac{-1}{2\mathbf{j}\pi k}(e^{-\mathbf{j}k\omega_0\tau/2} - e^{\mathbf{j}\omega_0\tau/2}) = \frac{1}{\pi k}\sin(k\omega_0\tau/2)\end{aligned}$$

Therefore,

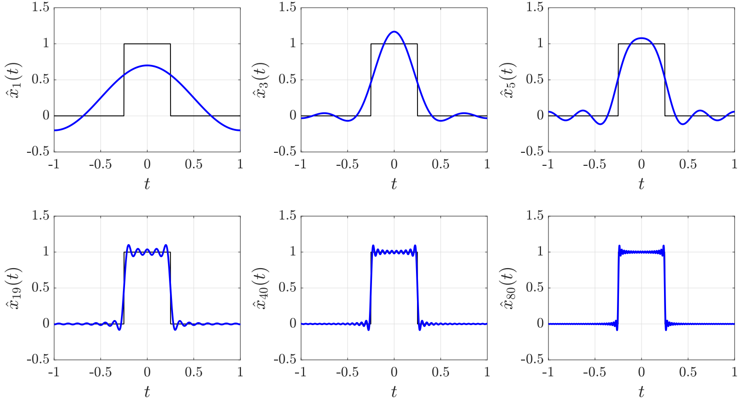
$$\hat{x}_K(t) = \frac{\tau}{T_0} + \sum_{k=-K}^{-1}\frac{\sin(k\omega_0\tau/2)}{\pi k}e^{\mathbf{j}k\omega_0 t} + \sum_{k=1}^{K}\frac{\sin(k\omega_0\tau/2)}{\pi k}e^{\mathbf{j}k\omega_0 t}.$$

Since the coefficients are even functions of k, this simplifies nicely to

$$\begin{aligned}\hat{x}_K(t) &= \frac{\tau}{T_0} + \sum_{k=1}^{K}\frac{\sin(k\omega_0\tau/2)}{\pi k}\left(e^{\mathbf{j}k\omega_0 t} + e^{-\mathbf{j}k\omega_0 t}\right) \\ &= \frac{\tau}{T_0} + \sum_{k=1}^{K}\frac{2\sin(k\omega_0\tau/2)}{\pi k}\cos(k\omega_0 t)\end{aligned}$$

(success! same formula as before)

Example: a square wave (revisited)



More terms = closer approximation of original signal

"*It takes high frequencies to make sharp corners*"

The CTFS and signal manipulations

- suppose we have the CTFS coefficients $\{\alpha_k\}_{k=-\infty}^{\infty}$ of some signal x
- how can we find the CTFS coefficients of signals obtained by simple manipulations, e.g., $3x(t)+2$, $x(t-5)$, $\mathrm{Re}\{x\}$, $\frac{\mathrm{d}x}{\mathrm{d}t}(t)$ ...?

There are many useful ***properties of the CTFS*** that you can use as ***shortcuts*** to find the CTFS coefficients for these transformed signals

- for example, if $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\mathbf{j}k\omega_0 t}$, note that

$$x(t-t_0) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\mathbf{j}k\omega_0(t-t_0)} = \sum_{k=-\infty}^{\infty} \left(\alpha_k e^{-\mathbf{j}k\omega_0 t_0}\right) e^{\mathbf{j}k\omega_0 t}$$

so the CTFS coefficients of $x(t-t_0)$ must be $\beta_k = \alpha_k e^{-\mathbf{j}k\omega_0 t_0}$

Table of CTFS properties

We let x be an T_0-periodic CT signal with CTFS coefficients α_k, and $\omega_0 = 2\pi/T_0$.

Name	$x(t)$	$\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-\mathbf{j}k\omega_0 t}\,\mathrm{d}t$
Time-shift by t_0	$x(t-t_0)$	$e^{-\mathbf{j}k\omega_0 t_0}\alpha_k$
Frequency-shift by k_0	$e^{\mathbf{j}k_0\omega_0 t}x(t)$	α_{k-k_0}
Conjugation	$x^*(t)$	α^*_{-k}
Time-reversal	$x(-t)$	α_{-k}
Differentiation	$\dot{x}(t)$	$(\mathbf{j}k\omega_0)\alpha_k$
Convolution	$\frac{1}{T_0}\int_0^{T_0} x(\tau)y(t-\tau)\,\mathrm{d}\tau$	$\alpha_k\beta_k$
Multiplication	$x(t)y(t)$	$\sum_{\ell=-\infty}^{\infty}\alpha_\ell\beta_{k-\ell}$
Real Part	$\mathrm{Re}\{x(t)\}$	$\frac{1}{2}(\alpha_k + \alpha^*_{-k})$
Imag Part	$\mathrm{Im}\{x(t)\}$	$\frac{1}{2\mathbf{j}}(\alpha_k - \alpha^*_{-k})$

Conjugate-symmetry for real signals

- suppose that x is a ***real-valued*** T_0-periodic signal
- the CTFS coefficients are given by

$$\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-\mathbf{j}k\omega_0 t}\,\mathrm{d}t, \qquad \omega_0 = \frac{2\pi}{T_0}, \qquad k \in \mathbb{Z}.$$

- from this it follows that

$$\begin{aligned}\alpha_k^* &= \frac{1}{T_0}\int_0^{T_0} x(t)^*(e^{-\mathbf{j}k\omega_0 t})^*\,\mathrm{d}t \\ &= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-\mathbf{j}(-k)\omega_0 t}\,\mathrm{d}t = \alpha_{-k}.\end{aligned}$$

So $\alpha_{-k} = \alpha_k^*$, and in particular then, $|\alpha_{-k}| = |\alpha_k|$. The ***magnitude*** of the CTFS coefficients is an ***even function of*** k.

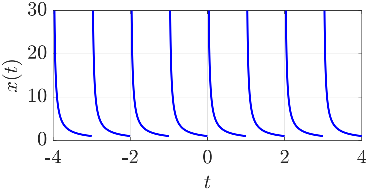
Summary

- we have now seen that
  - (i) we can "approximate/represent" a periodic CT signal with an infinite sum of CT complex exp. signals
  - (ii) the coefficients $\{\alpha_k\}_{k=-\infty}^{\infty}$ provide a *frequency-domain* representation of the signal, and give us insight into the important harmonics
  - (iii) properties of the signal are transferred into properties of the coefficients (e.g., a real-valued signal leads to conjugate-symmetric coefficients)
- some things we still need to understand are
  - (i) for what kinds of signals does Fourier series "work"?
  - (ii) in what sense does the approximation $\hat{x}_K$ "converge" to x?
  - (iii) what can we do for DT signals?

Does every periodic signal have a Fourier series? No.

- consider the periodization of the signal

$$x_{\text{fin}}(t) = \begin{cases} \frac{1}{t}, & \text{if } 0 < t \leq 1 \\ 0, & \text{else} \end{cases}$$



- compute the 0th Fourier coefficient to be

$$\alpha_0 = \frac{1}{1} \int_0^1 \frac{1}{t} \, \mathrm{d}t = \ln(1) - \ln(0) = +\infty$$

So unfortunately, some periodic signals simply ***cannot*** be represented using Fourier series. Under what assumptions do things work as expected?

Existence of the CTFS

Theorem 3.5. If x has finite action, i.e., $x \in \mathsf{L}_1^{\mathrm{per}}$, then the Fourier coefficients $\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-\mathbf{j}k\omega_0 t}\,\mathrm{d}t$ are well-defined and satisfy

$$\lim_{k\to\pm\infty} |\alpha_k| = 0.$$

Proof: Since x has finite action, we can bound the Fourier coefficients as

$$\begin{aligned} |\alpha_k| = \frac{1}{T_0}\left|\int_0^{T_0} x(t)e^{-\mathbf{j}k\omega_0 t}\,\mathrm{d}t\right| &\leq \frac{1}{T_0}\int_0^{T_0} |x(t)|\cdot|e^{-\mathbf{j}k\omega_0 t}|\,\mathrm{d}t \\ &= \frac{1}{T_0}\int_0^{T_0} |x(t)|\,\mathrm{d}t = \frac{\|x\|_1}{T_0} < \infty \end{aligned}$$

so all coefficients are well-defined. The proof of the second statement is outside our scope, but can be found by searching for "Riemann-Lebesgue Lemma".

Convergence of the CTFS

- if $x \in \mathsf{L}_1^{\text{per}}$, it *at least* makes sense to form the approximation

$$\hat{x}_K(t) = \sum_{k=-K}^{K} \alpha_k e^{\mathbf{j}k\omega_0 t}$$

- how do we capture the idea that $\hat{x}_K$ converges to x?

Definition 3.1. Let $e_K = \hat{x}_K - x$ denote the error between the approximation and the signal x. We say $\hat{x}_K$ converges to x

(i) ***pointwise at time $t_0 \in \mathbb{R}$*** if $\lim_{K\to\infty} e_K(t_0) = 0$;

(ii) ***uniformly*** if $\lim_{K\to\infty} \|e_K\|_\infty = 0$;

(iii) ***in energy*** if $\lim_{K\to\infty} \|e_K\|_2 = 0$.

- uniformly = amplitude of error goes to zero
- in energy = energy of error goes to zero

Pointwise convergence of the CTFS

Theorem 3.6. Suppose that x has finite action, i.e., $x \in \mathsf{L}_1^{\mathrm{per}}$.

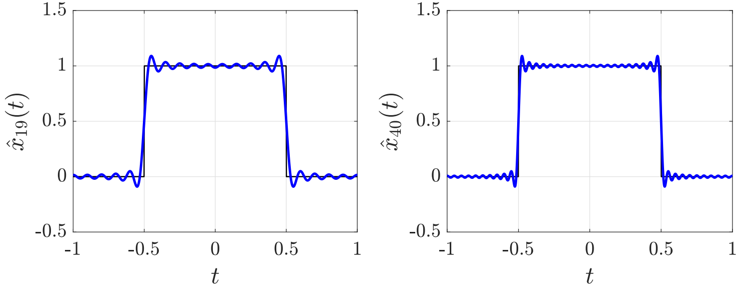
(i) If x has a continuous derivative at time $t_0 \in \mathbb{R}$, then $\hat{x}_K$ converges to x pointwise at t_0.

(ii) If the left-side limits $x(t_0^-)$, $\frac{\mathrm{d}x}{\mathrm{d}t}(t_0^-)$ and the right-side limits $x(t_0^+)$, $\frac{\mathrm{d}x}{\mathrm{d}t}(t_0^+)$ all exist at time $t_0 \in \mathbb{R}$, then

$$\lim_{K\to\infty} \hat{x}_K(t_0) = \tfrac{1}{2}(x(t_0^-) + x(t_0^+)).$$

- **point (i):** over time intervals where the signal is smooth, the approximation converges ***pointwise***
- **point (ii):** at a finite jump discontinuity, the approximation converges to the ***mid-point*** of the discontinuity

Example: a square wave ($\tau = 1, T_0 = 2$)

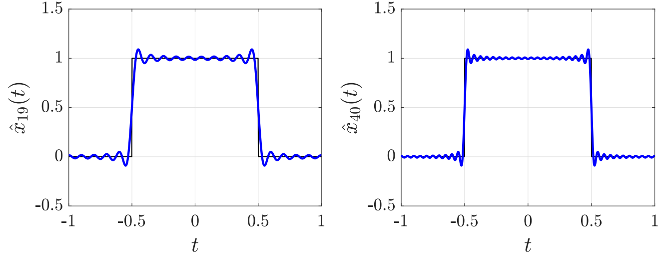


- approximation converges pointwise at all $t \neq \pm 0.5$
- approximation converges to discontinuity midpoint at $t = \pm 0.5$
- the ripple near $t = \pm 0.5$ is called the ***Gibb's phenomenon***; it never goes away, it just becomes more concentrated around $t = \pm 0.5$.

Uniform convergence of the CTFS

- uniform convergence requires stronger assumptions; one assumption that works is that the signal has a continuous derivative everywhere

Theorem 3.7. Suppose that $x \in \mathsf{L}_1^{\mathrm{per}}$. If x has a derivative which is continuous everywhere, then $\hat{x}_K$ converges to x uniformly.



- in this example we *do not* have uniform convergence

Convergence in energy of the CTFS

- signals with finite energy are often the nicest case to consider

Theorem 3.8. If x has finite energy, i.e., $x \in \mathsf{L}_2^{\text{per}}$, then

(i) $\hat{x}_K$ converges to x in energy;

(ii) the CTFS coefficients $\{\alpha_k\}_{k=-\infty}^{\infty}$ have finite energy, i.e., $\alpha \in \ell_2$;

(iii) the signal and the coefficients satisfy ***Parseval's relation***

$$\frac{1}{T_0}\int_0^{T_0} |x(t)|^2 \, \mathrm{d}t = \sum_{k=-\infty}^{\infty} |\alpha_k|^2 \qquad \text{or} \qquad \frac{1}{T_0}\|x\|_2^2 = \|\alpha\|_2^2.$$

A beautiful and surprising relationship between the energy of the signal and the energy of the CTFS coefficients.

Proof of Part (iii) of Theorem 3.8

Proof: Begin by computing that

$$|x(t)|^2 = x(t)x(t)^* = \left[\sum_{k=-\infty}^{\infty} \alpha_k e^{\mathbf{j}k\omega_0 t}\right] \left[\sum_{\ell=-\infty}^{\infty} \alpha_\ell e^{\mathbf{j}\ell\omega_0 t}\right]^*$$

$$= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \alpha_k \alpha_\ell^* e^{\mathbf{j}\omega_0 (k-\ell)t}$$

Recall that

$$\frac{1}{T_0} \int_0^{T_0} e^{\mathbf{j}(k-\ell)\omega_0 t} = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{otherwise} \end{cases}$$

Therefore we have

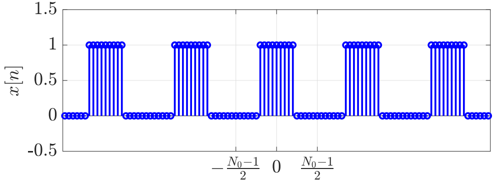
$$\frac{1}{T_0} \int_0^{T_0} |x(t)|^2 \, \mathrm{d}t = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \alpha_k \alpha_\ell^* \frac{1}{T_0} \int_0^{T_0} e^{\mathbf{j}(k-\ell)\omega_0 t} \, \mathrm{d}t$$

$$= \sum_{k=-\infty}^{\infty} |\alpha_k|^2.$$

The discrete-time Fourier series (DTFS)

Everything we just did also holds (with some minor changes) for ***discrete-time*** signals!

- let x be a DT periodic signal with fundamental period N_0



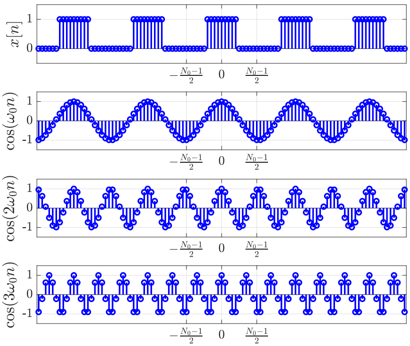
- we want to express x as $x[n] = \sum_k \alpha_k \phi_k[n]$
- as building blocks $\phi_k[n]$ we use the DT exponential signals

$$\phi_k[n] = e^{\mathbf{j}k\omega_0 n}, \qquad \omega_0 = 2\pi/N_0$$

The discrete-time Fourier series (DTFS)

- we therefore look for a representation of x of the form

$$x[n] = \sum_{k=0}^{N_0} \alpha_k e^{\mathbf{j}k\omega_0 n}, \qquad \omega_0 = \frac{2\pi}{N_0}$$



note: all the exponentials of different frequencies

$$0, \omega_0, 2\omega_0, \ldots, (N_0 - 1)\omega_0$$

are periodic and fit nicely within our fundamental period N_0 of x. Moreover,

$$e^{\mathbf{j}(N_0\omega_0)n} = e^{\mathbf{j}(2\pi)n} = e^{\mathbf{j}(0)n}$$

Orthogonality of complex exponentials

- DT exponentials satisfy the following orthogonality relationship

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1} e^{\mathbf{j}m\omega_0 n} e^{-\mathbf{j}\ell\omega_0 n} = \begin{cases} 1 & \text{if } m = \ell + kN_0,\ k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

For $m = \ell + kN_0$, we have $\frac{1}{N_0}\sum_{n=0}^{N_0-1} e^{\mathbf{j}(m-\ell)\omega_0 n} = \frac{1}{N_0}\sum_{n=0}^{N_0-1} e^{\mathbf{j}kN_0\frac{2\pi}{N_0}n} = 1$.
For the other case, we have

$$\begin{aligned}
\frac{1}{N_0}\sum_{n=0}^{N_0-1} e^{\mathbf{j}(m-\ell)\omega_0 n} &= \frac{1}{N_0}\sum_{n=0}^{N_0-1} (e^{\mathbf{j}(m-\ell)\omega_0})^n \\
&= \frac{1}{N_0}\frac{1 - e^{\mathbf{j}(m-\ell)\omega_0 N_0}}{1 - e^{\mathbf{j}(m-\ell)\omega_0}} \\
&= \frac{1}{N_0}\frac{1 - e^{\mathbf{j}(m-\ell)2\pi}}{1 - e^{\mathbf{j}(m-\ell)\omega_0}} = \frac{1}{N_0}\frac{1-1}{1 - e^{\mathbf{j}(m-\ell)\omega_0}} = 0
\end{aligned}$$

where we used the geometric series formula and $e^{\mathbf{j}2\pi n} = 1$ for all integers n.

The discrete-time Fourier series (DTFS)

Theorem 3.9. If x is a periodic DT signal with fundamental period N_0, with $\omega_0 = 2\pi/N_0$, then the ***discrete-time Fourier series*** of x is the sum

$$x[n] = \sum_{k=0}^{N_0-1} \alpha_k e^{\mathbf{j}k\omega_0 n}$$

where the ***Fourier coefficients*** α_k are given by

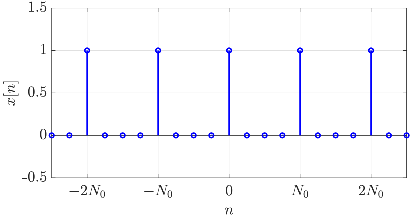
$$\alpha_k = \frac{1}{N_0} \sum_{l=0}^{N_0-1} x[l] e^{-\mathbf{j}k\omega_0 l}.$$

- **note:** both sums are finite!
- **note:** the sums can be taken over ***any*** window of length N_0

Example: the repeated unit impulse

Let $N_0 \in \mathbb{Z}_{\geq 2}$ be a desired period. Consider (the periodization of)

$$x[n] = \delta[n] \text{ for } 0 \leq n \leq N_0 - 1$$



- the DTFS coefficients are

$$\alpha_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} \delta[n] e^{-\mathbf{j}k\omega_0 n} = \frac{1}{N_0}(e^{-\mathbf{j}k\omega_0 \cdot 0} + 0 + \cdots + 0) = \frac{1}{N_0}$$

- we say the spectrum is ***flat*** or "***white***", since all the Fourier coefficients are equal to a constant; an impulse contains *equal contributions from all frequencies*.

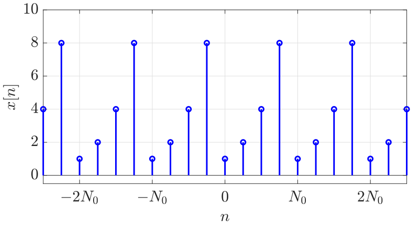
- the Fourier series is therefore

$$x[n] = \frac{1}{N_0}\sum_{k=0}^{N_0-1} e^{\mathbf{j}k\omega_0 n}$$

Example: a growing signal

Let $N_0 = 4$ be the period, and consider the periodic extension of

$$x[n] = 2^n \text{ for } 0 \leq n \leq 3$$



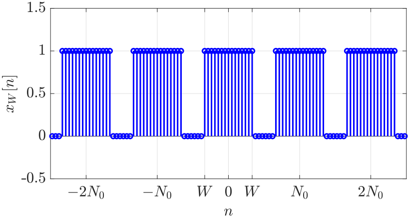
- note that $\omega_0 = 2\pi/4 = \pi/2$. we can compute the DTFS coefficients as

$$\begin{aligned}
\alpha_k &= \frac{1}{4}\sum_{n=0}^{3} 2^n e^{-\mathbf{j}k\omega_0 n} = \frac{1}{4}\sum_{n=0}^{3}(2e^{-\mathbf{j}k\frac{\pi}{2}})^n \\
&= \frac{1}{4}\frac{1-(2e^{-\mathbf{j}k\frac{\pi}{2}})^4}{1-(2e^{-\mathbf{j}k\frac{\pi}{2}})} \\
&= \frac{1}{4}\frac{1-16}{1-2(-\mathbf{j})^k} = -\frac{15}{4}\frac{1}{1-2(-\mathbf{j})^k}
\end{aligned}$$

Example: the windowing signal

Let $N_0 \in \mathbb{Z}_{\geq 3}$ be odd, and let $W \in \mathbb{Z}$ satisfy $0 \leq W \leq \frac{N_0-1}{2}$. Consider the periodization of

$$x[n] = \begin{cases} 1 & \text{if } |n| \leq W \\ 0 & \text{if } W < |n| \leq \frac{N_0-1}{2} \end{cases}$$



- to compute the coefficients in this example, it's convenient to take the sum as running over the window of length N_0 running from $-(N_0-1)/2$ to $+(N_0-1)/2$.
- for the dc coefficient α_0 we have

$$\alpha_0 = \frac{1}{N_0} \sum_{n=-(N_0-1)/2}^{(N_0-1)/2} x[n] = \frac{1}{N_0} \sum_{n=-W}^{W} 1 = \frac{2W+1}{N_0}$$

Example: the windowing signal

- for $k \neq 0$ we can compute that

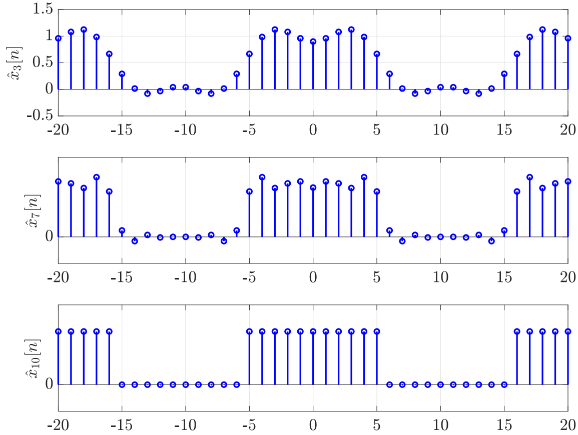
$$\alpha_k = \frac{1}{N_0} \sum_{n=-(N_0-1)/2}^{(N_0-1)/2} x[n] e^{-\mathbf{j}k\omega_0 n} = \frac{1}{N_0} \sum_{n=-W}^{W} e^{-\mathbf{j}k\omega_0 n}$$

- making the change of variable $m = n + W$, we can rewrite this as

$$\begin{aligned}
\alpha_k = \frac{1}{N_0} \sum_{m=0}^{2W} e^{-\mathbf{j}k\omega_0 (m-W)} &= \frac{e^{\mathbf{j}k\omega_0 W}}{N_0} \sum_{m=0}^{2W} e^{-\mathbf{j}k\omega_0 m} \\
&= \frac{e^{\mathbf{j}k\omega_0 W}}{N_0} \frac{1 - e^{-\mathbf{j}\omega_0 k(2W+1)}}{1 - e^{-\mathbf{j}k\omega_0}} \\
&= \frac{1}{N_0} \frac{e^{\mathbf{j}k\omega_0 W} - e^{-\mathbf{j}\omega_0 k(W+1)}}{1 - e^{-\mathbf{j}k\omega_0}} \\
&= \frac{1}{N_0} \frac{e^{-\mathbf{j}\omega_0 \frac{1}{2} k}}{e^{-\mathbf{j}\omega_0 \frac{1}{2} k}} \frac{e^{\mathbf{j}\omega_0 (W+\frac{1}{2})k} - e^{-\mathbf{j}\omega_0 (W+\frac{1}{2})k}}{e^{\mathbf{j}\omega_0 \frac{1}{2} k} - e^{-\mathbf{j}\omega_0 \frac{1}{2} k}} \\
&= \frac{1}{N_0} \frac{\sin(\omega_0 (W + \frac{1}{2})k)}{\sin(\omega_0 \frac{1}{2} k)}
\end{aligned}$$

Example: the windowing signal ($N_0 = 21$, $W = 2$)

approximations: $\hat{x}_K[n] = \sum_{k=-K}^{K} \alpha_k e^{\mathbf{j}k\omega_0 n}$



Periodicity of DTFS coefficients

- ***interesting observation:*** the coefficients α_k are themselves periodic with period N_0! That is, $\alpha_{k+N_0} = \alpha_k$ for all k.

- *Proof:* we compute directly that

$$\begin{aligned}
\alpha_{k+N_0} &= \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-\mathbf{j}(k+N_0)\omega_0 n} \\
&= \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-\mathbf{j}k\omega_0 n}e^{-\mathbf{j}N_0\omega_0 n} \\
&= \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-\mathbf{j}k\omega_0 n}e^{-\mathbf{j}N_0\frac{2\pi}{N_0}n} \\
&= \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-\mathbf{j}k\omega_0 n} \\
&= \alpha_k
\end{aligned}$$

Comparison of CTFS vs. DTFS

CTFS: $$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\mathbf{j}k\omega_0 t}, \quad \alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-\mathbf{j}k\omega_0 t}\,\mathrm{d}t.$$

- represents periodic $x(t)$ as ***infinite discrete sum*** of CT complex exp.
- α_k captures the "amount" of harmonic $k\omega_0$ contained in x
- in general, the sequence $\{\alpha_k\}_{k=-\infty}^{\infty}$ is aperiodic

DTFS: $$x[n] = \sum_{k=0}^{N_0-1} \alpha_k e^{\mathbf{j}k\omega_0 n}, \quad \alpha_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-\mathbf{j}k\omega_0 n}.$$

- represents periodic $x[n]$ as a ***finite discrete sum*** of DT complex exp.
- α_k captures the "amount" of harmonic $k\omega_0$ contained in x
- the sequence $\{\alpha_k\}_{k=-\infty}^{\infty}$ is always N_0-periodic

More on DTFS

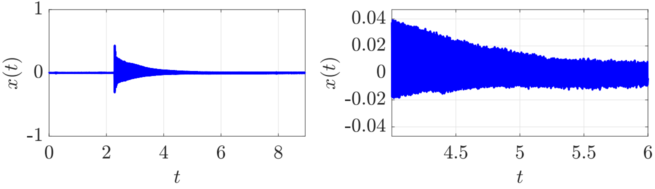
- in the appendix of these slides, you will find

  1. a derivation of the DTFS
  2. convergence of the approximations
  3. matrix-vector formulas for computing the DTFS
  4. a derivation showing that you can use the DTFS coefficients to numerically approximate the CTFS coefficients

If you take ECE431H1: Digital Signal Processing, you will spend much more time on the DTFS and the closely-related idea of the Fast Fourier Transform.

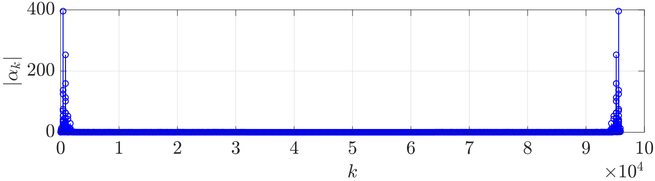
Application: analysis of a real signal

- remember the complicated CT signal from the start of this section:



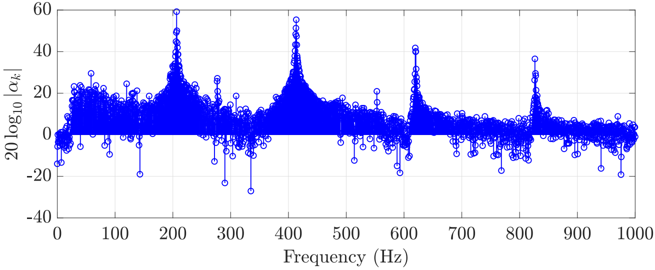
- sample at $f_\mathrm{s} = 48,000$ samples/sec to obtain a DT signal $x[n]$
- take the interval between 4 and 6 seconds, and repeat it over and over; this yields a periodic signal of period $N_0 = 96,001$
- compute the DTFS coefficients $\{\alpha_k\}_{k=0}^{96,000}$ and plot them!

Application: analysis of a real signal



- **note:** the plot is symmetric about the mid-point (why?)
- The horizontal axis is k, which is the *multiple* of $\omega_0 = 2\pi/N_0$. Since N_0 is number of samples per period, ω_0 has units *rad/sample*
- To convert horiz. axis units to Hz, we plot $k\frac{\omega_0}{2\pi}f_\mathrm{s} = k\frac{f_\mathrm{s}}{N_0}$ instead of k
- we will plot the vertical axis in ***decibels***, i.e., $20\log_{10}|\alpha_k|$

Application: analysis of a real signal



- we have strong spikes at
  - (i) $\approx$ 207Hz $= \mathrm{G}_3^\#$
  - (ii) $\approx$ 415Hz $= \mathrm{G}_4^\#$
  - (iii) $\approx$ 622Hz $= \mathrm{D}_5^\#$
  - (iv) $\approx$ 830Hz $= \mathrm{G}_5^\#$

This signal is a recording of a $\mathrm{G}_3^\#$ note played on an acoustic guitar! The DTFS analysis reveals the frequency content.

Table of DTFS properties

We let x be an N_0-periodic DT signal with DTFS coefficients α_k, and $\omega_0 = 2\pi/N_0$.

Name	$x[n]$	$\alpha_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-\mathbf{j}k\omega_0 n}$
Time-shift by n_0	$x[n-n_0]$	$e^{-\mathbf{j}\omega_0 n_0 k}\alpha_k$
Frequency-shift by k_0	$e^{\mathbf{j}k_0\omega_0 n}x[n]$	α_{k-k_0}
Conjugation	$x^*[n]$	α^*_{-k}
Time-reversal	$x[-n]$	α_{-k}
First-difference	$x[n]-x[n-1]$	$(1-e^{-\mathbf{j}\omega_0 k})\alpha_k$
Convolution	$\frac{1}{N_0}\sum_{m=0}^{N_0-1} x[m]y[n-m]$	$\alpha_k\beta_k$
Multiplication	$x[n]y[n]$	$\sum_{\ell=0}^{N_0-1}\alpha_\ell\beta_{k-\ell}$
Real Part	$\mathrm{Re}\{x[n]\}$	$\frac{1}{2}(\alpha_k+\alpha^*_{-k})$
Imag Part	$\mathrm{Im}\{x[n]\}$	$\frac{1}{2\mathbf{j}}(\alpha_k-\alpha^*_{-k})$

Relevant MATLAB commands

- compute the DTFS coefficients of a DT signal x

```
%% Define one period of the signal
N_0 = 50;
x = randn(N_0,1);

%% Compute DTFS coefficients
alpha = fft(x,N_0);

%% Plot magnitude of coefficients
stem(0:N_0-1, abs(alpha));
```

Supplementary reading

- **O-W:** A. V. Oppenheim and S. Willsky, *Signals and Systems*, 2nd Ed.
- **BB:** B. Boulet, *Fundamentals of Signals and Systems*.
- **BPL:** B. P. Lathi, *Signal Processing and Linear Systems*.
- **K-S:** H. Kwakernaak and R. Sivan, *Modern Signals and Systems*.
- **EL-PV:** E. Lee and P. Varaiya, *Structure and Interpretation of Signals and Systems*, 2nd Ed.
- **ADL:** A. D. Lewis, *The Mathematical Theory of Signals and Systems*.

Topic	**O-W**	**BB**	**BPL**	**K-S**	**EL-PV**	**ADL**
The CTFS	3.3	4	3.4, 3.5	6.1, 6.4	7.5, 10.2	V4 5.1–5.3
CTFS properties	3.5	4	3.4, 3.5	6.4	7.5	V4 5.1–5.3
Convergence results	3.4	4	3.4, 3.5	6.4	7.5	V4 5.2
The DFTS	3.6	12	10.1	6.4, 7.3	7.6, 10.3	V4 7.2
DTFS properties	3.7	12	10.3	6.4, 7.3	7.6	V4 7.2

Personal Notes

Personal Notes

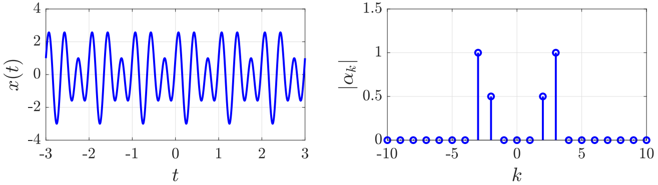
Personal Notes

4. The Fourier Transform

- extending the CTFS to aperiodic signals
- the continuous-time Fourier transform (CTFT)
- examples
- existence of the CTFT
- the CTFT of a complex exponential
- properties of the CTFT
- the discrete-time Fourier transform (DTFT)

Introduction

- we now know how to use the Fourier series to analyze ***periodic*** signals



(i) we *represent* a periodic signal as a sum of complex exponential signals

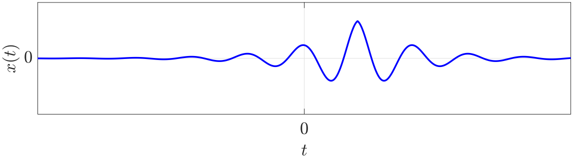
(ii) we *analyze* the frequency content of a periodic signal by examining α_k

- what if our signal of interest is ***aperiodic***? What can we do . . .

The ***Fourier transform*** is the *extension* of the Fourier series to aperiodic signals, and is one of the most powerful tools in all of engineering!

Derivation of the CT Fourier Transform

- suppose we have a general CT signal x, e.g.,



Key idea: an *aperiodic* signal is a *periodic* signal with infinite period . . .

Steps we will take:

(i) window the signal to $[-T, T)$, then periodize it

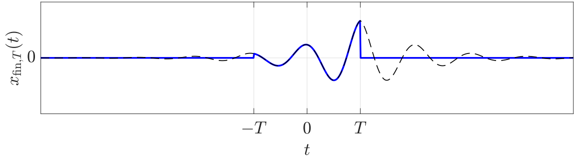
(ii) compute the CTFS of the periodized signal

(iii) take the limit as $T \to \infty$

Derivation of the CT Fourier Transform

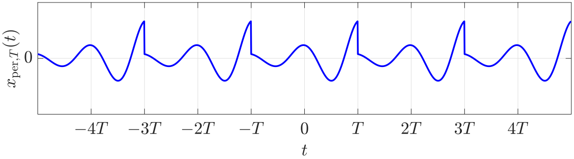
- we begin by ***windowing*** x to obtain a finite-duration signal

$$x_{\text{fin},T}(t) = x(t) \cdot [u(t+T) - u(t-T)]$$



- we can now ***periodize*** $x_{\text{fin},T}$ to obtain the $2T$-periodic signal

$$x_{\text{per},T}(t) = \sum_{m=-\infty}^{\infty} x_{\text{fin},T}(t - m(2T))$$



Derivation of the CT Fourier Transform

- since $x_{\text{per},T}$ is $2T$-periodic, we can ***represent it via the CTFS***

$$x_{\text{per},T}(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\mathbf{j}k\omega_0 t}$$

where $\omega_0 = (2\pi)/(2T) = \pi/T$ is the fundamental ang. frequency

- as always, the CTFS coefficients α_k are given by

$$\alpha_k = \frac{1}{2T} \int_{-T}^{T} x_{\text{per},T}(t) e^{-\mathbf{j}k\omega_0 t} \, \mathrm{d}t$$

- however, over the interval $[-T, T)$, we have $x_{\text{per},T}(t) = x(t)$, so

$$\alpha_k = \frac{1}{2T} \int_{-T}^{T} x(t) e^{-\mathbf{j}k\omega_0 t} \, \mathrm{d}t$$

Derivation of the CT Fourier Transform

- as some simplifying notation, if we define the function

$$X_T : \mathbb{R} \to \mathbb{C}, \qquad X_T(\mathbf{j}\omega) = \int_{-T}^{T} x(t)e^{-\mathbf{j}\omega t}\,\mathrm{d}t$$

  then the CTFS coefficients are simply ***samples*** of X_T

$$\alpha_k = \frac{1}{2T} X_T(\mathbf{j}k\omega_0), \qquad k \in \{-\infty, \ldots, \infty\}$$

- plugging this back into the CTFS, we find that

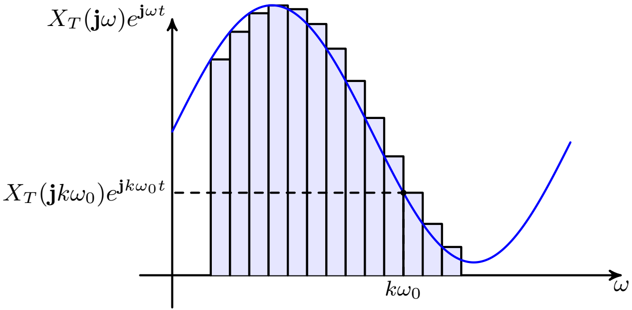
$$x_{\mathrm{per},T}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{2T} X_T(\mathbf{j}k\omega_0)e^{\mathbf{j}k\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X_T(\mathbf{j}k\omega_0)e^{\mathbf{j}k\omega_0 t}\omega_0$$

  since $T = \pi/\omega_0$.

Derivation of the CT Fourier Transform

Our Fourier Series: $$x_{\mathrm{per},T}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X_T(\mathbf{j}k\omega_0)e^{\mathbf{j}k\omega_0 t}\omega_0$$

- The sum here is a ***Riemann sum***: $X_T(\mathbf{j}k\omega_0)e^{\mathbf{j}k\omega_0 t}$ are ***samples*** of the function $\omega \mapsto X_T(\mathbf{j}\omega)e^{\mathbf{j}\omega t}$, spaced by a ***width*** of ω_0



Derivation of the CT Fourier Transform

Our Fourier Series: $$x_{\text{per},T}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X_T(\mathbf{j}k\omega_0)e^{\mathbf{j}k\omega_0 t}\omega_0$$

- as $T \to \infty$, $\omega_0 = \frac{2\pi}{T} \to 0$, and the sum becomes the ***integral***

$$\lim_{T\to\infty} x_{\text{per},T}(t) = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\mathbf{j}\omega)e^{\mathbf{j}\omega t}\,\mathrm{d}\omega$$

where

$$X(\mathbf{j}\omega) = \lim_{T\to\infty} X_T(\mathbf{j}\omega) = \int_{-\infty}^{\infty} x(t)e^{-\mathbf{j}\omega t}\,\mathrm{d}t$$

These last two formulas extend the CTFS to aperiodic signals!

The CT Fourier Transform (CTFT)

Definition 4.1. The ***continuous-time Fourier transform (CTFT)*** of a CT signal x is the complex-valued signal $X : \mathbb{R} \to \mathbb{C}$ defined pointwise by

$$X(\mathbf{j}\omega) = \int_{-\infty}^{\infty} x(t) e^{-\mathbf{j}\omega t} \, \mathrm{d}t.$$

We call X the ***Fourier transform*** or ***spectrum*** of x.

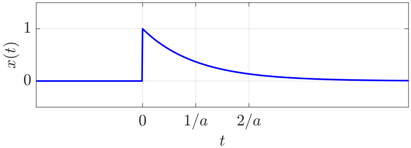
Definition 4.2. The ***inverse continuous-time Fourier transform (inverse CTFT)*** of a CT spectrum X is the CT signal $x : \mathbb{R} \to \mathbb{C}$ defined pointwise by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\mathbf{j}\omega) e^{\mathbf{j}\omega t} \, \mathrm{d}\omega.$$

CTFT Example: right-sided decaying exponential

for $a > 0$ consider the signal

$$x(t) = e^{-at}u(t)$$



- we can compute the spectrum to be

$$X(\mathbf{j}\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-\mathbf{j}\omega t}\,\mathrm{d}t = \int_{0}^{\infty} e^{-(a+\mathbf{j}\omega)t}\,\mathrm{d}t$$

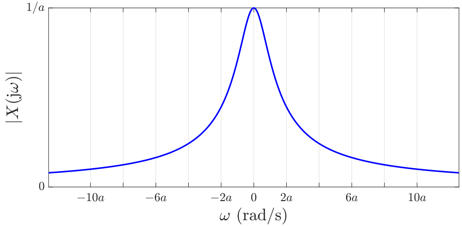
$$= -\frac{1}{a+\mathbf{j}\omega}e^{-(a+\mathbf{j}\omega)t}\Big|_{0}^{\infty} = \frac{1}{a+\mathbf{j}\omega}$$

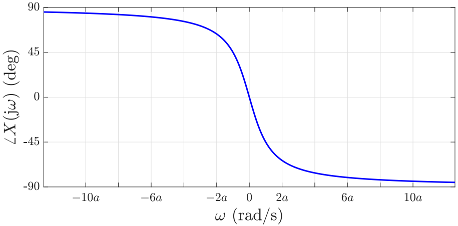
- the ***magnitude*** and ***phase*** of $X(\mathbf{j}\omega)$ are

$$|X(\mathbf{j}\omega)| = \frac{1}{\sqrt{\omega^2 + a^2}}, \quad \angle X(\mathbf{j}\omega) = -\tan^{-1}(\omega/a)$$

CTFT Example: right-sided decaying exponential

- let's plot $|X(\mathbf{j}\omega)|$ and $\angle X(\mathbf{j}\omega)$





Notes:

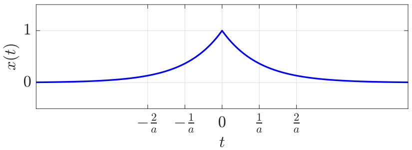
(i) magnitude of X is larger at ***small*** values of $|\omega|$; the signal x contains lots of "low-frequency content"

(ii) the plots have a nice symmetry; this is actually a ***general*** property for any *real-valued* signal x; we will show this soon

CTFT Example: two-sided decaying exponential

for $a > 0$ consider the signal

$$x(t) = e^{-a|t|}$$



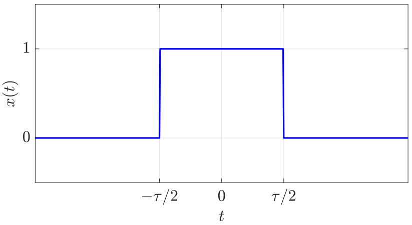
- we can compute the spectrum to be

$$\begin{aligned} X(\mathbf{j}\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-\mathbf{j}\omega t}\,\mathrm{d}t &= \int_{-\infty}^{0} e^{at} e^{-\mathbf{j}\omega t}\,\mathrm{d}t + \int_{0}^{\infty} e^{-at} e^{-\mathbf{j}\omega t}\,\mathrm{d}t \\ &= \frac{1}{a - \mathbf{j}\omega} + \frac{1}{a + \mathbf{j}\omega} \\ &= \frac{a + \mathbf{j}\omega - \mathbf{j}\omega + a}{(-a + \mathbf{j}\omega)(a + \mathbf{j}\omega)} \\ &= \frac{2a}{a^2 + \omega^2} \end{aligned}$$

CTFT Example: two-sided windowing signal

For $\tau > 0$ consider the signal

$$x(t) = u(t + \tfrac{\tau}{2}) - u(t - \tfrac{\tau}{2})$$
$$= \begin{cases} 1 & \text{if } -\tau/2 \le t < \tau/2 \\ 0 & \text{otherwise} \end{cases}$$

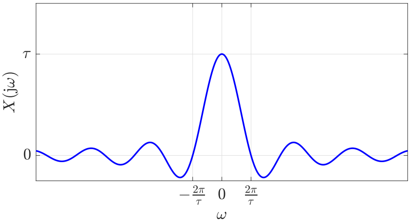


- we compute the CTFT of x to be

$$X(\mathbf{j}\omega) = \int_{-\infty}^{\infty} \left[u(t + \tfrac{\tau}{2}) - u(t - \tfrac{\tau}{2})\right] e^{-\mathbf{j}\omega t}\,\mathrm{d}t = \int_{-\tau/2}^{\tau/2} e^{-\mathbf{j}\omega t}\,\mathrm{d}t$$
$$= -\frac{1}{\mathbf{j}\omega} e^{-\mathbf{j}\omega t}\Big|_{-\tau/2}^{\tau/2} = \frac{2}{2\mathbf{j}\omega}(e^{\mathbf{j}\omega\tau/2} - e^{-\mathbf{j}\omega\tau/2})$$
$$= \frac{\sin(\omega\tau/2)}{\omega/2}$$

CTFT Example: two-sided windowing signal

$$x(t) = u(t + \tfrac{\tau}{2}) - u(t - \tfrac{\tau}{2}) \quad \overset{\text{CTFT}}{\Longleftrightarrow} \quad X(\mathbf{j}\omega) = \frac{\sin(\omega\tau/2)}{\omega/2}$$

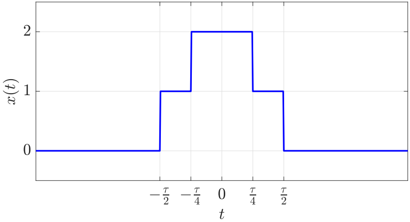


Signals of the form $\sin(x)/x$ are called "sinc" signal, and occurs frequently in signal processing theory.

Note: The window size in the time-domain was τ, but the spectrum is concentrated in an interval proportional to $1/\tau$.

CTFT Example: two-sided windowing signal

- how could we compute the CTFT of the signal shown below?



- we can express x as

$$x(t) = \left[u(t + \tfrac{\tau}{2}) - u(t - \tfrac{\tau}{2})\right] + \left[u(t + \tfrac{\tau}{4}) - u(t - \tfrac{\tau}{4})\right]$$

- we can combine the CTFTs of the two pieces to obtain

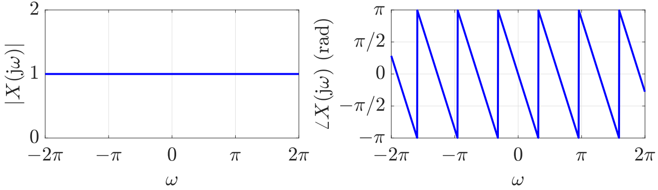
$$X(\mathbf{j}\omega) = \frac{\sin(\omega\tau/2)}{\omega/2} + \frac{\sin(\omega\tau/4)}{\omega/2}$$

CTFT Example: shifted impulse

- for $\tau \in \mathbb{R}$ consider the delayed-by-τ impulse $x(t) = \delta(t - \tau)$.
- the CTFT of this signal is

$$X(\mathbf{j}\omega) = \int_{-\infty}^{\infty} \delta(t-\tau)e^{-\mathbf{j}\omega t}\,\mathrm{d}t = e^{-\mathbf{j}\omega\tau}$$

and therefore $|X(\mathbf{j}\omega)| = 1$ and $\angle X(\mathbf{j}\omega) = -\omega\tau$



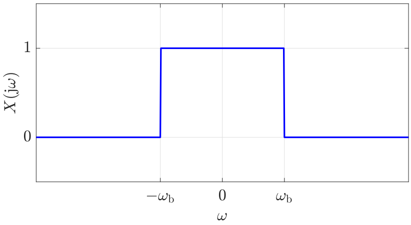
General principle: If x is very *concentrated* in the time domain, the spectrum X will be very *spread out* the in frequency domain.

i-CTFT Example: band limited spectrum

For $\omega_\mathrm{b} > 0$ consider the spectrum

$$X(\mathbf{j}\omega) = u(\omega + \omega_\mathrm{b}) - u(\omega - \omega_\mathrm{b})$$

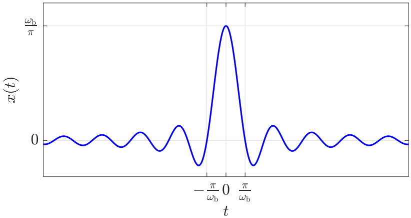
$$= \begin{cases} 1 & \text{if } -\omega_\mathrm{b} \leq \omega < \omega_\mathrm{b} \\ 0 & \text{otherwise} \end{cases}$$



$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\mathbf{j}\omega) e^{\mathbf{j}\omega t} \, \mathrm{d}\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_\mathrm{b}}^{\omega_\mathrm{b}} e^{\mathbf{j}\omega t} \, \mathrm{d}\omega$$

$$= \frac{1}{2\pi} \frac{1}{\mathbf{j}t} e^{\mathbf{j}\omega t} \bigg|_{-\omega_\mathrm{b}}^{\omega_\mathrm{b}} = \frac{\sin(\omega_\mathrm{b} t)}{\pi t}$$



i-CTFT Example: partial fraction expansion

Partial fraction expansion is a useful technique for doing i-CTFT calculations when you are dealing with ***rational*** functions of ω.

- suppose that some CT signal x has spectrum given by

$$X(\mathbf{j}\omega) = \frac{\mathbf{j}\omega}{(1+\mathbf{j}\omega)(2+\mathbf{j}\omega)}$$

- we rewrite this as

$$X(\mathbf{j}\omega) = \frac{A}{1+\mathbf{j}\omega} + \frac{B}{2+\mathbf{j}\omega} = \frac{(2A+B)+\mathbf{j}\omega(A+B)}{(1+\mathbf{j}\omega)(2+\mathbf{j}\omega)}$$

- comparing, we find that $A = -1$ and $B = 2$, so

$$X(\mathbf{j}\omega) = \frac{-1}{1+\mathbf{j}\omega} + \frac{2}{2+\mathbf{j}\omega} \quad \Longrightarrow \quad x(t) = -e^{-t}u(t) + 2e^{-2t}u(t).$$

i-CTFT Example: partial fraction expansion

When the numerator degree is equal to (or greater than) the denominator degree, this procedure needs to be modified a bit.

- consider the spectrum

$$X(\mathbf{j}\omega) = \frac{1 - \omega^2 + \mathbf{j}\omega}{2 - \omega^2 + 3\mathbf{j}\omega}$$

- the easiest way to proceed is to first note that

$$\lim_{\omega \to \infty} X(\mathbf{j}\omega) = 1$$

and to express X as $X(\mathbf{j}\omega) = 1 + \tilde{X}(\mathbf{j}\omega)$ where

$$\tilde{X}(\mathbf{j}\omega) = X(\mathbf{j}\omega) - 1 = \frac{1 - \omega^2 + \mathbf{j}\omega}{2 - \omega^2 + 3\mathbf{j}\omega} - 1 = \frac{-1 - 2\mathbf{j}\omega}{2 - \omega^2 + 3\mathbf{j}\omega}$$

i-CTFT Example: partial fraction expansion

- we therefore have that

$$X(\mathbf{j}\omega) = 1 - \frac{1 + 2\mathbf{j}\omega}{2 - \omega^2 + 3\mathbf{j}\omega} = 1 - \frac{1 + 2\mathbf{j}\omega}{(\mathbf{j}\omega + 1)(\mathbf{j}\omega + 2)}$$

- note that the second term has numerator degree *strictly* less than denominator degree; we can therefore apply partial fractions

$$\frac{1 + 2\mathbf{j}\omega}{(\mathbf{j}\omega + 1)(\mathbf{j}\omega + 2)} = \frac{A}{\mathbf{j}\omega + 2} + \frac{B}{\mathbf{j}\omega + 1}$$

  and quickly find that $A = 3$ and $B = -1$, therefore

$$X(\mathbf{j}\omega) = 1 + \frac{1}{\mathbf{j}\omega + 1} - \frac{3}{\mathbf{j}\omega + 2}$$

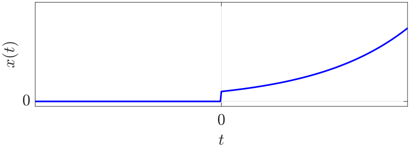
- based on our previous calculations, we finally obtain

$$x(t) = \delta(t) + e^{-t}u(t) - 3e^{-2t}u(t)$$

Does every CT signal have a CTFT? No.

for $a > 0$ consider the signal

$$x(t) = e^{at}u(t)$$



- let's try to compute the spectrum ...

$$X(\mathbf{j}\omega) = \int_{-\infty}^{\infty} e^{at}u(t)e^{-\mathbf{j}\omega t}\,\mathrm{d}t = \int_{0}^{\infty} e^{(a-\mathbf{j}\omega)t}\,\mathrm{d}t$$

$$= \frac{1}{a-\mathbf{j}\omega}e^{(a-\mathbf{j}\omega)t}\Big|_{0}^{\infty} = \infty$$

So unfortunately, some CT signals simply ***do not*** have a spectrum. Under what assumptions do things work as expected?

Existence of the CTFT for finite action signals

Theorem 4.1. If x has finite action, i.e., $x \in \mathsf{L}_1$, then the CTFT $X(\mathbf{j}\omega)$ is well-defined and satisfies $\lim_{\omega \to \pm\infty} |X(\mathbf{j}\omega)| = 0$.

Proof: Since x has finite action, we can bound the spectrum as

$$\begin{aligned} |X(\mathbf{j}\omega)| = \left| \int_{-\infty}^{\infty} x(t) e^{-\mathbf{j}\omega t} \, \mathrm{d}t \right| &\leq \int_{-\infty}^{\infty} |x(t)| \cdot |e^{-\mathbf{j}\omega t}| \, \mathrm{d}t \\ &= \int_{-\infty}^{\infty} |x(t)| \, \mathrm{d}t = \|x\|_1 < \infty \end{aligned}$$

so $X(\mathbf{j}\omega)$ is well-defined for all $\omega \in \mathbb{R}$. The proof of the second statement is outside our scope, but can be found by searching for "Riemann-Lebesgue Lemma".

This condition is ***not*** necessary; signals such as $e^{\mathbf{j}\omega_0 t}$ do not satisfy this, but nonetheless have well-defined CTFTs!

Existence of the CTFT for finite-energy signals

- it turns out that finite-energy signals are also CTFT-able
- as for finite-action signals, this condition is sufficient for existence, but not necessary

Theorem 4.2. If x has finite energy, i.e., $x \in \mathsf{L}_2$, then

(i) X exists and has finite energy, i.e., $X \in \mathsf{L}_2$, and

(ii) the signal and its spectrum satisfy ***Parseval's relation***

$$\int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\mathbf{j}\omega)|^2 \, \mathrm{d}\omega$$

A beautiful and surprising relationship between the energy of the signal and the energy of its spectrum.

CTFT Example: the complex exponential

- the complex exponential $x(t) = e^{\mathbf{j}\omega_0 t}$ does not have finite action nor finite energy, but is nonetheless CTFT-able
- consider the spectrum $X(\mathbf{j}\omega) = 2\pi\delta(\omega - \omega_0)$; an impulse placed at the frequency $\omega_0 \in \mathbb{R}$
- we compute the inverse CTFT of X to be

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{\mathbf{j}\omega t}\,\mathrm{d}\omega = e^{\mathbf{j}\omega_0 t}$$

so $e^{\mathbf{j}\omega_0 t}$ and $2\pi\delta(\omega - \omega_0)$ must be CTFT pairs!

$$x(t) = \underbrace{e^{\mathbf{j}\omega_0 t}}_{\text{spread out in time}} \quad \overset{\text{CTFT}}{\Longleftrightarrow} \quad X(\mathbf{j}\omega) = \underbrace{2\pi\delta(\omega - \omega_0)}_{\text{concentrated in frequency}}$$

CTFT Example: a periodic signal

We can also apply the CTFT to periodic signals

- let x be a periodic CT signal with fundamental period T_0, and let $\omega_0 = 2\pi/T_0$ be the fundamental angular frequency
- we represent x using the continuous-time Fourier ***series***

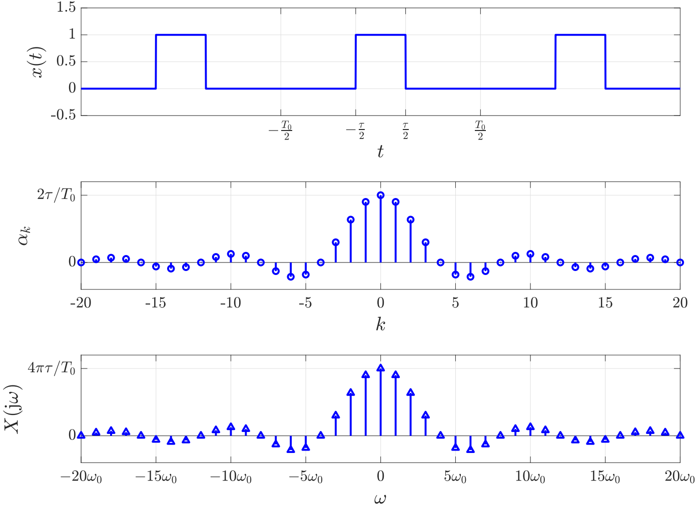
$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\mathbf{j}k\omega_0 t}.$$

- going term by term, the CTFT spectrum must be

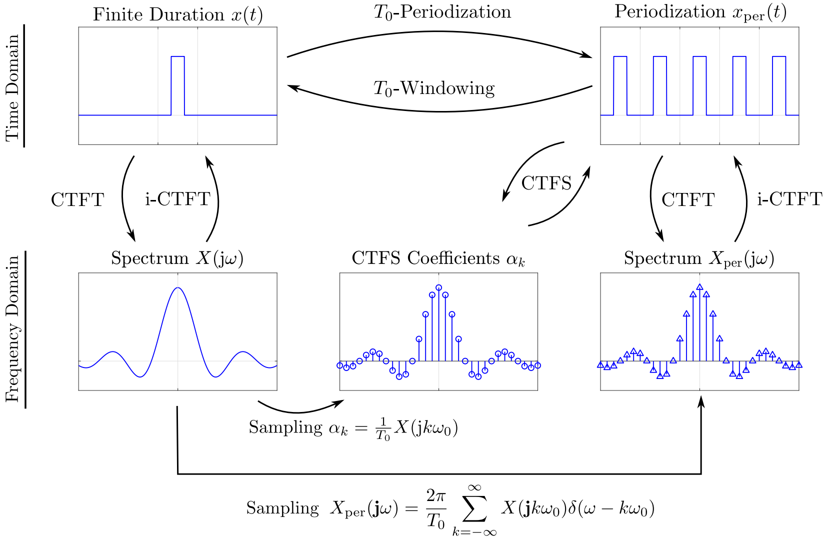
$$X(\mathbf{j}\omega) = 2\pi \sum_{k=-\infty}^{\infty} \alpha_k \delta(\omega - k\omega_0)$$

The CTFT of a periodic signal is a sum of impulse functions located at multiples of the fundamental frequency ω_0!

Example: the CTFT of a periodic signal



Relationships between CTFT and CTFS



The CTFT and signal manipulations

- suppose we have calculated the spectrum X of some signal x
- how can we find the spectra of signals obtained by simple manipulations, e.g., $x(t-1)$, $x(2t)$, $\mathrm{Re}\{x\}$, $\frac{\mathrm{d}x}{\mathrm{d}t}(t)$, $\ldots$?

There are many useful ***properties of the CTFT*** that you can use as ***shortcuts*** to find the spectra for these related signals

- **example:** since $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\mathbf{j}\omega)e^{\mathbf{j}\omega_0 t}$, note that

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\mathbf{j}\omega)e^{\mathbf{j}\omega_0(t-t_0)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(X(\mathbf{j}\omega)e^{-\mathbf{j}\omega_0 t_0}\right) e^{\mathbf{j}\omega_0 t}$$

so the spectrum of $x(t-t_0)$ is $X(\mathbf{j}\omega)e^{-\mathbf{j}\omega t_0}$

Table of CTFT properties

We let x be a CT signal with CTFT spectrum X.

Name	$x(t)$	$X(\mathbf{j}\omega)$
Time-shift by t_0	$x(t-t_0)$	$e^{-\mathbf{j}\omega t_0}X(\mathbf{j}\omega)$
Frequency-shift by ω_0	$e^{\mathbf{j}\omega_0 t}x(t)$	$X(\mathbf{j}(\omega-\omega_0))$
Conjugation	$x(t)^*$	$X(-\mathbf{j}\omega)^*$
Time-scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X(\mathbf{j}\omega/a)$
Differentiation	$\dot{x}(t)$	$(\mathbf{j}\omega)X(\mathbf{j}\omega)$
Time Multiplication	$tx(t)$	$\mathbf{j}\frac{\mathrm{d}X(\mathbf{j}\omega)}{\mathrm{d}\omega}$
Convolution	$\int_{-\infty}^{\infty} x(\tau)y(t-\tau)\,\mathrm{d}\tau$	$X(\mathbf{j}\omega)Y(\mathbf{j}\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\mathbf{j}\nu)Y(\mathbf{j}(\omega-\nu))\,\mathrm{d}\nu$
Real Part	$\mathrm{Re}\{x(t)\}$	$\frac{1}{2}(X(\mathbf{j}\omega)+X(-\mathbf{j}\omega)^*)$
Imag Part	$\mathrm{Im}\{x(t)\}$	$\frac{1}{2\mathbf{j}}(X(\mathbf{j}\omega)-X(-\mathbf{j}\omega)^*)$

CTFT Example: Importance of Magnitude vs. Phase

Conjugation	$x(t)^*$	$X(-\mathbf{j}\omega)^*$
Time-scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X(\mathbf{j}\omega/a)$

- suppose that $s(t)$ represents your favourite (real-valued) song
- if we play it ***backwards*** $b(t) = s(-t)$, what happens to the spectrum?

$$B(\mathbf{j}\omega) = S(-\mathbf{j}\omega) = S(\mathbf{j}\omega)^* \quad \Longrightarrow \quad |B(\mathbf{j}\omega)| = |S(\mathbf{j}\omega)|$$

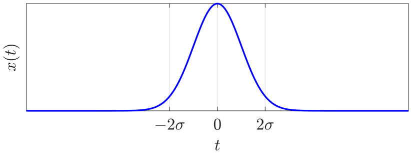
- the songs $s(t)$ and $b(t)$ would sound ***completely different*** to your ear, but their spectra have equal magnitudes at all frequencies ...

While we focus in ECE216 mostly on magnitude, because it's simpler, much of the important information is actually in the phase!

CTFT Example: Gaussian distribution

For $\sigma > 0$ consider the signal

$$x(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(t/\sigma)^2}$$



- **Fact:** $\int_{-\infty}^{\infty} x(t)\,\mathrm{d}t = 1$ (unit area)
- first, note that $x(t)$ satisfies the ***ordinary differential equation***

$$\dot{x}(t) = \frac{1}{\sigma\sqrt{2\pi}}(-\tfrac{t}{\sigma^2})e^{-\frac{1}{2}(t/\sigma)^2} = -\frac{t}{\sigma^2}x(t)$$

- from our table, we know that

$$\text{CTFT of } \dot{x}(t) = (\mathbf{j}\omega)X(\mathbf{j}\omega), \qquad \text{CTFT of } tx(t) = \mathbf{j}\frac{\mathrm{d}}{\mathrm{d}\omega}X(\mathbf{j}\omega)$$

- equating both sides, the CTFT $X(\mathbf{j}\omega)$ satisfies the ODE

$$\omega X(\mathbf{j}\omega) = -\frac{1}{\sigma^2}\frac{\mathrm{d}X(\mathbf{j}\omega)}{\mathrm{d}\omega} \qquad \Longleftrightarrow \qquad \frac{\mathrm{d}X(\mathbf{j}\omega)}{\mathrm{d}\omega} = -\sigma^2\omega X(\mathbf{j}\omega)$$

CTFT Example: Gaussian distribution

- this is the same ODE, just with the replacement $\sigma \to 1/\sigma$
- the solution therefore has the form

$$X(\mathbf{j}\omega) = ce^{-\frac{1}{2}(\sigma\omega)^2}, \qquad c = \text{unknown constant}$$

- however, since we know that

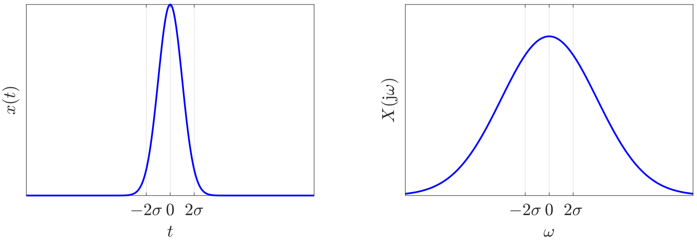
$$X(0) = \int_{-\infty}^{\infty} x(t)e^{-\mathbf{j}(0)t}\,\mathrm{d}t = \int_{-\infty}^{\infty} x(t)\,\mathrm{d}t = 1$$

it must be that $c = 1$

$$x(t) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(t/\sigma)^2} \quad \overset{\text{CTFT}}{\Longleftrightarrow} \quad X(\mathbf{j}\omega) = e^{-\frac{1}{2}(\sigma\omega)^2}$$

CTFT Example: Gaussian distribution

The CTFT of a Gaussian of width σ is a Gaussian of width $1/\sigma$!



General principle: If x is very *concentrated* in the time domain, the spectrum X will be very *spread out* the in frequency domain.

CTFT Properties: conjugate symmetry

If x is a real signal, meaning $x(t) \in \mathbb{R}$ for all $t \in \mathbb{R}$, then the spectrum $X(\mathbf{j}\omega)$ is conjugate symmetric, meaning $X(\mathbf{j}\omega)^* = X(-\mathbf{j}\omega)$ for all $\omega \in \mathbb{R}$.

Proof: From the definition $X(\mathbf{j}\omega) = \int_{-\infty}^{\infty} x(t)e^{-\mathbf{j}\omega t}\,\mathrm{d}t$ we have

$$X(\mathbf{j}\omega)^* = \int_{-\infty}^{\infty} x(t)^*(e^{-\mathbf{j}\omega t})^* = \int_{-\infty}^{\infty} x(t)e^{-(-1)\mathbf{j}\omega t} = X(-\mathbf{j}\omega).$$

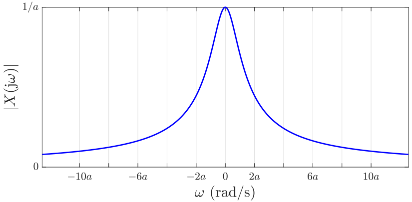
In polar form: $X(\mathbf{j}\omega) = |X(\mathbf{j}\omega)|e^{\mathbf{j}\angle X(\mathbf{j}\omega)}$, so we have

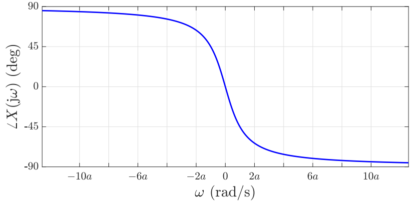
$$X(\mathbf{j}\omega)^* = |X(\mathbf{j}\omega)|e^{-\mathbf{j}\angle X(\mathbf{j}\omega)}$$
$$X(-\mathbf{j}\omega) = |X(-\mathbf{j}\omega)|e^{\mathbf{j}\angle X(-\mathbf{j}\omega)}$$

so the magnitude $|X(\mathbf{j}\omega)|$ is an ***even function*** of $\omega \in \mathbb{R}$ and the phase $\angle X(\mathbf{j}\omega)$ is an ***odd*** function of $\omega \in \mathbb{R}$.

CTFT Example: right-sided decaying exponential

$$x(t) = e^{-at}u(t) \quad \overset{\text{CTFT}}{\Longleftrightarrow} \quad X(\mathbf{j}\omega) = \frac{1}{\mathbf{j}\omega + a}$$





CTFT Properties: multiplication

- suppose we multiply two CT signals $x, \tilde{x}$ and set $y(t) = x(t)\tilde{x}(t)$
- we can calculate the CTFT of y

$$\begin{aligned} Y(\mathbf{j}\omega) &= \int_{-\infty}^{\infty} x(t)\tilde{x}(t)e^{-\mathbf{j}\omega t}\,\mathrm{d}t = \int_{-\infty}^{\infty} x(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{X}(\mathbf{j}\nu)e^{\mathbf{j}\nu t}\,\mathrm{d}\nu\right)e^{-\mathbf{j}\omega t}\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{X}(\mathbf{j}\nu)\left(\int_{-\infty}^{\infty} x(t)e^{-\mathbf{j}(\omega-\nu)t}\,\mathrm{d}t\right)\mathrm{d}\nu \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{X}(\mathbf{j}\nu)X(\mathbf{j}(\omega-\nu))\,\mathrm{d}\nu \end{aligned}$$

- this kind of integral is called a ***convolution*** between X and $\tilde{X}$

Multiplication in time-domain ⇔ convolution in frequency-domain.

Example: amplitude modulation

- in communication systems one often ***modulates*** a signal by multiplying it by a harmonic signal:

$$y(t) = \underbrace{\cos(\omega_c t)}_{\text{“carrier”}} \cdot \underbrace{x(t)}_{\text{“baseband”}} = \frac{1}{2}\left(e^{\mathbf{j}\omega_c t} + e^{-\mathbf{j}\omega_c t}\right) x(t)$$

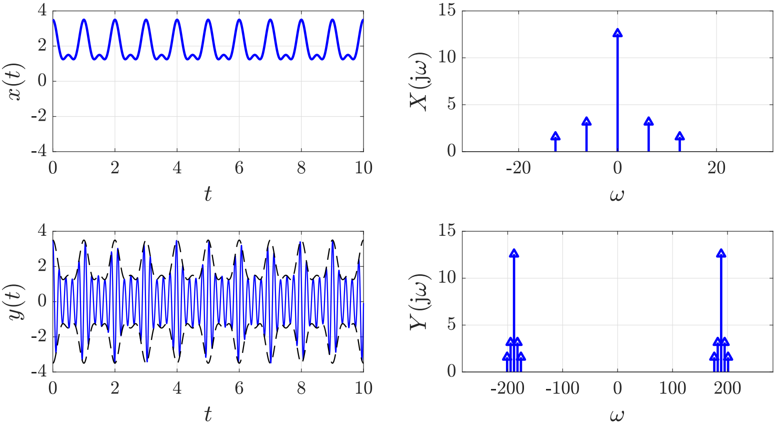
- the spectrum of cosine is $C(\mathbf{j}\omega) = 2\pi \frac{\delta(\omega-\omega_c)+\delta(\omega+\omega_c)}{2}$
- the spectrum of y is therefore given by

$$Y(\mathbf{j}\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\mathbf{j}\nu)X(\mathbf{j}(\omega-\nu))\,\mathrm{d}\nu = \frac{1}{2}\left(X(\mathbf{j}(\omega-\omega_c) + X(\mathbf{j}(\omega+\omega_c))\right)$$

- modulation ***shifts*** the entire frequency spectrum of X!
- in practice, this can allow for easier signal transmission and simultaneous transmission of multiple signals

Example: amplitude modulation

- $x(t) = 2 + \cos(2\pi t) + \frac{1}{2}\cos(4\pi t), \quad \omega_c = 60\pi \approx 188$



CTFT Properties: convolution

- the ***convolution*** of two CT signals x and $\tilde{x}$ is defined to be

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\tilde{x}(t-\tau)\,\mathrm{d}\tau.$$

- let's compute the CTFT of y

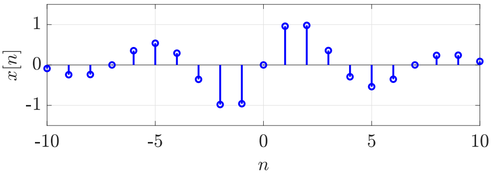
$$\begin{aligned}
Y(\mathbf{j}\omega) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\tilde{x}(t-\tau)\,\mathrm{d}\tau\right] e^{-\mathbf{j}\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} \tilde{x}(t-\tau)e^{-\mathbf{j}\omega t}\,\mathrm{d}t\right]\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} \tilde{x}(\sigma)e^{-\mathbf{j}\omega(\sigma+\tau)}\,\mathrm{d}\sigma\right]\,\mathrm{d}\tau \\
&= \left[\int_{-\infty}^{\infty} x(\tau)e^{-\mathbf{j}\omega\tau}\,\mathrm{d}\tau\right] \left[\int_{-\infty}^{\infty} \tilde{x}(\sigma)e^{-\mathbf{j}\omega\sigma}\,\mathrm{d}\sigma\right] = X(\mathbf{j}\omega)\tilde{X}(\mathbf{j}\omega)
\end{aligned}$$

Convolution in time-domain $\Leftrightarrow$ multiplication in frequency-domain.

The discrete-time Fourier transform (DTFT)

Everything we just did also works (with some minor changes) for aperiodic ***discrete-time*** signals!

- let x be an aperiodic DT signal



- you can directly mirror all the arguments we did for CT signals
- **Steps:** (i) window the signal to a duration of $2N + 1$, (ii) periodize, (iii) compute the ***DTFS***, and (iv) take the limit as $N \to \infty$ ...

The DT Fourier Transform (DTFT)

Definition 4.3. The ***discrete-time Fourier transform (DTFT)*** of a DT signal x is the function $X : \mathbb{R} \to \mathbb{C}$ defined pointwise by

$$X(e^{\mathbf{j}\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-\mathbf{j}\omega n}.$$

We call X the ***Fourier transform*** or ***spectrum*** of x.

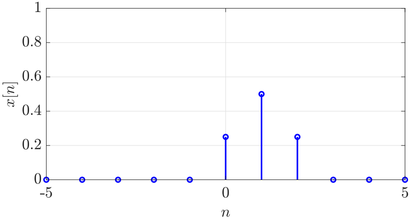
Definition 4.4. The ***inverse discrete-time Fourier transform (inverse DTFT)*** of a DT spectrum X is the DT signal $x : \mathbb{Z} \to \mathbb{C}$ defined pointwise by

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{\mathbf{j}\omega}) e^{\mathbf{j}\omega n} \, \mathrm{d}\omega.$$

DTFT Example: a sum of impulses

Consider the signal

$$x[n] = \tfrac{1}{4}\delta[n] + \tfrac{1}{2}\delta[n-1] + \tfrac{1}{4}\delta[n-2]$$

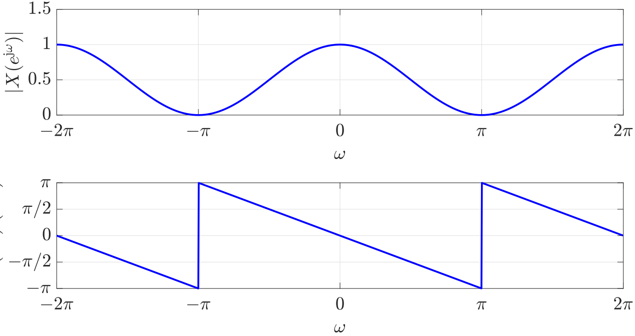


- we compute the DTFT to be

$$\begin{aligned}
X(e^{\mathbf{j}\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-\mathbf{j}\omega n} = \sum_{n=-\infty}^{\infty} (\tfrac{1}{4}\delta[n] + \tfrac{1}{2}\delta[n-1] + \tfrac{1}{4}\delta[n-2])e^{-\mathbf{j}\omega n} \\
&= \tfrac{1}{4}e^{-\mathbf{j}\omega(0)} + \tfrac{1}{2}e^{-\mathbf{j}\omega(1)} + \tfrac{1}{4}e^{-\mathbf{j}\omega(2)} \\
&= \tfrac{1}{4}\left(1 + 2e^{-\mathbf{j}\omega} + e^{-\mathbf{j}2\omega}\right) \\
&= \tfrac{1}{4}e^{-\mathbf{j}\omega}(e^{\mathbf{j}\omega} + 2 + e^{-\mathbf{j}\omega}) \\
&= \tfrac{1}{4}e^{-\mathbf{j}\omega}(2 + 2\cos(\omega))
\end{aligned}$$

DTFT Example: a sum of impulses

- magnitude is $|X(e^{\mathbf{j}\omega})| = \frac{1}{4}(2 + 2\cos(\omega))$, phase is $\angle X(e^{\mathbf{j}\omega}) = -\omega$

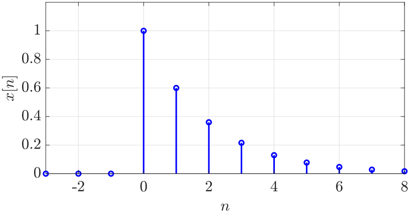


- the magnitude plot is 2π-periodic
- the phase plot is linear, but since phase is an angle, it is always 2π-periodic; in the plot above, the phase is "wrapped"

Example: Right-sided decaying exponential

For $0 < a < 1$, consider the signal

$$x[n] = a^n u[n]$$



- we compute the DTFT to be

$$X(e^{\mathbf{j}\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-\mathbf{j}\omega n} = \sum_{n=0}^{\infty} a^n e^{-\mathbf{j}\omega n}$$

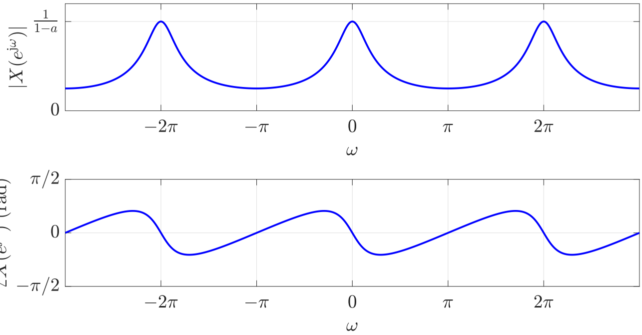
$$= \sum_{n=0}^{\infty} (ae^{-\mathbf{j}\omega})^n$$

$$= \frac{1}{1 - ae^{-\mathbf{j}\omega}}$$

Example: Right-sided decaying exponential

- The spectrum has magnitude

$$|X(e^{\mathbf{j}\omega})| = \left|\frac{1}{1 - a\cos(\omega) + \mathbf{j}a\sin(\omega)}\right| = \frac{1}{\sqrt{(1 - a\cos(\omega))^2 + a^2\sin(\omega)^2}}$$



The DTFT spectrum is always 2π-periodic

The spectrum $X(e^{\mathbf{j}\omega})$ of a DT signal x is a 2π-periodic function of ω.

Proof: Using $X(e^{\mathbf{j}\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-\mathbf{j}\omega n}$, we compute that

$$\begin{aligned} X(e^{\mathbf{j}(\omega+2\pi)}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-\mathbf{j}(\omega+2\pi)n} = \sum_{n=-\infty}^{\infty} x[n]e^{-\mathbf{j}\omega n}e^{-\mathbf{j}(2\pi)n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-\mathbf{j}\omega n}(e^{-\mathbf{j}(2\pi)})^n \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-\mathbf{j}\omega n}(1)^n \\ &= X(e^{\mathbf{j}\omega}). \end{aligned}$$

Comparison of CTFT vs. DTFT

CTFT: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\mathbf{j}\omega)e^{\mathbf{j}\omega t}, \quad X(\mathbf{j}\omega) = \int_{-\infty}^{\infty} x(t)e^{-\mathbf{j}\omega t}\,\mathrm{d}t.$

- represents aperiodic $x(t)$ as *continuous sum* of CT complex exp.
- spectrum $X(\mathbf{j}\omega)$ captures "amount" of frequency ω contained in x
- in general, the spectrum $X(\mathbf{j}\omega)$ is aperiodic

DTFT: $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{\mathbf{j}\omega})e^{\mathbf{j}\omega n}\,\mathrm{d}\omega, \quad X(e^{\mathbf{j}\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-\mathbf{j}\omega n}.$

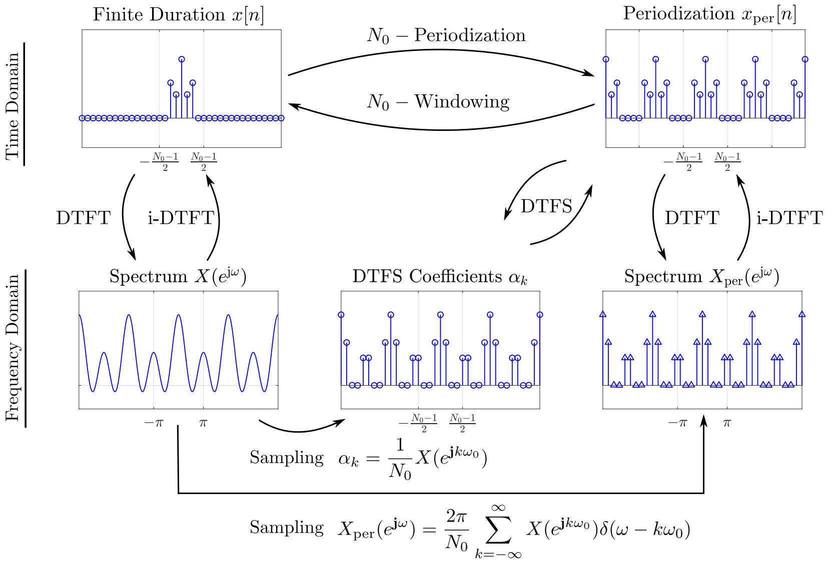
- represents aperiodic $x[n]$ as *continuous sum* of DT complex exp.
- spectrum $X(e^{\mathbf{j}\omega})$ captures "amount" of frequency ω contained in x
- the spectrum $X(e^{\mathbf{j}\omega})$ is always 2π-periodic

Table of DTFT properties

We let x be a DT signal with 2π-periodic DTFT spectrum X.

Name	$x[n]$	$X(e^{\mathbf{j}\omega})$
Time-shift by n_0	$x[n-n_0]$	$e^{-\mathbf{j}\omega n_0}X(e^{\mathbf{j}\omega})$
Frequency-shift by ω_0	$e^{\mathbf{j}\omega_0 n}x[n]$	$X(e^{\mathbf{j}(\omega-\omega_0)})$
Conjugation	$x[n]^*$	$X(e^{-\mathbf{j}\omega})^*$
Time-reversal	$x[-n]$	$X(e^{-\mathbf{j}\omega})$
Differencing	$x[n]-x[n-1]$	$(1-e^{-\mathbf{j}\omega})X(e^{\mathbf{j}\omega})$
Convolution	$\sum_{m=-\infty}^{\infty} x[m]y[n-m]$	$X(e^{\mathbf{j}\omega})Y(e^{\mathbf{j}\omega})$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{\mathbf{j}\nu})Y(e^{\mathbf{j}(\omega-\nu)})\,\mathrm{d}\nu$
Real Part	$\mathrm{Re}\{x[n]\}$	$\frac{1}{2}(X(e^{\mathbf{j}\omega})+X(e^{-\mathbf{j}\omega})^*)$
Imag Part	$\mathrm{Im}\{x[n]\}$	$\frac{1}{2\mathbf{j}}(X(e^{\mathbf{j}\omega})-X(e^{-\mathbf{j}\omega})^*)$

Relationships between DTFT and DTFS



Extra: proof of inverse CTFT relationship

Proposition 4.1. If x is a CT signal with CTFT X, then the i-CTFT of X recovers the original signal x.

Proof: The i-CTFT of X is

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\mathbf{j}\omega)e^{\mathbf{j}\omega t}\,\mathrm{d}\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)e^{-\mathbf{j}\omega\tau}\,\mathrm{d}\tau\right]e^{\mathbf{j}\omega t}\,\mathrm{d}\omega \\ &= \int_{-\infty}^{\infty} x(\tau)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathbf{j}\omega t}e^{-\mathbf{j}\omega\tau}\,\mathrm{d}\omega\right]\mathrm{d}\tau.\end{aligned}$$

By our previous results, we know the term in brackets is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathbf{j}\omega t}e^{-\mathbf{j}\omega\tau}\,\mathrm{d}\omega = \delta(t-\tau)$$

so we find that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\mathbf{j}\omega)e^{\mathbf{j}\omega t}\,\mathrm{d}\omega = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,\mathrm{d}\tau = x(t).$$

Relevant MATLAB commands

- MATLAB has some symbolic tools for FT computations

```
%% Define symbolic variables
syms t w

%% Define signal
x = [heaviside(t+1)-heaviside(t-1)]*exp(-abs(t));

%% Compute CTFT
X = fourier(x,t,w);

%% Compute inverse CTFT
x_recovered = ifourier(X,w,t);
```

Supplementary reading

- **O-W:** A. V. Oppenheim and S. Willsky, *Signals and Systems*, 2nd Ed.
- **BB:** B. Boulet, *Fundamentals of Signals and Systems*.
- **BPL:** B. P. Lathi, *Signal Processing and Linear Systems*.
- **K-S:** H. Kwakernaak and R. Sivan, *Modern Signals and Systems*.
- **EL-PV:** E. Lee and P. Varaiya, *Structure and Interpretation of Signals and Systems*, 2nd Ed.
- **ADL:** A. D. Lewis, *The Mathematical Theory of Signals and Systems*.

Topic	**O-W**	**BB**	**BPL**	**K-S**	**EL-PV**	**ADL**
The CTFT	4.1	5	4.1	7.1, 7.2, 7.4	10.5	V4 6.1–6.3
CTFT properties	4.3	5	4.2, 4.3, 4.6	7.4	10.7	V4 6.1–6.3
The DTFT	5.1	12	10.2	7.4	10.4	V4 7.1
DTFT properties	5.3	12	10.3, 10.4	7.4	10.7	V4 7.1

Personal Notes

Personal Notes

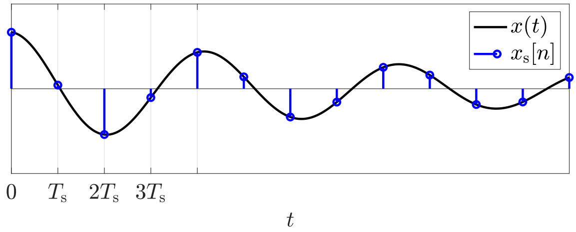
Personal Notes

5. Sampling, Aliasing, and Interpolation

- introduction
- the sampling function
- sampling theorem for band-limited signals
- reconstruction and interpolation

Introduction to sampling

- to ***sample*** some quantity of interest means to collect a measurement



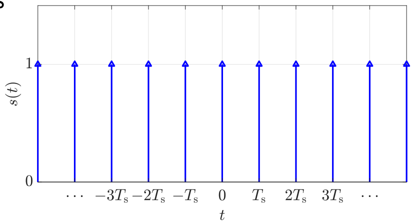
- from "very few" samples, there is likely little we can learn
- if we collect "enough" samples, we may be able to conclude more
- we will study the sampling of CT signals, and try to formulate these ideas mathematically

The sampling function

The ***sampling function*** is defined as

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_{\mathrm{s}})$$

where $T_{\mathrm{s}} > 0$ is the sampling period



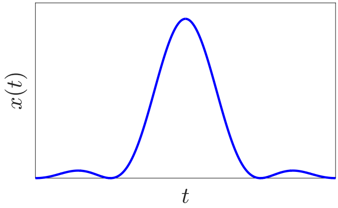
- this is an infinite "train" of CT impulses, spaced by T_{s} seconds
- multiplication by s "samples" the values of x

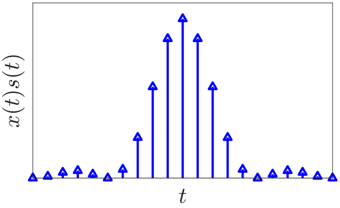
$$s(t)x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_{\mathrm{s}})x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_{\mathrm{s}})x(nT_{\mathrm{s}})$$

resulting in a ***weighted sum of impulses***

The sampling function

$$s(t)x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_\mathrm{s})x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_\mathrm{s})x(nT_\mathrm{s})$$





The CT Fourier Series of the sampling function

- with fund. frequency $\omega_{\mathrm{s}} = \frac{2\pi}{T_{\mathrm{s}}}$, the CTFS coefficients α_k of s are given by

$$\begin{aligned}\alpha_k = \frac{1}{T_{\mathrm{s}}}\int_{-T_{\mathrm{s}}/2}^{T_{\mathrm{s}}/2} s(t)e^{-\mathbf{j}k\omega_{\mathrm{s}}t}\,\mathrm{d}t &= \frac{1}{T_{\mathrm{s}}}\int_{-T_{\mathrm{s}}/2}^{T_{\mathrm{s}}/2}\left[\sum_{n=-\infty}^{\infty}\delta(t-nT_{\mathrm{s}})\right]e^{-\mathbf{j}k\omega_{\mathrm{s}}t}\,\mathrm{d}t \\ &= \sum_{n=-\infty}^{\infty}\frac{1}{T_{\mathrm{s}}}\left[\int_{-T_{\mathrm{s}}/2}^{T_{\mathrm{s}}/2}\delta(t-nT_{\mathrm{s}})e^{-\mathbf{j}k\omega_{\mathrm{s}}t}\,\mathrm{d}t\right]\end{aligned}$$

- the impulse $\delta(t-nT_{\mathrm{s}})$ is only in the interval $[-\frac{T_{\mathrm{s}}}{2}, \frac{T_{\mathrm{s}}}{2}]$ if $n = 0$:

$$\int_{-T_{\mathrm{s}}/2}^{T_{\mathrm{s}}/2}\delta(t-nT_{\mathrm{s}})e^{-\mathbf{j}k\omega_{\mathrm{s}}t}\,\mathrm{d}t = \begin{cases} e^{-\mathbf{j}k\omega_{\mathrm{s}}(0)} = 1 & \text{if } n = 0 \\ 0 & \text{otherwise}\end{cases}$$

- the CTFS coefficients are therefore $\alpha_k = \frac{1}{T_{\mathrm{s}}}$, so we find that

$$s(t) = \frac{1}{T_{\mathrm{s}}}\sum\nolimits_{k=-\infty}^{\infty} e^{\mathbf{j}k\omega_{\mathrm{s}}t} \qquad \text{(note: wow!)}$$

The CT Fourier Transform of the sampling function

- we previously studied how to take the CTFT of a periodic signal:

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\mathbf{j}k\omega_0 t} \quad \Longrightarrow \quad X(\mathbf{j}\omega) = 2\pi \sum_{k=-\infty}^{\infty} \alpha_k \delta(\omega - k\omega_0)$$

- therefore

$$s(t) = \frac{1}{T_{\mathrm{s}}} \sum\nolimits_{k=-\infty}^{\infty} e^{\mathbf{j}k\omega_{\mathrm{s}} t} \quad \Longrightarrow \quad S(\mathbf{j}\omega) = \frac{2\pi}{T_{\mathrm{s}}} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_{\mathrm{s}})$$

The CTFT of the sampling function is a sampling function!

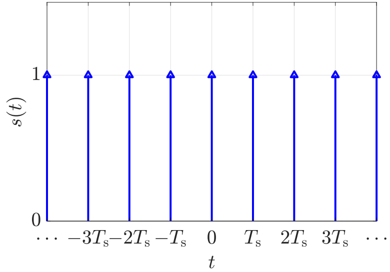
- the impulses are spaced in the frequency domain by ω_{s} rad/s

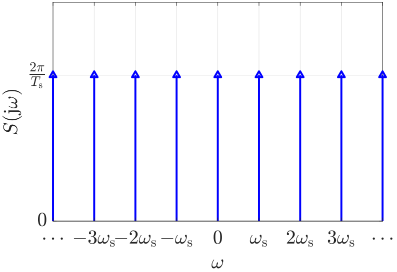
The "picket fence miracle"

The CTFT of the sampling function is a sampling function!

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_\mathrm{s})$$

$$S(\mathbf{j}\omega) = \frac{2\pi}{T_\mathrm{s}} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_\mathrm{s})$$

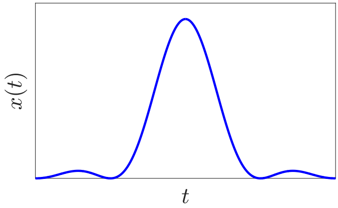


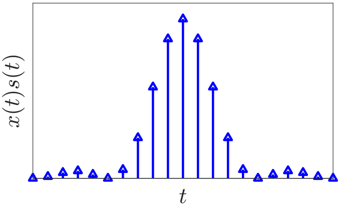


Band-limited signals

Definition 5.1. A CT signal x is ***bandlimited*** with ***bandwidth*** $B > 0$ if its CTFT spectrum X satisfies $X(\mathbf{j}\omega) = 0$ for all $|\frac{\omega}{2\pi}| \geq B$.

- the spectrum is zero outside the interval $[-2\pi B, 2\pi B]$
- bandlimited signals are an important ***model***; many signals in practice are bandlimited, or are filtered such that they become bandlimited
- for bandlimited x, suppose we sample with sampling period T_s





Sampling of band-limited signals

- what is the spectrum of the sampled signal $x_{\mathrm{s}}(t) = x(t)s(t)$?

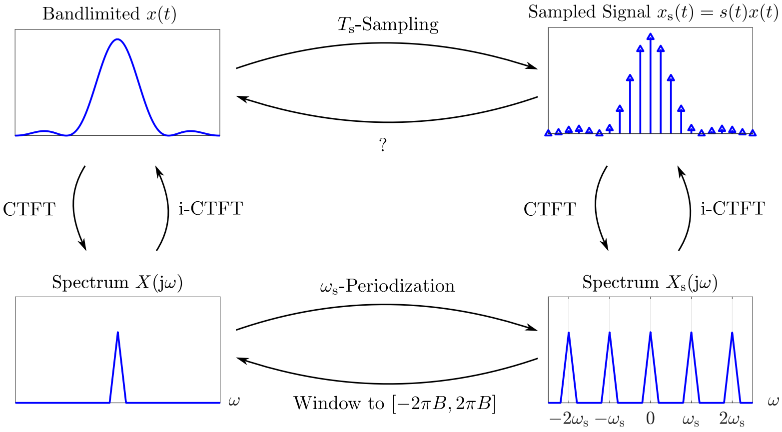
Multiplication in time-domain ⇔ convolution in frequency-domain.

$$\begin{aligned} X_{\mathrm{s}}(\mathbf{j}\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\mathbf{j}\nu)X(\mathbf{j}(\omega-\nu))\,\mathrm{d}\nu \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{2\pi}{T_{\mathrm{s}}}\sum_{k=-\infty}^{\infty}\delta(\nu-k\omega_{\mathrm{s}})\right]X(\mathbf{j}(\omega-\nu))\,\mathrm{d}\nu \\ &= \frac{1}{T_{\mathrm{s}}}\sum_{k=-\infty}^{\infty} X(\mathbf{j}(\omega-k\omega_{\mathrm{s}})) \end{aligned}$$

- the spectrum of the *sampled* signal is a ***periodized*** version of the spectrum of *original* signal, with period ω_{s} in the frequency domain!

Sampling of band-limited signals

- if $\omega_{\rm s} \geq 4\pi B$, or equivalently $T_{\rm s} \leq \frac{1}{2B}$ then the periodization is nicely spread out in the frequency domain, and the picture looks like this



Sampling of band-limited signals

- from ***only the samples***, we can obtain the periodized spectrum and window it to recover the original spectrum $X(\mathbf{j}\omega)$

$$X(\mathbf{j}\omega) = T_{\mathrm{s}} \cdot X_{\mathrm{s}}(\mathbf{j}\omega) \cdot \underbrace{[u(\omega + 2\pi B) - u(\omega - 2\pi B)]}_{\triangleq W(\mathbf{j}\omega)}$$

- now we can recover ***entire original signal*** $x(t)$ using the i-CTFT!

Theorem 5.1 (Nyquist-Shannon Theorem). Let x be a bandlimited CT signal with bandwidth B. If x is sampled with sampling period T_{s} satisfying $T_{\mathrm{s}} \leq \frac{1}{2B}$, then x can be exactly recovered from the samples.

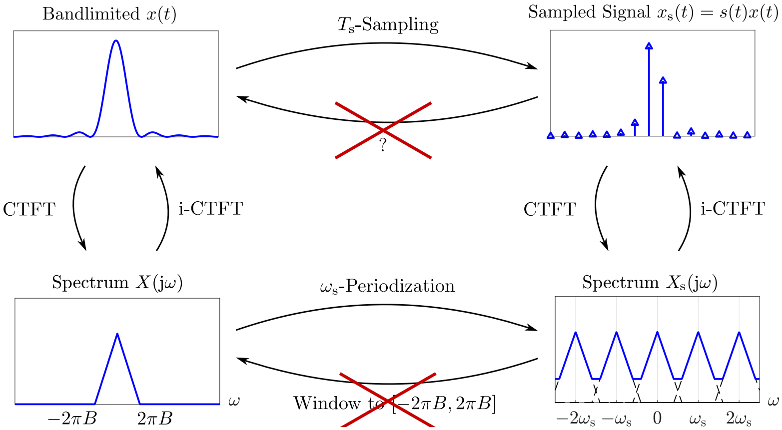
- the minimum sampling frequency ω_{s} which satisfies this is equal to $4\pi B$, and is called the ***Nyquist frequency***

Application: digital audio recording

- the human ear cannot perceive audio frequencies above 20kHz
- when recording audio, one can therefore filter out frequencies above 20kHz, producing a band-limited signal with $B = 20$kHz
- to perfectly reconstruct this bandlimited signal from samples, one needs to sample at $2B \approx 40$kHz
- this logic is how the standard rate of 44.1kHz was chosen for high-quality audio
- the extra 2.05kHz added to the bandwidth allows for some wiggle room in the design of the filter which produces the bandlimited signal

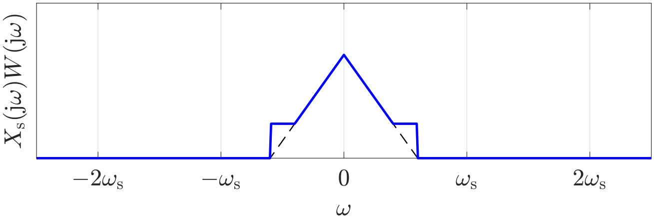
Aliasing

- what happens if $\omega_s < 4\pi B$?



Aliasing

- windowing the sampled spectrum $X_s(\mathbf{j}\omega)$ no longer recovers the original spectrum $X(\mathbf{j}\omega)$



- performing the i-CTFT on this will obviously ***not*** recover the original signal x. Instead, you will obtain a *distorted* version of the original signal; this distortion is called ***aliasing***

- to avoid aliasing, you must sample faster!

Extra: reconstruction and interpolation

- we can derive a more direct formula to express the original signal $x(t)$ in terms of the samples
- if $T_{\mathrm{s}} \leq \frac{1}{2B}$, our formula for the spectrum of x is

$$X(\mathbf{j}\omega) = T_{\mathrm{s}} \cdot X_{\mathrm{s}}(\mathbf{j}\omega) \cdot \underbrace{[u(\mathbf{j}(\omega + 2\pi B)) - u(\mathbf{j}(\omega - 2\pi B))]}_{\triangleq W(\mathbf{j}\omega)}$$

Multiplication in frequency-domain $\Leftrightarrow$ convolution in time-domain.

- we already know that

$$x_{\mathrm{s}}(t) = \sum_{n=-\infty}^{\infty} x(nT_{\mathrm{s}})\delta(t - nT_{\mathrm{s}})$$

Extra: reconstruction and interpolation

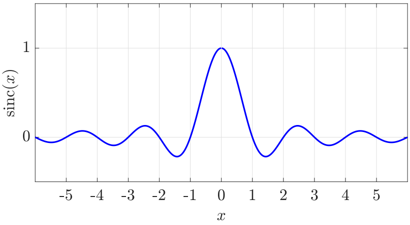
- from our previous calculations, the i-CTFT of $W(\mathbf{j}\omega)$ is

$$w(t) = \frac{\sin(2\pi Bt)}{\pi t} = 2B\frac{\sin(2\pi Bt)}{(2\pi Bt)} \triangleq 2B\,\mathrm{sinc}(2Bt)$$

Note: $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$

(i) $\mathrm{sinc}(0) = 1$, and

(ii) $\mathrm{sinc}(x) = 0$ for all $x \in \{\pm 1, \pm 2, \pm 3, \ldots\}$.



Extra: reconstruction and interpolation

- we can now recover x as

$$
\begin{aligned}
x(t) &= T_\mathrm{s} \int_{-\infty}^{\infty} x_\mathrm{s}(\tau) w(t-\tau)\,\mathrm{d}\tau \\
&= T_\mathrm{s} \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x(nT_\mathrm{s}) \delta(\tau - nT_\mathrm{s}) \right] \left[2B\mathrm{sinc}(2B(t-\tau)) \right] \mathrm{d}\tau \\
&= 2BT_\mathrm{s} \sum_{n=-\infty}^{\infty} x(nT_\mathrm{s}) \int_{-\infty}^{\infty} \delta\left(\tau - nT_\mathrm{s}\right) \mathrm{sinc}(2B(t-\tau))\,\mathrm{d}\tau \\
&= 2BT_\mathrm{s} \sum_{n=-\infty}^{\infty} x(nT_\mathrm{s}) \mathrm{sinc}(2B(t-nT_\mathrm{s}))
\end{aligned}
$$

- the reconstruction is a sum of scaled and time-shifted sinc signals
- if we assume that $T_\mathrm{s} = \frac{1}{2B}$, things simplify

Extra: reconstruction and interpolation

Time-domain reconstruction: $$x(t) = \sum_{n=-\infty}^{\infty} x(\tfrac{n}{2B})\mathrm{sinc}(2Bt - n)$$

- the samples of x are at times $t_\ell = \frac{\ell}{2B}$ for $\ell \in \mathbb{Z}$
- at the sampling instants, we have

$$\begin{aligned} x(t_\ell) = \sum_{n=-\infty}^{\infty} x(\tfrac{n}{2B})\mathrm{sinc}(2B\tfrac{\ell}{2B} - n) &= \sum_{n=-\infty}^{\infty} x(\tfrac{n}{2B})\mathrm{sinc}(\ell - n) \\ &= x(\tfrac{\ell}{2B}) \\ &= x(t_\ell) \end{aligned}$$

  since only one term in the sum gives a non-zero contribution

- for each sample, there is a sinc function centred directly at the sample
- we achieve ***exact*** reconstruction at the sampling instants

Extra: convolution with the sampling function

- implicit in our previous arguments is the following nice fact

Periodization $\iff$ convolution with $s(t)$

$$
\begin{aligned}
(x * s)(t) &= \int_{-\infty}^{\infty} x(\tau)s(t-\tau)\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_{\mathrm{s}} - \tau)\right]\mathrm{d}\tau \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\delta(t - nT_{\mathrm{s}} - \tau)\,\mathrm{d}\tau \\
&= \sum_{n=-\infty}^{\infty} x(t - nT_{\mathrm{s}})
\end{aligned}
$$

Supplementary reading

- **O-W:** A. V. Oppenheim and S. Willsky, *Signals and Systems*, 2nd Ed.
- **BB:** B. Boulet, *Fundamentals of Signals and Systems*.
- **BPL:** B. P. Lathi, *Signal Processing and Linear Systems*.
- **K-S:** H. Kwakernaak and R. Sivan, *Modern Signals and Systems*.
- **EL-PV:** E. Lee and P. Varaiya, *Structure and Interpretation of Signals and Systems*, 2nd Ed.
- **ADL:** A. D. Lewis, *The Mathematical Theory of Signals and Systems*.

Topic	**O-W**	**BB**	**BPL**	**K-S**	**EL-PV**	**ADL**
The sampling theorem	7.1	15	5.1	9.2	11.3	
Aliasing	7.3	15	5.1	9.2	11.1, 11.3	
Signal reconstruction	7.2	15	5.1	9.2	11.2	

Personal Notes

Personal Notes

Personal Notes

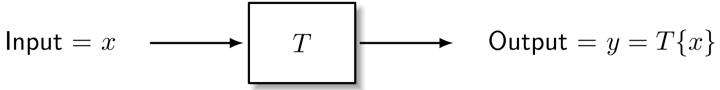
6. Fundamentals of Continuous-Time Systems

- definition of a CT system and examples
- linearity, time-invariance, causality
- memory, invertibility, stability
- linear time-invariant (LTI) systems
- impulse response of a LTI system
- response of a LTI system and convolution
- convolution
- LTI system properties and the impulse response
- more on LTI systems and causality
- series, parallel, and feedback combinations of LTI systems
- differential equations and CT LTI systems

What is a CT system

- a ***system*** is any entity that interacts with its external environment though input and output signals

- in this course, we focus on systems which produce a unique output for a given input, i.e., an ***input-output mapping*** system



- **notation:** $T\{x\}(t)$ is the *value* of the output signal at time $t \in \mathbb{R}$.

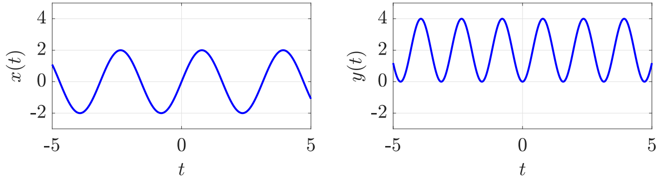
- CT systems often (but not always) model ***physical systems with inputs and outputs***, such as circuits, electromechanical systems, aerodynamics, thermodynamics, biological, chemical, social, . . .

Pointwise definition of a CT system

- a CT system takes a CT input and produces a CT output

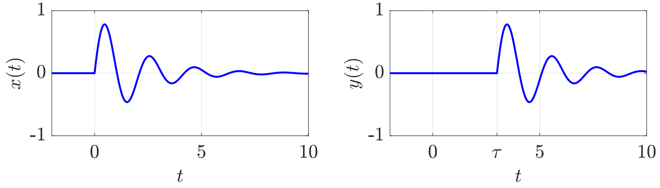
One way to define a CT system is ***pointwise***: give a formula for $y(t) = T\{x\}(t)$ in terms of (potentially) *all* the input signal values $\{x(t)\}_{t\in\mathbb{R}}$. This yields an ***explicit*** formula for the output.

- **example:** the system T_{sq} defined by $y(t) = (x(t))^2$ produces an output which is the squared value of the input at each $t \in \mathbb{R}$

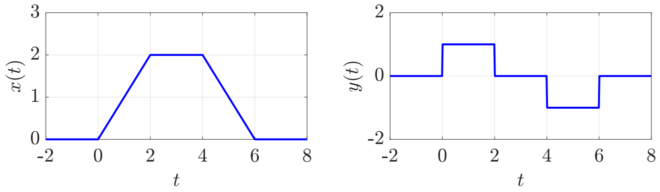


Pointwise definition of a CT system

- **example:** the τ-delay system T_{delay} defined by $y(t) = x(t - \tau)$.

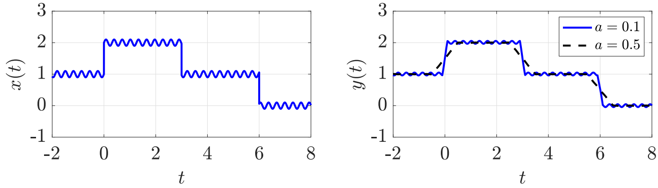


- **example:** the derivative system T_{diff} defined by $y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$

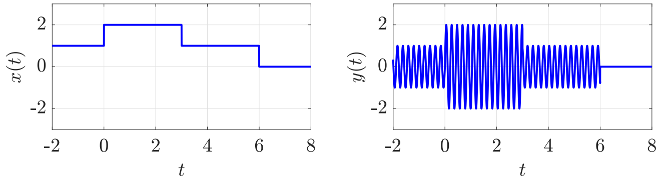


Pointwise definition of a CT system

- **example:** for $a > 0$ the system $T_{\text{avg}}\{x\}(t) = \frac{1}{2a}\int_{t-a}^{t+a} x(\tau)\,\mathrm{d}\tau$ averages the input signal over the time window $[t-a, t+a]$

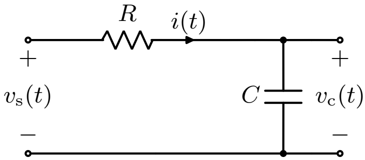


- **example:** the system T_{mod} defined by $y(t) = x(t) \cdot A\sin(\omega_0 t)$



Example: the RC circuit

Capacitor: $i(t) = C\frac{\mathrm{d}v_\mathrm{c}}{\mathrm{d}t}$, Ohm's Law: $i(t) = \frac{1}{R}[v_\mathrm{s}(t) - v_\mathrm{c}(t)]$



1st-order linear c.c. ODE

$$RC\frac{\mathrm{d}v_\mathrm{c}}{\mathrm{d}t} + v_\mathrm{c}(t) = v_\mathrm{s}(t)$$

- differential equations do not have unique solutions, but a system ***must*** produce a unique output for any given input
- to ensure uniqueness, we will impose that the input v_s and the desired response v_c should both be ***right-sided***

Example: the RC circuit

- let's compute $v_\mathrm{c}(t)$ via the method of ***integrating factors***
- multiply both sides by $e^{t/RC}$ to obtain

$$e^{t/RC}\left[\frac{\mathrm{d}v_\mathrm{c}}{\mathrm{d}t}+\frac{1}{RC}v_\mathrm{c}(t)\right]=e^{t/RC}\frac{1}{RC}v_\mathrm{s}(t)$$

which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(e^{\tau/RC}v_\mathrm{c}(\tau)\right)=\frac{1}{RC}e^{\tau/RC}v_\mathrm{s}(\tau)$$

- integrating from $-\infty$ to t, the left-hand side simplifies to

$$\int_{-\infty}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(e^{\tau/RC}v_\mathrm{c}(\tau)\right)\,\mathrm{d}t=e^{t/RC}v_\mathrm{c}(t)-\underbrace{\lim_{\tau\to-\infty}e^{\tau/RC}v_\mathrm{c}(\tau)}_{=0\text{ by right-sidedness}}=e^{t/RC}v_\mathrm{c}(t)$$

so the equation becomes

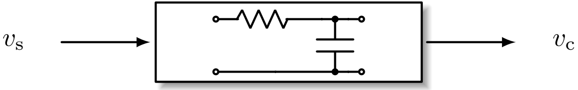
$$e^{t/RC}v_\mathrm{c}(t)=\int_{-\infty}^{t}\tfrac{1}{RC}e^{\tau/RC}v_\mathrm{s}(\tau)\,\mathrm{d}\tau$$

Example: the RC circuit

- rearranging, we obtain the point-wise system representation

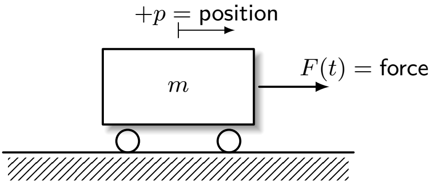
$$v_{\mathrm{c}}(t) = \int_{-\infty}^{t} \frac{1}{RC} e^{-(t-\tau)/RC} v_{\mathrm{s}}(\tau)\,\mathrm{d}\tau$$

- our RC circuit defines a system T_{RC}, which transforms a voltage source signal into a capacitor voltage signal



We will more systematically examine systems defined by ODEs at the end of the chapter.

Example: a point mass



Newton's 2nd Law

$$m\frac{\mathrm{d}}{\mathrm{d}t}v(t) = F(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}p(t) = v(t)$$

- we again assume that the force $F(t)$ and responses $p(t), v(t)$ are right-sided
- integrating, we obtain

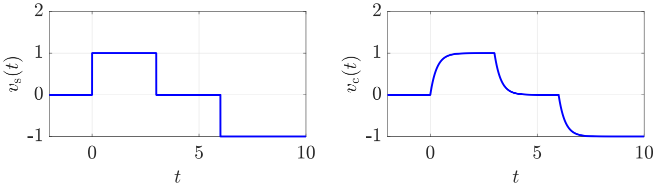
$$v(\tau) = \frac{1}{m}\int_{-\infty}^{\tau} F(\sigma)\,\mathrm{d}\sigma, \qquad p(t) = \int_{-\infty}^{t} v(\tau)\,\mathrm{d}\tau$$

and we again get a pointwise system representation

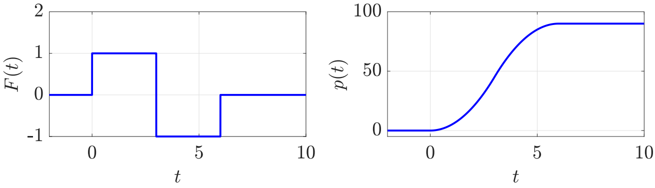
$$p(t) = \frac{1}{m}\int_{-\infty}^{t}\int_{-\infty}^{\tau} F(\sigma)\,\mathrm{d}\sigma\mathrm{d}\tau \qquad \text{or simply} \quad p = T_{\mathrm{Newton}}\{F\}$$

Response of RC circuit and point mass

- response of the RC circuit to a voltage input

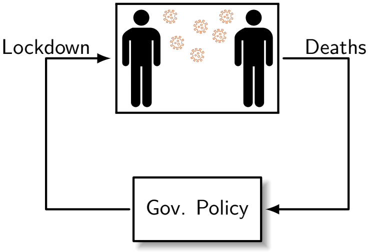


- response of the point mass to a force input



More complex example: control of COVID-19

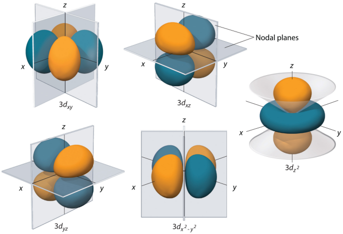
- systems can also model very large-scale phenomena



- **COVID-19/population system:** complicated differential equations modelling interaction between disease and population
- **Government policy system:** decision-making rules for when to enact/lift restrictions and lockdowns (i.e., a feedback controller)

More complex example: systems described by PDEs

- ***partial differential equations*** are also key sources of CT systems
- **examples:** fluid flow (Navier-Stokes), electromagnetic fields/waves (Maxwell), and *quantum mechanics* (Schrödinger)



$$-\frac{\hbar^2}{2m}\nabla^2\psi(\boldsymbol{r},t) + V(\boldsymbol{r})\psi(\boldsymbol{r},t) = i\hbar\frac{\partial}{\partial t}\psi(t,\boldsymbol{r})$$

- **Input:** photon excitation, **Output:** quantum state of atom/molecule
- more PDEs in APM 384

System properties: linearity

Definition 6.1. A CT system T is ***linear*** if for any two input signals $x, \tilde{x}$ and any two constants $\alpha, \tilde{\alpha} \in \mathbb{C}$ it holds that

$$T\{\alpha x + \tilde{\alpha}\tilde{x}\} = \alpha T\{x\} + \tilde{\alpha} T\{\tilde{x}\}.$$

- **in words:** if the input is a *linear combination* of different signals, you can apply T to *each piece* and add the results ("***superposition***")
- for a linear system, zero input always produces zero output (why?)
- superposition extends to discrete and continuous sums (integrals)

$$T\left\{\sum_{j=1}^{n} \alpha_j x_j\right\} = \sum_{j=1}^{n} \alpha_j T\{x_j\}, \quad T\left\{\int \alpha(\tau) x_\tau \, \mathrm{d}\tau\right\} = \int \alpha(\tau) T\{x_\tau\} \, \mathrm{d}\tau.$$

Example: the squaring system

- **recall:** the squaring system T_{sq} defined by $y(t) = (x(t))^2$
- let $x(t)$ and $\tilde{x}(t)$ be two inputs with corresponding outputs

$$y(t) = T_{\text{sq}}\{x\}(t) = (x(t))^2, \qquad \tilde{y}(t) = T_{\text{sq}}\{\tilde{x}\}(t) = (\tilde{x}(t))^2.$$

- for constants $\alpha, \tilde{\alpha}$, we have

$$\begin{aligned} T_{\text{sq}}\{\alpha x + \tilde{\alpha}\tilde{x}\}(t) &= (\alpha x(t) + \tilde{\alpha}\tilde{x}(t))^2 \\ &= \alpha^2(x(t))^2 + \tilde{\alpha}^2(\tilde{x}(t))^2 + 2\alpha\tilde{\alpha}x(t)\tilde{x}(t). \end{aligned}$$

- on the other hand

$$\alpha T_{\text{sq}}\{x\}(t) + \tilde{\alpha}T_{\text{sq}}\{\tilde{x}\}(t) = \alpha(x(t))^2 + \tilde{\alpha}(\tilde{x}(t))^2$$

Therefore, the system T_{sq} is ***not*** linear

Example: the RC circuit

- **recall:** the RC circuit $T_{\mathrm{RC}}\{v_{\mathrm{s}}\}(t) = \int_{-\infty}^{t} \frac{1}{RC} e^{-(t-\tau)/RC} v_{\mathrm{s}}(\tau)\,\mathrm{d}\tau$
- let $v_{\mathrm{c}}, \tilde{v}_{\mathrm{c}}$ be two input signals and let $\alpha, \tilde{\alpha}$ be constants
- we calculate that

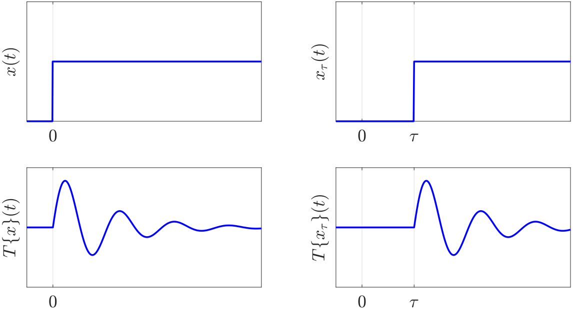
$$
\begin{aligned}
T_{\mathrm{RC}}\{\alpha v_{\mathrm{s}} + \tilde{\alpha}\tilde{v}_{\mathrm{s}}\}(t) &= \int_{-\infty}^{t} \tfrac{1}{RC} e^{-(t-\tau)/RC} (\alpha v_{\mathrm{s}}(\tau) + \tilde{\alpha}\tilde{v}_{\mathrm{s}}(\tau))\,\mathrm{d}\tau \\
&= \alpha \int_{-\infty}^{t} \tfrac{1}{RC} e^{-(t-\tau)/RC} v_{\mathrm{s}}(\tau)\,\mathrm{d}\tau + \tilde{\alpha} \int_{-\infty}^{t} \tfrac{1}{RC} e^{-(t-\tau)/RC} \tilde{v}_{\mathrm{s}}(\tau)\,\mathrm{d}\tau \\
&= \alpha T_{\mathrm{RC}}\{v_{\mathrm{s}}\}(t) + \tilde{\alpha} T_{\mathrm{RC}}\{\tilde{v}_{\mathrm{s}}\}(t)
\end{aligned}
$$

Therefore, the system T_{RC} is linear.

System properties: time-invariance

- ***notation:*** let $x_\tau(t) = x(t-\tau)$ be short form notation for a time-delayed signal

Definition 6.2. A CT system T is ***time-invariant*** if for any input x with output $y = T\{x\}$, it holds that $y_\tau = T\{x_\tau\}$ for all time shifts $\tau \in \mathbb{R}$.



physical meaning: an experiment on the system *tomorrow* will produce the same results as an experiment on the system *today*.

Example: the squaring system

To check if a system is time-invariant, we compute y_τ, compute $T\{x_\tau\}$, and see if they are equal.

- for the squaring system, let x be an input with corresponding output

$$y(t) = T_{\rm sq}\{x\}(t) = (x(t))^2$$

- if we simply shift the obtained output, we obtain

$$y_\tau(t) = (x(t-\tau))^2$$

- if we instead shift the input signal to be $x_\tau(t) = x(t-\tau)$, we compute

$$T_{\rm sq}\{x_\tau\}(t) = (x_\tau(t))^2 = (x(t-\tau))^2$$

These two calculations agree, so $T_{\rm sq}$ is time-invariant.

Example: the modulation system

- for the system T_{mod} let x be an input with output

$$y(t) = T_{\text{mod}}\{x\}(t) = x(t) \cdot A\sin(\omega_0 t)$$

- if we simply shift the obtained output, we obtain

$$y_\tau(t) = x(t-\tau) \cdot A\sin(\omega_0(t-\tau))$$

- if we instead shift the input signal to be $x_\tau(t) = x(t-\tau)$, we compute

$$T_{\text{mod}}\{x_\tau\}(t) = x(t-\tau)A\sin(\omega_0 t)$$

These two calculations disagree, so T_{mod} is not time-invariant.

Example: the RC circuit

- for T_{RC} we have $y(t) = \int_{-\infty}^{t} \frac{1}{RC} e^{-(t-\sigma)/RC} x(\sigma)\,\mathrm{d}\sigma$
- if we simply shift the obtained output by $\tau \in \mathbb{R}$, we have

$$y_\tau(t) = y(t-\tau) = \int_{-\infty}^{t-\tau} \tfrac{1}{RC} e^{-(t-\tau-\sigma)/RC} x(\sigma)\,\mathrm{d}\sigma.$$

- if we instead shift the input signal by τ, the output $T\{x_\tau\}$ is

$$\begin{aligned} T_{\mathrm{RC}}\{x_\tau\}(t) &= \int_{-\infty}^{t} \tfrac{1}{RC} e^{-(t-\xi)/RC} x(\xi-\tau)\,\mathrm{d}\xi \\ &= \int_{-\infty}^{t-\tau} \tfrac{1}{RC} e^{-(t-\tau-\sigma)/RC} x(\sigma)\,\mathrm{d}\sigma, \qquad \text{where } \sigma = t-\tau. \end{aligned}$$

These two calculations agree, so T_{RC} is time-invariant. Physically, the circuit is time-invariant because R and C are constant.

System properties: causality

Definition 6.3. A CT system T is ***causal*** if for all $t \in \mathbb{R}$, the output value $y(t)$ depends only on the *past and present* input values $\{x(\tau)\}_{\tau \leq t}$.

- ***physical meaning:*** the system does not "look into the future"

You can often check causality by just inspecting the formula for $y(t)$.

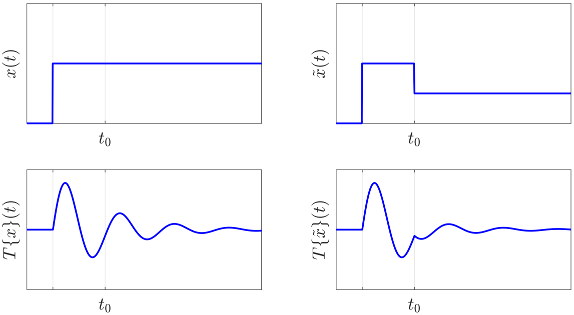
- the RC circuit $y(t) = \int_{-\infty}^{t} \frac{1}{RC} e^{-(t-\tau)/RC} x(\tau)\,\mathrm{d}\tau$ is causal, because the integral uses only the values $\{x(\tau)\}$ for $-\infty \leq \tau \leq t$.

- the averaging system $y(t) = \frac{1}{2a} \int_{t-a}^{t+a} x(\tau)\,\mathrm{d}\tau$ is ***not*** causal, because the integral uses the values $\{x(\tau)\}$ for $t \leq \tau \leq t + a$

System properties: causality (equivalent definition)

- the following equivalent definition is easy to visualize

Definition 6.4. A CT system T is causal if for any time t_0 and any two inputs $x, \tilde{x}$ that satisfy $x(t) = \tilde{x}(t)$ for all $t \leq t_0$, the corresponding outputs $y = T\{x\}$ and $\tilde{y} = T\{\tilde{x}\}$ satisfy $y(t) = \tilde{y}(t)$ for all $t \leq t_0$.



interpretation: if two inputs agree up to some time, then the corresponding outputs must also agree up to that time.

Proof that Def. 6.3 and Def. 6.4 are the same

Proof: Suppose the system is causal in the sense of Definition 6.3. Let x be an input with corresponding output $y = T\{x\}$. Now let $t_0 \in \mathbb{R}$, and suppose $\tilde{x}$ is another input satisfying $\tilde{x}(\tau) = x(\tau)$ for all $\tau \leq t_0$. For any time $t \leq t_0$, we have that $\tilde{y}(t) = T\{\tilde{x}\}(t)$ depends only on $\tilde{x}$ up to time t, which is identical to x up to time t. Thus,

$$\tilde{y}(t) = T\{\tilde{x}\}(t) = T\{x\}(t) = y(t),$$

and so the two outputs agree at time t. Since this holds for all times $t \leq t_0$, the system is causal in the sense of Definition 6.4.

Conversely, suppose the system is causal in the sense of Definition 6.4. Let $t_0 \in \mathbb{R}$ and let x be an input. Define a new input $\tilde{x}$ via

$$\tilde{x}(t) = \begin{cases} x(t) & \text{if } t \leq t_0 \\ 0 & \text{if } t > t_0. \end{cases}$$

Obviously, $\tilde{x}$ just contains the values of x up to time t_0. Therefore, $T\{x\}$ and $T\{\tilde{x}\}$ also agree up to time t_0, so $y(t) = T\{x\}(t) = T\{\tilde{x}\}$ depends only on the input values of x up to time t_0, so the system is causal in the sense of Definition 6.3.

System properties: memoryless

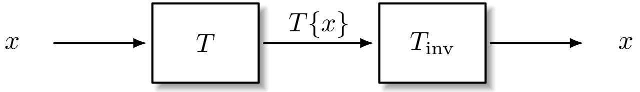
Definition 6.5. A CT system T is ***memoryless*** if for all times $t \in \mathbb{R}$, the output value $y(t)$ depends only on the input value $x(t)$ at that same time.

- ***meaning:*** only the current time matters, not the past or future
- this is a special case of causality

- the squaring system $T\{x\}(t) = (x(t))^2$ is memoryless
- the modulation system $T\{x\}(t) = A\sin(\omega_0 t) \cdot x(t)$ is memoryless
- the derivative system $T\{x\}(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$ is not memoryless
- the RC circuit $y(t) = \int_{-\infty}^{t} \frac{1}{RC} e^{-(t-\tau)/RC} x(\sigma)\,\mathrm{d}\sigma$ is not memoryless

System properties: invertibility

Definition 6.6. A CT system T is ***invertible*** if there exists another CT system T_{inv} such that $T_{\text{inv}}\{T\{x\}\} = T\{T_{\text{inv}}\{x\}\} = x$ for all inputs x.



- ***physical meaning:*** you can "undo" the operation of T
- **example:** the delay system $T\{x\}(t) = x(t - \tau)$ is invertible, with its inverse being $T_{\text{inv}}\{x\}(t) = x(t + \tau)$.
- **example:** the squaring system $T\{x\}(t) = (x(t))^2$ is not invertible, for the same reason the function $y = x^2$ is not invertible.

System properties: stability (linear systems only)

- **recall:** a CT signal x has ***finite amplitude*** or is ***bounded*** if $\|x\|_\infty = \max_{t\in\mathbb{R}} |x(t)|$ is finite, and if so, we write $x \in \mathsf{L}_\infty$
- **note:** if we label $A = \|x\|_\infty$, then this means that $|x(t)| \leq A$ for all $t \in \mathbb{R}$; the signal's magnitude is always less than A

Definition 6.7. A linear CT system T is ***Bounded-Input Bounded-Output (BIBO) stable*** if there is a constant $K \geq 0$ such that $\|y\|_\infty \leq K\|x\|_\infty$ for all bounded inputs x and outputs $y = T\{x\}$.

- **roughly speaking:** bounded inputs produce bounded outputs
- stability is a critical property in many engineering applications

Example: the RC circuit

- **recall:** the RC circuit $T_{\mathrm{RC}}\{x\}(t) = \int_{-\infty}^{t} \frac{1}{RC} e^{-(t-\tau)/RC} x(\tau)\,\mathrm{d}\tau$
- let x be a bounded input, i.e., $x \in \mathsf{L}_\infty$. We try to bound y:

$$
\begin{aligned}
|y(t)| &= \left| \int_{-\infty}^{t} \tfrac{1}{RC} e^{-(t-\tau)/RC} x(\tau)\,\mathrm{d}\tau \right| \\
&\leq \int_{-\infty}^{t} \tfrac{1}{RC} e^{-(t-\tau)/RC} |x(\tau)|\,\mathrm{d}\tau \\
&\leq \int_{-\infty}^{t} \tfrac{1}{RC} e^{-(t-\tau)/RC} \cdot \left(\max_{\sigma \in \mathbb{R}} |x(\sigma)| \right) \mathrm{d}\tau \\
&= \|x\|_\infty \int_{-\infty}^{t} \tfrac{1}{RC} e^{-(t-\tau)/RC}\,\mathrm{d}\tau = \|x\|_\infty .
\end{aligned}
$$

- so $\|y\|_\infty = \max_{t \in \mathbb{R}} |y(t)| \leq \|x\|_\infty$.

Since x was arbitrarily chosen, T_{RC} is BIBO stable with $K = 1$.

Example: integrator system

- consider the ***integrator*** system $T_{\text{int}}\{x\}(t) = \int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau$

To show that a system is ***not*** BIBO stable, you need to give an example of a bounded input which leads to an ***unbounded*** output.

- let's try the unit step input $x(t) = u(t)$. We compute that

$$y(t) = T_{\text{int}}\{u\}(t) = \int_{-\infty}^{t} u(\tau)\,\mathrm{d}\tau = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases} = t \cdot u(t)$$

- this output is unbounded, even though the input was bounded!

Therefore, T_{int} is not BIBO stable.

Example engineering applications of system properties

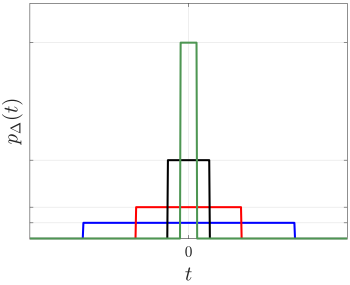
- **linearity**: if I double the applied voltage signal, I will get double the response from my circuit
- **time-invariance**: if I design my circuit to my liking today, it will behave the same way tomorrow
- **causality**: important for real-time signal processing, filtering, and control
- **memoryless**: many important nonlinear phenomena (e.g., signal saturation, deadband, quantization, friction) are memoryless.
- **invertibility**: "CTRL+Z". Also, many feedback control problems are implicitly about approximately inverting the system model
- **stability**: unstable systems are often dangerous in practice, will lead to component damage and failure.

Linear Time-Invariant (LTI) systems

- we now focus on systems which are ***both*** linear and time-invariant
- remember that, roughly speaking
  1. linearity: "the superposition principle holds"
  2. time-invariance: "the system will be the same tomorrow as it is today"
- LTI systems are important for several reasons:
  - (i) many real physical processes are reasonably modelled as LTI systems
  - (ii) many useful engineering algorithms are described by LTI systems
  - (iii) we have good theoretical tools for analyzing LTI systems
  - (iv) we have good procedures available for designing LTI systems
- you will use LTI systems in ***many*** other courses: control, communications, signal processing, energy systems, robotics, . . .

Recall: the CT unit impulse signal

- the CT unit impulse $\delta(t)$ is defined as $\delta(t) = \lim_{\Delta \to 0} p_\Delta(t)$
- an ***idealized*** very fast pulse at $t = 0$
- notation for shifted impulse: $\delta_\tau(t) = \delta(t - \tau)$



Unit Area: $\int_{-\infty}^{\infty} \delta_\tau(t)\,\mathrm{d}t = 1$

Sifting: $\int_{-\infty}^{\infty} x(t)\delta_\tau(t)\,\mathrm{d}t = x(\tau)$

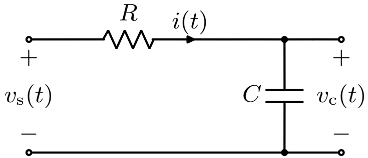
The impulse response of a LTI system

Definition 6.8. The ***impulse response*** h of a CT LTI system T is the response to a unit CT impulse input applied at $t = 0$, i.e., $h = T\{\delta\}$.

- **key idea:** we "hit the system sharply", and observe how it responds
- **example:** the integrating system $T_{\text{int}}\{x\}(t) = \int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau$ is LTI, and we have $h(t) = \int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau = u(t)$
- **example:** the delay system $T_{\text{delay}}\{x\}(t) = x(t - \tau)$ is LTI, and we have $h(t) = \delta(t - \tau) = \delta_\tau(t)$, a shifted impulse function.
- **example:** the averaging system $y(t) = \frac{1}{2a}\int_{t-a}^{t+a} x(\tau)\,\mathrm{d}\tau$ is LTI, and we have $h(t) = \frac{1}{2a}(u(t + a) - u(t - a))$.

Example: impulse response of RC circuit

- the RC circuit is linear, time-invariant, and causal



$$v_c(t) = \int_{-\infty}^{t} \frac{1}{RC} e^{-(t-\tau)/RC} v_s(\tau)\,\mathrm{d}\tau$$

$$v_c = T_{RC}\{v_s\}$$

$$h(t) = T_{RC}\{\delta\}(t) = \int_{-\infty}^{t} \frac{1}{RC} e^{-(t-\tau)/RC} \delta(\tau)\,\mathrm{d}\tau = \begin{cases} \frac{1}{RC} e^{-t/RC} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

- **physically:** we apply a big voltage spike to the circuit at $t = 0$, the capacitor quickly charges and develops a voltage $\frac{1}{RC}$, then the capacitor discharges exponentially with rate $\frac{1}{RC}$

The response of an LTI system

Remarkably, the impulse response enable us to compute the output of the LTI system for ***any*** input signal.

- consider the CT system $y = T\{x\}$ where T is LTI
- from the sifting formula, we have that

$$x(\tau) = \int_{-\infty}^{\infty} x(t)\delta_\tau(t)\,\mathrm{d}t$$

- if we just switch the labels t and τ, then we obtain

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta_t(\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} x(\tau)\delta_\tau(t)\,\mathrm{d}\tau$$

since $\delta_t(\tau) = \delta(\tau - t) = \delta(t - \tau) = \delta_\tau(t)$

The response of an LTI system

- this gives us an ***impulse representation*** of any signal x:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,\mathrm{d}\tau \qquad \text{or} \qquad x = \int_{-\infty}^{\infty} x(\tau)\delta_\tau\,\mathrm{d}\tau$$

interpretation: we are expressing x as a (continuous) *linear combination* of the "basis"' signals δ_τ with weighting coefficients $x(\tau)$

- let's compute the output $y = T\{x\}$:

$$y = T\{x\} = T\left\{\int_{-\infty}^{\infty} x(\tau)\delta_\tau\,\mathrm{d}\tau\right\} = \int_{-\infty}^{\infty} x(\tau)T\{\delta_\tau\}\,\mathrm{d}\tau$$

where we used ***linearity*** (i.e., superposition) of the system T

The response of an LTI system

- to go further, we need to calculate $T\{\delta_\tau\}$
- since T is ***time-invariant***, the response $T\{\delta_\tau\}$ to an impulse at time τ is equal to the impulse response $h = T\{\delta\}$ shifted by τ time units:

$$T\{\delta_\tau\} = h_\tau \qquad \text{or} \qquad T\{\delta_\tau\}(t) = h(t-\tau).$$

- we therefore have that

$$y(t) = T\{x\}(t) = \int_{-\infty}^{\infty} h_\tau(t)x(\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,\mathrm{d}\tau$$

This very important formula is called the ***convolution*** of the signals h and x, and is denoted by $y = h * x$

Some observations about convolution

Convolution: $$y(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,\mathrm{d}\tau.$$

- **Observation #1:** convolution is ***not*** a pointwise operation. For any time $t \in \mathbb{R}$, the value $y(t)$ depends on ***the entire signal*** x and ***the entire signal*** h.
- **Observation #2:** computing the value $y(t)$ involves a few steps:
  - (i) time-reverse h
  - (ii) shift the result forward by t seconds
  - (iii) multiply the result pointwise by x
  - (iv) integrate over all τ

Properties of CT convolution

- we can think of convolution as a general operation which takes two signals v, w and returns another signal $v * w$ defined as

$$(v * w)(t) = \int_{-\infty}^{\infty} v(t-\tau)w(\tau)\,\mathrm{d}\tau.$$

For any CT signals v, w, x and any constants α, β the following hold:

(i) ***superposition:*** $x * (\alpha v + \beta w) = \alpha(x * v) + \beta(x * w)$

(ii) ***commutative:*** $v * w = w * v$

(iii) ***time-invariance:*** $v * w_\sigma = (v * w)_\sigma$

(iv) ***identity element:*** $\delta * x = x$

(v) ***differentiation:*** $D(v * w) = (Dv) * w = v * (Dw)$ where "D" denotes differentiation

Proof of commutative property of convolution

The convolution operation is ***commutative***, meaning that $v * w = w * v$. More explicitly, we have that

$$\int_{-\infty}^{\infty} v(t-\tau)w(\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} v(\tau)w(t-\tau)\,\mathrm{d}\tau \qquad \text{for all } t \in \mathbb{R}.$$

Proof: For any fixed t, let $\sigma = t - \tau$, so that $\tau = t - \sigma$. Then

$$(v * w)(t) = \int_{-\infty}^{\infty} v(t-\tau)w(\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} v(\sigma)w(t-\sigma)\,\mathrm{d}\sigma = (w * v)(t)$$

which shows the equality.

- the other properties are proved very similarly by applying the definition and doing basic calculus; we will skip the details.

Can we always convolve any two signals? No.

- convolution is a tool, and all tools have limitations
- **example:** if $v(t) = 1$ for all $t \in \mathbb{R}$, then

$$(v * v)(t) = \int_{-\infty}^{\infty} (1)(1)\,\mathrm{d}\tau = +\infty, \qquad \text{for all } t \in \mathbb{R}.$$

- **question:** when is $v * w$ well-defined?

$$\text{When is } (v * w)(t) = \int_{-\infty}^{\infty} v(t-\tau)w(\tau)\,\mathrm{d}\tau \text{ finite for all } t?$$

- ***intuitively*** we would get a finite result if
  (a) $\tau \mapsto v(t-\tau)w(\tau)$ has finite duration for all t, or
  (b) v or w (or possibly both) tend to zero fast enough as $\tau \to \pm\infty$

Conditions for existence of convolution

Proposition 6.1. The following statements hold:

(i) If v and w are ***right-sided***, then $v * w$ exists and is also right-sided;

(ii) If v or w has ***finite duration***, then $v * w$ exists. Moreover, if v and w both have finite duration, then $v * w$ also has finite duration;

(iii) If v, w have ***finite action***, then $v * w$ exists and also has finite action. Moreover, $\|v * w\|_1 \leq \|v\|_1 \|w\|_1$;

(iv) If v has finite action and w has finite energy, then $v * w$ exists and has finite energy. Moreover, $\|v * w\|_2 \leq \|v\|_1 \|w\|_2$;

(v) If v has finite action and w has finite amplitude, then $v * w$ exists and has finite amplitude. Moreover, $\|v * w\|_\infty \leq \|v\|_1 \|w\|_\infty$.

In cases (i), (ii), or (iii), convolution is also ***associative***, meaning that

$$x * (v * w) = (x * v) * w, \quad \text{for all signals } x, v, w.$$

Partial Proof of Proposition 6.1

Proof: Our convolution formula is

$$z(t) = \int_{-\infty}^{\infty} v(t-\tau)w(\tau)\,\mathrm{d}\tau. \tag{1}$$

(i): If v, w are both right-sided, then

- there is a time τ_1 such that $w(\tau) = 0$ for all $\tau < \tau_1$, and
- there is a time τ_2 such that $v(\tau) = 0$ for all $\tau < \tau_2$, so therefore $v(t-\tau) = 0$ for all $\tau > t - \tau_2$

and we therefore have that

$$z(t) = \int_{-\infty}^{\infty} v(t-\tau)w(\tau)\,\mathrm{d}\tau = \int_{\tau_1}^{t-\tau_2} v(t-\tau)w(\tau)\,\mathrm{d}\tau.$$

This is now a finite integral, so we will get a finite result as long as v and w are "nice enough" signals. To see that z is also right-sided, note that if $t < \tau_1 + \tau_2$, then $v(t-\tau)w(\tau)$ will always be zero for all τ, and the integral will yield zero. Thus, z is right sided from time $\tau_1 + \tau_2$.

Partial Proof of Proposition 6.1

(ii): Similar reasoning to part (i).

(iii): We can calculate directly that

$$\begin{aligned}
\|z\|_1 = \int_{-\infty}^{\infty} |z(t)|\,\mathrm{d}t &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} v(t-\tau)w(\tau)\,\mathrm{d}\tau \right| \mathrm{d}t \\
&\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |v(t-\tau)| \cdot |w(\tau)|\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |v(t-\tau)|\,\mathrm{d}t \right] \cdot |w(\tau)|\,\mathrm{d}\tau = \|v\|_1 \|w\|_1 .
\end{aligned}$$

(iv): Omitted because it's a bit tricky to prove.

(v): Omitted, but it's an easy calculation; try it.

Convolution example: integrator system

- the integrator system T_{int} has $h(t) = u(t)$. Let's compute the output to input $x(t) = u(t)$ via convolution:

$$y(t) = \int_{-\infty}^{\infty} u(t-\tau)u(\tau)\,\mathrm{d}\tau$$

$$\text{for } t < 0: \quad y(t) = \int_{-\infty}^{0} u(t-\tau)\underbrace{u(\tau)}_{=0}\,\mathrm{d}\tau + \int_{0}^{\infty} \underbrace{u(t-\tau)}_{=0} u(\tau)\,\mathrm{d}\tau = 0$$

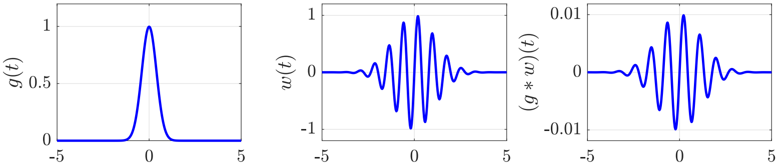
$$\text{for } t \geq 0: \quad y(t) = \int_{-\infty}^{0} u(t-\tau)\underbrace{u(\tau)}_{=0}\,\mathrm{d}\tau + \int_{0}^{\infty} \underbrace{u(t-\tau)}_{=1 \text{ if } \tau < t} \underbrace{u(\tau)}_{=1}\,\mathrm{d}\tau$$

$$= \int_{0}^{t} \mathrm{d}\tau = t$$

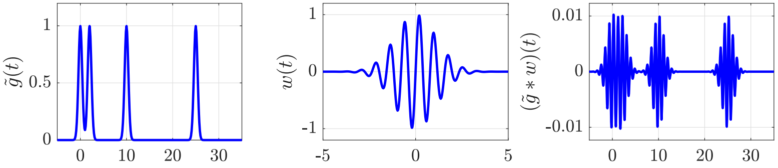
so we again obtain the output $y(t) = t \cdot u(t)$!

Convolution example: Gaussian and wavelet

- you can use convolution as an operation to build interesting signals
- consider the Gaussian, the "wavelet", and their convolution:



- if we instead convolve $w(t)$ with various shifted Gaussians ...



LTI system properties and the impulse response

- **summary:** the output of any LTI system is given by the convolution of the impulse response with the input signal

$$y = h * x \qquad\qquad y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,\mathrm{d}\tau$$

- h is very useful for ***understanding properties*** of LTI systems

Theorem 6.1. A CT LTI system T with impulse response $h = T\{\delta\}$ is

(i) causal if and only if $h(t) = 0$ for all $t < 0$;

(ii) memoryless if and only if $h(t) = \alpha\delta(t)$ for some $\alpha \in \mathbb{C}$;

(iii) BIBO stable if h has finite action, i.e., $h \in \mathsf{L}_1$, in which case

$$\|y\|_\infty \leq \|h\|_1 \|x\|_\infty, \quad \text{for all } x \in \mathsf{L}_\infty \text{ with } y = T\{x\}.$$

Proof of Theorem 6.1 (sufficiency only)

Proof: Our convolution formula for the output is

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,\mathrm{d}\tau. \tag{2}$$

(i): A system is causal if $y(t)$ depends only on $\{x(\tau)\}_{\tau \leq t}$. If $h(t) = 0$ for all $t < 0$, then (2) simplifies to

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)\,\mathrm{d}\tau$$

which shows that T is causal.

(ii): Try to make the argument yourself using the convolution formula.

(iii): Let $x \in \mathsf{L}_\infty$ be a bounded input. Then

$$|y(t)| = \left|\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau\right| \leq \int_{-\infty}^{\infty} |h(\tau)| \cdot |x(t-\tau)|\,\mathrm{d}\tau \leq \|x\|_\infty \int_{-\infty}^{\infty} |h(\tau)|\mathrm{d}\tau$$

so $\|y\|_\infty = \max_{t\in\mathbb{R}} |y(t)| \leq \|h\|_1 \|x\|_\infty$, and hence the system is BIBO stable.

More on LTI systems and causality

- models of ***physical systems*** are always causal, so it is particularly important to understand causal LTI systems
- **recall:** a linear system always produces zero output for zero input
- this simplifies the interpretation of causality from Definition 6.4

Proposition 6.2. A linear CT system T is causal if and only if for any time t_0 and any input x such that $x(t) = 0$ for all $t \leq t_0$, the output $y = T\{x\}$ satisfies $y(t) = 0$ for all $t \leq t_0$.

Equivalently: if the input is right-sided from time t_0, then the output will also be right-sided from time t_0.

More on LTI systems and causality

- the case of an input starting at time 0 comes up quite often

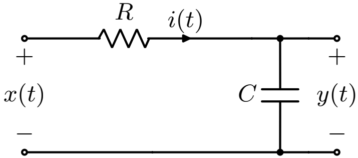
Theorem 6.2. Let T be a causal LTI system with impulse response $h = T\{\delta\}$. If x is right-sided from time 0, then $y = T\{x\}$ is right-sided from time 0 and

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,\mathrm{d}\tau = \int_{0^-}^{t} h(t-\tau)x(\tau)\,\mathrm{d}\tau, \qquad t \geq 0.$$

Proof: Suppose $t < 0$. When $-\infty < \tau < t < 0$, $x(\tau)$ is zero, since x is right-sided from time 0. When $t < \tau < \infty$, $h(t-\tau) = 0$, by causality. So the integrand is zero over all τ, so $y(t) = 0$ for any $t < 0$. Now suppose $t \geq 0$. Since $x(\tau) = 0$ for $\tau < 0$, we can start the integral at $\tau = 0^-$ instead of $-\infty$. Since $h(t-\tau) = 0$ for $\tau > t$, we can stop the integral at time t instead of $+\infty$.

Convolution example: RC circuit

- the RC circuit is LTI, causal, and BIBO stable



$$h(t) = \tfrac{1}{RC}e^{-t/RC}u(t)$$

- **our goal:** compute the response $y(t)$ to the unit pulse input

$$x(t) = \frac{1}{\Delta}\left[u(t) - u(t-\Delta)\right] \quad \text{or} \quad x = \frac{1}{\Delta}(u - u_\Delta)$$

- **note:** since this input is right-sided from time 0, and the system is *LTI and causal*, we know immediately that $y(t) = 0$ for all $t < 0$

Example: RC circuit

$$h(t) = \tfrac{1}{RC} e^{-t/RC} u(t), \qquad x(t) = \frac{1}{\Delta}\left[u(t) - u(t-\Delta)\right]$$

Use the properties of LTI systems and convolution to simplify.

$$\begin{aligned} y &= h * (\tfrac{1}{\Delta}(u - u_\Delta)) && \text{(convolution)} \\ &= \tfrac{1}{\Delta}(h * u) - \tfrac{1}{\Delta}(h * u_\Delta) && \text{(superposition)} \\ &= \tfrac{1}{\Delta}(h * u) - \tfrac{1}{\Delta}(h * u)_\Delta && \text{(time-invariance)} \end{aligned}$$

- therefore, we can just focus on calculating $h * u$ for $t \geq 0$:

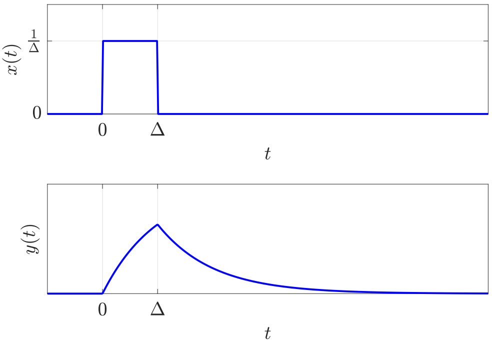
$$(h * u)(t) = \int_0^t \tfrac{1}{RC} e^{-(t-\tau)/RC} \underbrace{u(\tau)}_{=1} \, \mathrm{d}\tau = -e^{-(t-\tau)/RC}\Big|_0^t = 1 - e^{-t/RC}$$

- therefore for all t: $\quad (h * u)(t) = (1 - e^{-t/RC})u(t)$

Example: RC circuit

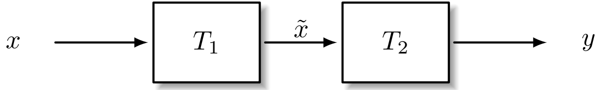
- putting it all together, we find that

$$
\begin{aligned}
y(t) &= \frac{1}{\Delta}(h * u)(t) - \frac{1}{\Delta}(h * u)(t-\Delta) \\
&= \frac{1}{\Delta}(1 - e^{-t/RC})u(t) - \frac{1}{\Delta}(1 - e^{-(t-\Delta)/RC})u(t-\Delta)
\end{aligned}
$$



Series combinations of LTI systems

- consider the ***series*** combination of two LTI systems



- if h_1 and h_2 are the associated impulse responses, then

$$y = h_2 * \tilde{x} = h_2 * (h_1 * x) = (h_2 * h_1) * x$$

where we used the associative property of convolution. Therefore

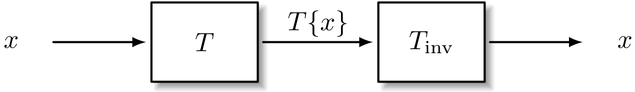
(i) the series combination has impulse response $h_2 * h_1$

(ii) by the commutative property $h_2 * h_1 = h_1 * h_2$, so ***we can always change the order*** of two LTI systems and still get the same result.

Since the above requires the associative property, we are implicitly assuming we are in cases (i), (ii), or (iii) of Proposition 6.1.

LTI systems and invertibility

- **recall:** a system T is invertible if we can "undo" the operation by applying some other system T_{inv}



- for LTI systems, we can say more about invertibility

Lemma 6.1. Suppose that T is a LTI system with impulse response h. Then T is invertible if and only if there exists another impulse response h_{inv} such that $h * h_{\text{inv}} = \delta$.

- **immediate consequence:** the inverse of a LTI system (if there is one) is ***also*** a LTI system!

Example: invertibility

- consider the LTI system T with impulse response

$$h(t) = \delta(t) + (a-b)e^{-bt}u(t), \qquad a, b \in \mathbb{R}, a \neq b.$$

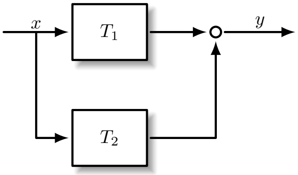
- **claim:** T is invertible and inverse has impulse response

$$h_{\text{inv}}(t) = \delta(t) - (a-b)e^{-at}u(t).$$

$$\begin{aligned}
h * h_{\text{inv}}(t) &= \int_{0^-}^{t} h(t-\tau)h_{\text{inv}}(\tau)\,\mathrm{d}\tau \\
&= \int_{0^-}^{t} (\delta(t-\tau) + (a-b)e^{-b(t-\tau)})(\delta(\tau) - (a-b)e^{-a\tau})\,\mathrm{d}\tau \\
&= \delta(t) - (a-b)e^{-at} + (a-b)e^{-bt} - (a-b)^2 e^{-bt} \int_{0}^{t} e^{-(a-b)\tau}\,\mathrm{d}\tau \\
&= \delta(t) - (a-b)e^{-at} + (a-b)e^{-bt} + (a-b)e^{-bt}[e^{-at}e^{bt} - 1] \\
&= \delta(t)
\end{aligned}$$

Parallel combinations of LTI systems

the ***parallel*** combination of two LTI systems is given by this diagram



- if h_1 and h_2 are the associated impulse responses, then

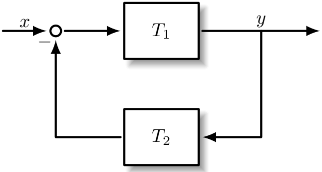
$$y = (h_1 * x) + (h_2 * x) = (h_1 + h_2) * x$$

where we used the linear property of convolution.

The impulse response of the parallel combination is $h_1 + h_2$

Feedback combinations of LTI systems

the ***negative feedback*** combination of two LTI systems is given by the diagram



- $y = h_1 * (x - h_2 * y) = h_1 * x - h_1 * (h_2 * y)$
- since $\delta * y = y$, we can rearrange to find that

$$(\delta + h_1 * h_2) * y = h_1 * x \qquad \text{(can we solve for } y\text{?)}$$

If the LTI system with impulse response $(\delta + h_1 * h_2)$ is ***invertible***, then $y = h * x$ where $h = (\delta + h_1 * h_2)^{-1} * h_1$.

Differential equations and CT systems

- many physical phenomena are described by differential equations; the RC circuit and a point mechanical mass are two simple examples
- when studying those two, we ***explicitly solved*** the ODE to obtain a pointwise system definition; this isn't always easy to do . . .
- **fact:** under mild technical conditions, ODEs with inputs still define CT systems, even if you cannot solve the ODE explicitly!

We will now study a special case of this general fact, and argue that ***linear, inhomogeneous, constant-coefficient ordinary differential equations*** can define causal CT LTI systems.

LICC-ODEs and CT systems

- we consider an n^{th} order, linear, inhomogeneous, constant-coefficient ordinary differential equation (LICC-ODE)

$$a_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y(t)}{\mathrm{d}t^{n-1}} + \cdots + a_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t)$$
$$= b_m \frac{\mathrm{d}^m x(t)}{\mathrm{d}t^m} + b_{m-1}\frac{\mathrm{d}^{m-1} x(t)}{\mathrm{d}t^{m-1}} + \cdots + b_1 \frac{\mathrm{d}x(t)}{\mathrm{d}t} + b_0 x(t)$$

  which must hold for all times $t \in \mathbb{R}$. In this equation

  (i) x is a given CT signal,
  (ii) y is the unknown CT signal to be solved for,
  (iii) n, m are nonnegative integers, and
  (iv) the coefficients $a_0, \ldots, a_n, b_0, \ldots, b_m$ are real constants.

- without loss of generality, we can always assume that $a_n = 1$

LICC-ODEs and CT systems

- we will use the short-form notation $D^k y(t) = \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k}$, and

$$Q(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$$
$$P(D) = b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0$$

- with this, we can write our ODE succinctly as

$$Q(D)y(t) = P(D)x(t) \qquad \text{(LICC-ODE)}$$

When does (LICC-ODE) define a (causal, LTI) CT system?

- we need to think carefully about
  - (i) the possibility of multiple outputs for one input
  - (ii) the issue of "initial conditions"
  - (iii) the issues of time-invariance and causality

LICC-ODEs and CT systems

(i) **Issue of multiple solutions:** Consider $n = 1$, $m = 0$, with input $x(t) = \delta(t)$, i.e., $\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \delta(t)$. This equation has ***infinitely many*** solutions

$$y(t) = u(t) + c, \qquad c \in \mathbb{R}.$$

However, this is not allowed: by our definition, a system ***must*** produce only one output for each input.

(ii) **Issue of "initial" conditions:** Sometimes you can use ***initial conditions*** at some time, e.g., $t = 0$, to fix the non-uniqueness issue. But what if $y(t)$ contains a CT impulse at $t = 0$? Then $y(0)$ is not even well-defined. So if we want to include impulsive signals, initial conditions are not the answer.

(iii) **Issue of causality:** Nothing in the ODE stipulates that $y(t)$ must only depend on $\{x(\tau)\}_{\tau \leq t}$; we should not automatically expect causality

LICC-ODEs define LTI CT systems

- it turns out that *all these issues will be resolved* if we
  (i) restrict our attention to ***right-sided inputs*** $x(t)$, and
  (ii) restrict our attention ***right-sided solutions*** $y(t)$.

Theorem 6.3. For each right-sided input $x(t)$, the LICC-ODE $Q(D)y(t) = P(D)x(t)$ possesses exactly one right-sided solution $y(t)$, and therefore defines a system $y = T\{x\}$. Moreover

- the system T is linear, time-invariant, and causal;
- the system T is BIBO stable if and only if $m \leq n$ and all roots of

$$Q(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

in the complex variable $s \in \mathbb{C}$ have negative real part.

Proof sketch for Theorem 6.3

Existence and Uniqueness of a Right-Sided Solution: Establishing this is beyond our scope.

Linearity: Take two right-sided inputs x_1, x_2, with corresponding right-sided outputs y_1, y_2. Now consider the right-sided input $x = c_1x_1 + c_2x_2$ for $c_1, c_2 \in \mathbb{R}$. We want to establish that $y = c_1y_1 + c_2y_2$ is the corresponding right-sided output. This is verified by direct substitution into $Q(D)y(t) = P(D)x(t)$ as follows:

$$\begin{aligned} Q(D)y = Q(D) \cdot (c_1y_1 + c_2y_2) &= c_1Q(D)y_1 + c_2Q(D)y_2 \\ &= c_1P(D)x_1 + c_2P(D)x_2 \\ &= P(D) \cdot (c_1x_1 + c_2x_2) \\ &= P(D)x \end{aligned}$$

We must also establish that if $x = 0$ is the zero input, then the response y is zero. The zero-input response is determined by the homogeneous ODE $Q(D)y = 0$:

$$\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y(t)}{\mathrm{d}t^{n-1}} + \cdots + a_1\frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0y(t) = 0.$$

Proof sketch for Theorem 6.3

Linearity (Continued): We know from the theory of ODEs that the general solution to this linear constant coefficient ODE has the form

$$y(t) = \sum_{k=1}^{n} b_k t^{m_k} e^{p_k t}$$

where $b_1, \ldots, b_n$ are free constants, $m_1, \ldots, m_n$ are nonnegative integers, and $p_1, \ldots, p_n$ are the roots of the polynomial $Q(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$. (Aside: as a case that might be more familiar, if the roots of $Q(s)$ are all distinct, then the constants m_k are all zero, and you just have a sum of exponentials).

However, by the existence/uniqueness result, we also know that $y(t)$ must be right-sided, so there must exist a time $\tau^\star$ such that $y(t)$ and all its derivatives are zero for all times $t < \tau^\star$. This is only possible if all constants b_k are zero, and thus $y(t) = 0$ for all time, which shows that zero input produces zero output.

Proof sketch for Theorem 6.3

Time-Invariance: Let x be a right-sided input with corresponding right-sided output y satisfying $Q(D)y(t) = P(D)x(t)$. We want to consider the effect of a shifted input. Let $\tau \in \mathbb{R}$ and consider the shifted input $x_\tau(t) = x(t-\tau)$; since x is right-sided, so is x_τ. The system's response $\eta(t) = T\{x_\tau\}(t)$ to $x_\tau(t)$ must be right-sided as well, and satisfy $Q(D)\eta(t) = P(D)x_\tau(t)$. But because acting with the differential operators $P(D)$ and $Q(D)$ only consists of taking derivatives and multiplying by constants, and adding, both operations commute with time-shifting. Therefore

$$Q(D)\eta(t) = P(D)x_\tau(t) = [P(D)x]_\tau(t) = [Q(D)y]_\tau(t) = Q(D)y_\tau(t)$$

so we conclude that $Q(D)(\eta(t) - y_\tau(t)) = 0$. Note we can also write this as

$$Q(D) \cdot (\eta(t) - y_\tau(t)) = P(D) \cdot (0).$$

By uniqueness of the response, it follows that $\eta(t) - y_\tau(t)$ must be the response of the system to the zero input. However, the system is linear, so the response to zero input must be zero. We conclude that $\eta(t) = y_\tau(t)$, so the system is time-invariant.

Proof sketch for Theorem 6.3

Causality: Let $t_0 \in \mathbb{R}$ be an arbitrary time, and suppose that $x(t)$ is an input that is right sided from time t_0. We need to show that the response $y(t)$, determined by $Q(D)y(t) = P(D)x(t)$, is also right-sided from t_0.

From existence/uniqueness, we know that the response $y(t)$ is right-sided. Let $\tau^\star$ be the largest time such that $y(t)$ is right-sided from time $\tau^\star$, i.e., the response y starts from time $\tau^\star$. By contradiction, let's assume that $\tau^\star < t_0$. Then for any time t in the range $(\tau^\star, t_0)$, we know that x and all its derivatives must be zero, and thus $Q(D)y(t) = 0$ for all $t \in (\tau^\star, t_0)$. In fact, $Q(D)y(t) = 0$ for all $t \in (-\infty, t_0)$, due to right-sidedness of y. Similar to our previous argument for linearity, this can only be true if $y(t) = 0$ for all $t \in (-\infty, t_0)$, but this contradicts the assumption that y begins from time $\tau^\star < t_0$. Thus, $y(t)$ is right-sided from time t_0, which shows causality.

Proof sketch for Theorem 6.3

BIBO Stability Part 1: Suppose that $m > n$. As an example of such a system, consider $n = 0$ and $m = 1$, i.e., $y(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$. The right-sided input $x(t) = \sin(t^2)u(t)$ is bounded, but with this input we have

$$\begin{aligned} y(t) = \frac{\mathsf{d}}{\mathsf{d}t}\left[\sin(t^2)u(t)\right] &= 2t\cos(t^2)u(t) + \sin(t^2)\delta(t) \\ &= 2t\cos(t^2)u(t) + \sin(0)\delta(t) \\ &= 2t\cos(t^2)u(t) \end{aligned}$$

which is unbounded. This argument can be generalized to any situation where $m > n$; we omit the details. Thus, if $m > n$, the system will never be BIBO stable.

Proof sketch for Theorem 6.3

BIBO Stability Part 2: Suppose then that $m \leq n$. The impulse response h of the system is determined by applying the input $x(t) = \delta(t)$, i.e., as the solution to the ODE

$$Q(D)h(t) = P(D)\delta(t).$$

We now apply the Laplace transform to both sides of this equation, and use the fact that all initial conditions are zero. Using the derivative rule, we have that

$$\text{Laplace}\{Q(D)h(t)\} = s^n H(s) + a_{n-1}s^{n-1}H(s) + \cdots + a_0 H(s) = Q(s)H(s)$$

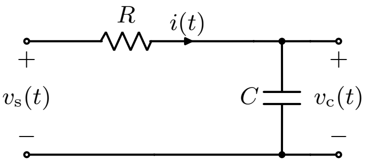
and since $\text{Laplace}\{\delta(t)\} = 1$, we have $\text{Laplace}\{P(D)\delta(t)\} = P(s)$, so $H(s) = \frac{P(s)}{Q(s)}$.

So $H(s)$ is a ratio of two polynomials, with the order of the denominator greater than or equal to the order of the numerator. Using partial fraction expansion and the inverse Laplace transform will give us an expression of the form

$$h(t) = \left[c_0\delta(t) + \sum_{k=1}^{n} c_k t^{m_k} e^{p_k t}\right] u(t)$$

where $c_0, \ldots, c_n$ are constants, $m_1, \ldots, m_n$ are nonnegative integers, and $p_1, \ldots, p_n$ are the roots of $Q(s)$. The only way we $h(t)$ will have finite action is if these are decaying exponentials, which will occur if and only if $\mathrm{Re}\{p_k\} < 0$ for all roots.

Example: the RC circuit



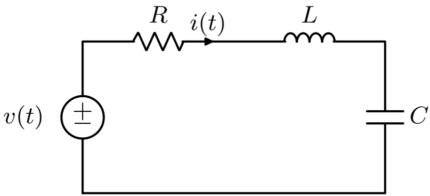
$$\frac{\mathrm{d}v_{\mathrm{c}}}{\mathrm{d}t} + \tfrac{1}{RC}v_{\mathrm{c}}(t) = \tfrac{1}{RC}v_{\mathrm{s}}(t)$$

- we now know that any right-sided input v_{s} will lead to a right-sided output v_{c}, and the ODE defines a causal LTI system
- the single root of the polynomial $Q(s) = s + \frac{1}{RC}$ is

$$s = -\frac{1}{RC}$$

which always has negative real part, so the circuit is BIBO stable

Example: the series RLC circuit



$$LC\frac{\mathrm{d}^2 i(t)}{\mathrm{d}t^2} + RC\frac{\mathrm{d}i(t)}{\mathrm{d}t} + i(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t}$$

- we now know that any right-sided input $v(t)$ will lead to a right-sided output $i(t)$, and the ODE defines a causal LTI system
- the roots of the polynomial $Q(s) = s^2 + \frac{R}{L}s + \frac{1}{LC}$ are

$$s = -\frac{R}{2L} \pm \frac{1}{2LC}\sqrt{R^2C^2 - 4LC} = -\frac{R}{2L} \pm \frac{R}{2L}\sqrt{1 - \frac{4L}{R^2C}}$$

Example: the series RLC circuit

$$s = -\frac{R}{2L} \pm \frac{R}{2L}\sqrt{1 - \frac{4L}{R^2C}}$$

- **case #1:** if $\frac{4L}{R^2C} > 1$, then the second term is imaginary, and both roots have real part $-\frac{R}{2L}$, so the circuit is BIBO stable
- **case #2:** if $0 < \frac{4L}{R^2C} < 1$, then the entire square root is strictly between 0 and 1, so the first term always out-weighs the second term, and the circuit is again BIBO stable

Final comment: In cases where $Q(s)$ is a more complicated polynomial, you must try to factor $Q(s)$ into simple pieces. In ECE311 you will learn other, more general techniques for assessing stability.

From time-domain to frequency-domain analysis

- we now understand a good amount about CT systems in general, and even more about CT LTI and causal systems
- the next two chapters are devoted to analyzing CT LTI systems in the ***frequency domain***, via two different methods
- the two methods have different assumptions and strengths

Fourier Trans. Method

- two-sided CTFT-able inputs
- causality not required
- used in signal processing, communications, ...

Laplace Trans. Method

- LT-able inputs which are right-sided from time 0
- causal systems only
- used in control, energy, robotics, aerospace, ...

Relevant MATLAB commands

- computing a "CT" convolution

```
%Define Time
T_max = 10; step = 0.001; t = -T_max:step:T_max;

%%Define impulse response of RC circuit
R = 0.3; C = 1; h = heaviside(t).*exp(-t/R/C)/R/C;

%% Define input signal
x = heaviside(t) - heaviside(t-3) - heaviside(t-6);

%% Compute convolution
y = step*conv(x,h,'same');

%% Plot
plot(t,x); hold on; plot(t,y); hold off;
```

Supplementary reading

- **O-W:** A. V. Oppenheim and S. Willsky, *Signals and Systems*, 2nd Ed.
- **BB:** B. Boulet, *Fundamentals of Signals and Systems*.
- **BPL:** B. P. Lathi, *Signal Processing and Linear Systems*.
- **K-S:** H. Kwakernaak and R. Sivan, *Modern Signals and Systems*.
- **EL-PV:** E. Lee and P. Varaiya, *Structure and Interpretation of Signals and Systems*, 2nd Ed.
- **ADL:** A. D. Lewis, *The Mathematical Theory of Signals and Systems*.

Topic	**O-W**	**BB**	**BPL**	**K-S**	**EL-PV**	**ADL**
Definition of a system	1.5	1	1.6	3.1, 3.2	1.2, 2.3	V5 1.1–1.2
System properties	1.6, 2.3	1	1.7	3.2, 3.3, 3.6		
LTI systems, impulse response	2.2	2	2.1–2.8	3.3, 3.4		
Convolution	2.2	2	2.3	3.5	9.1	V4 4.1–4.2
System interconnections	1.5, 9.8	1		3.9		
Differential equations	2.4	3	2.5	4.1–4.6	13.7	

Personal Notes

Personal Notes

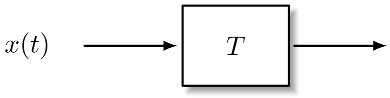
Personal Notes

7. Analysis of Continuous-Time Systems using the Fourier Transform

- response of a LTI system to a complex exponential input
- response of a LTI system to a periodic input
- response of a LTI system to a general two-sided input
- frequency response conditions for invertibility
- filtering in the frequency domain using LTI systems

Warm-up: LTI systems with complex exponential inputs

- let T be a LTI system with impulse response h



$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau$$

- if the input x is a complex exponential $x(t) = e^{\mathbf{j}\omega_0 t}$, then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{\mathbf{j}\omega_0(t-\tau)}\,\mathrm{d}\tau = \underbrace{\left[\int_{-\infty}^{\infty} h(\tau)e^{-\mathbf{j}\omega_0\tau}\,\mathrm{d}\tau\right]}_{\text{CTFT of } h \text{ at } \omega=\omega_0} e^{\mathbf{j}\omega_0 t}$$

If the input is a complex exp., then the output is ***also*** a complex exp. with the ***same frequency***, but ***scaled by the CTFT of h***!

Frequency response of a LTI system

The CTFT of the impulse response is known as the system's ***frequency response***, and is denoted by $H(\mathbf{j}\omega) = \int_{-\infty}^{\infty} h(t)e^{-\mathbf{j}\omega t}\,\mathrm{d}t$.

- write $H(\mathbf{j}\omega) \in \mathbb{C}$ in polar form as $H(\mathbf{j}\omega) = |H(\mathbf{j}\omega)|e^{\mathbf{j}\angle H(\mathbf{j}\omega)}$

$$x(t) = e^{\mathbf{j}\omega_0 t} \quad \Longrightarrow \quad y(t) = H(\mathbf{j}\omega_0)e^{\mathbf{j}\omega_0 t} = |H(\mathbf{j}\omega_0)|e^{\mathbf{j}(\angle H(\mathbf{j}\omega_0))}e^{\mathbf{j}\omega_0 t}$$

(i) ***"eigenfunction" property:*** if $x(t) = e^{\mathbf{j}\omega_0 t}$, then $y(t) \propto e^{\mathbf{j}\omega_0 t}$

(ii) ***amplitude scaling:*** output amplitude is scaled by $|H(\mathbf{j}\omega_0)|$

(iii) ***phase shifting:*** output phase is shifted by $\angle H(\mathbf{j}\omega_0)$

A LTI system amplitude-scales and phase-shifts any complex exp. input!

Example: RC circuit

- the RC circuit has impulse response $h(t) = \frac{1}{RC}e^{-t/RC}u(t)$
- the frequency response is

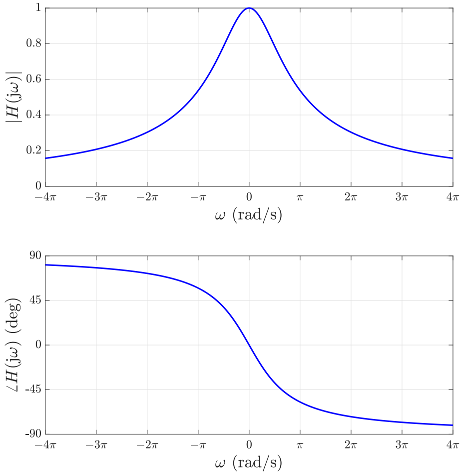
$$H(\mathbf{j}\omega) = \int_{-\infty}^{\infty} h(t)e^{-\mathbf{j}\omega t}\,\mathrm{d}t = \frac{1}{1 + RC(\mathbf{j}\omega)}$$

- in terms of magnitude and phase, we have

$$\begin{aligned} |H(\mathbf{j}\omega)| &= \frac{|1|}{\sqrt{1^2 + (RC\omega)^2}} \\ &= \frac{1}{\sqrt{(RC)^2\omega^2 + 1}} \end{aligned} \qquad \begin{aligned} \angle H(\mathbf{j}\omega) &= \angle 1 - \angle(1 + RC(\mathbf{j}\omega)) \\ &= 0 - \tan^{-1}(RC\omega) \\ &= -\tan^{-1}(RC\omega) \end{aligned}$$

Example: RC circuit ($RC = \frac{1}{2}$)

- it is often very useful to plot these as a function of ω



Notes:

(i) mag $\to 0$ as $\omega \to \pm\infty$

(ii) phase $\to \pm 90°$ as $\omega \to \mp\infty$

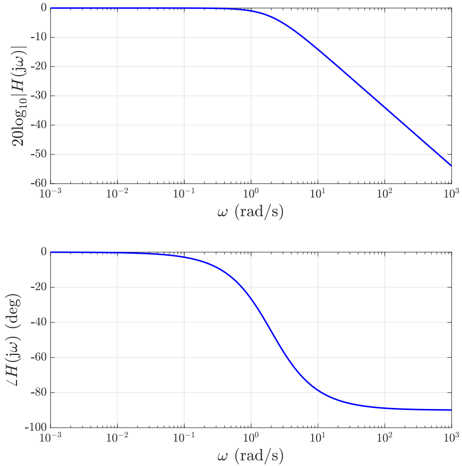
(iii) low frequencies are passed, high frequencies are attenuated

Example: RC circuit

- more often, we plot this information with slightly different axes:
  (i) log (base 10) scale for frequency;
  (ii) log (base 10) scale for magnitude, multiplied by 20 (***decibels***)
  (iii) only plot for positive frequencies $\omega > 0$ (why?)
- with a log-log scale, the key features of the magnitude plot often become more obvious
- such a plot is called a ***Bode plot*** of the frequency response
- note that if $H(\mathbf{j}\omega) = H_{\text{num}}(\mathbf{j}\omega)/H_{\text{den}}(\mathbf{j}\omega)$ then

$$20\log_{10}|H(\mathbf{j}\omega)| = 20\log_{10}|H_{\text{num}}(\mathbf{j}\omega)| - 20\log_{10}|H_{\text{den}}(\mathbf{j}\omega)|$$

Example: RC circuit ($RC = \frac{1}{2}$)



Notes:

(i) magnitude max. value of 0 dB

(ii) magnitude shows "break point" at ≈ 2 rad/s, transition from flat to linear

(iii) frequencies above ≈ 2 rad/s are attenuated

(iv) frequencies below ≈ 2 rad/s are passed

We call such a system a ***low-pass filter***

Response of a LTI system to a periodic input

- we previously studied how to take the CTFT of a periodic signal:

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\mathbf{j}k\omega_0 t} \quad \Longleftrightarrow \quad X(\mathbf{j}\omega) = 2\pi \sum_{k=-\infty}^{\infty} \alpha_k \delta(\omega - k\omega_0)$$

- we can now compute the output of the LTI system as

$$Y(\mathbf{j}\omega) = H(\mathbf{j}\omega)X(\mathbf{j}\omega) = 2\pi \sum_{k=-\infty}^{\infty} \alpha_k H(\mathbf{j}k\omega_0)\delta(\omega - k\omega_0)$$

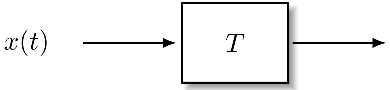
and therefore, taking the inverse CTFT:

$$y(t) = \sum_{k=-\infty}^{\infty} \underbrace{\alpha_k H(\mathbf{j}k\omega_0)}_{\text{Fourier coefficients}} e^{\mathbf{j}k\omega_0 t} \qquad \text{(Fourier series of output!)}$$

- ***amazing:*** the output is $T_0 = \frac{2\pi}{\omega_0}$ periodic, just like the input!

Response of a LTI system and the CTFT

These ideas generalize to more general input signals



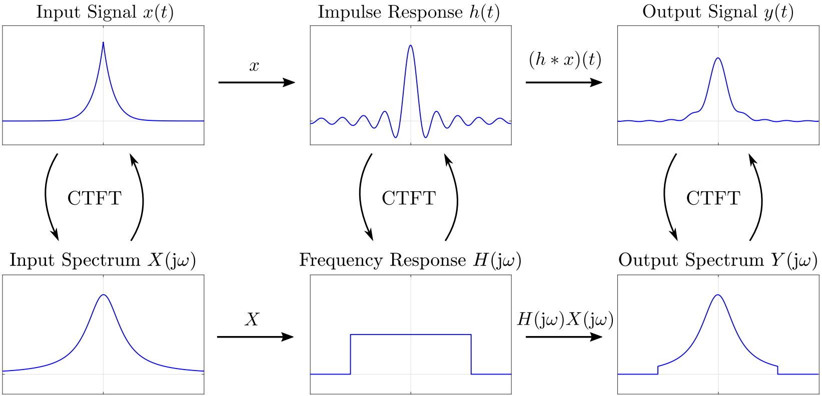
$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,\mathrm{d}\tau$$

- **recall:** convolution in time $\Longleftrightarrow$ multiplication in frequency

$$y(t) = (h * x)(t) \qquad \Longleftrightarrow \qquad Y(\mathbf{j}\omega) = H(\mathbf{j}\omega)X(\mathbf{j}\omega)$$

In the frequency domain, the output spectrum $Y(\mathbf{j}\omega)$ is given by the ***product*** of the frequency response $H(\mathbf{j}\omega)$ and the input spectrum $X(\mathbf{j}\omega)$. Much simpler than convolution!

Response of a LTI system and the CTFT



Example: computing the output for the RC circuit

- the RC circuit has impulse response $h(t) = \frac{1}{RC}e^{-t/RC}u(t)$
- the frequency response is

$$\begin{aligned} H(\mathbf{j}\omega) &= \int_{-\infty}^{\infty} h(t)e^{-\mathbf{j}\omega t}\,\mathrm{d}t = \int_{0}^{\infty} \tfrac{1}{RC}e^{-t/RC}e^{-\mathbf{j}\omega t}\,\mathrm{d}t \\ &= \tfrac{1}{RC}\int_{0}^{\infty} e^{-(\frac{1}{RC}+\mathbf{j}\omega)t}\,\mathrm{d}t = -\frac{\frac{1}{RC}}{\frac{1}{RC}+\mathbf{j}\omega}e^{-(\frac{1}{RC}+\mathbf{j}\omega)t}\Big|_{0}^{\infty} \\ &= \frac{1}{1+RC(\mathbf{j}\omega)} \end{aligned}$$

- for $c > 0$ consider the input signal

$$x(t) = e^{-ct}u(t) \qquad \Longleftrightarrow \qquad X(\mathbf{j}\omega) = \frac{1}{c+\mathbf{j}\omega}$$

Let's compute the output using the FT method

Example: computing the output for the RC circuit

- the spectrum of the output signal y is

$$Y(\mathbf{j}\omega) = H(\mathbf{j}\omega)X(\mathbf{j}\omega) = \frac{1}{1+RC(\mathbf{j}\omega)}\frac{1}{c+\mathbf{j}\omega}$$

- using the method of partial fractions, we write

$$\begin{aligned} Y(\mathbf{j}\omega) = \frac{A}{1+RC(\mathbf{j}\omega)} + \frac{B}{c+\mathbf{j}\omega} &= \frac{A(c+\mathbf{j}\omega)+B(1+RC(\mathbf{j}\omega))}{[1+RC(\mathbf{j}\omega)][c+\mathbf{j}\omega]} \\ &= \frac{Ac+B+(A+BRC)(\mathbf{j}\omega)}{[1+RC(\mathbf{j}\omega)][c+\mathbf{j}\omega]} \end{aligned}$$

so $Ac+B=1$ and $A+BRC=0$, yielding

$$A = \frac{-RC}{1-cRC}, \qquad B = \frac{1}{1-cRC} \qquad \text{(assuming } cRC \neq 1\text{)}$$

- inverting $Y(\mathbf{j}\omega)$ term by term, we obtain

$$y(t) = \frac{-RC}{1-cRC}\frac{1}{RC}e^{-t/RC}u(t) + \frac{1}{1-cRC}e^{-ct}u(t)$$

LTI systems and invertibility

- **recall:** a LTI system with impulse response h is invertible if there exists an impulse response h_{inv} such that $h * h_{\text{inv}} = \delta$.

Let's look at what this says in the frequency domain ...

- take CTFT of both sides and use convolution property:

$$h * h_{\text{inv}} = \delta \qquad \Longleftrightarrow \qquad H(\mathbf{j}\omega)H_{\text{inv}}(\mathbf{j}\omega) = 1$$

so we can always solve for $H_{\text{inv}}(\mathbf{j}\omega)$ ***as long as*** $\boldsymbol{H(\mathbf{j}\omega) \neq 0}$

Proposition 7.1. A LTI system T with impulse response h is invertible if and only if $H(\mathbf{j}\omega) \neq 0$ for all $\omega \in \mathbb{R}$.

Example: invertibility

- **recall:** the LTI system T with impulse response

$$h(t) = \delta(t) + (a-b)e^{-bt}u(t), \qquad a, b \in \mathbb{R},\ a \neq b.$$

- this system has frequency response

$$H(\mathbf{j}\omega) = 1 + \frac{a-b}{\mathbf{j}\omega + b} = \frac{\mathbf{j}\omega + a}{\mathbf{j}\omega + b}$$

- since $H(\mathbf{j}\omega) \neq 0$ for all $\omega \in \mathbb{R}$, the inverse T_{inv} of T exists and has freq. response

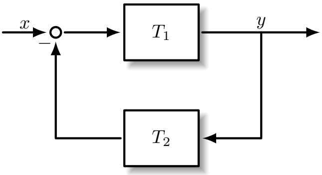
$$H_{\text{inv}}(\mathbf{j}\omega) = \frac{\mathbf{j}\omega + b}{\mathbf{j}\omega + a} \quad \Longleftrightarrow \quad h_{\text{inv}}(t) = \delta(t) + (b-a)e^{-at}u(t)$$

  which agrees with Chapter 6.

Example: invertibility and feedback interconnections

the ***negative feedback*** combination of two LTI systems is the LTI system T with impulse response

$$h = (\delta + h_1 * h_2)^{-1} * h_1.$$



- we now know that $(\delta + h_1 * h_2)$ is invertible if and only if

$$1 + H_1(\mathbf{j}\omega)H_2(\mathbf{j}\omega) \neq 0 \qquad \text{for all } \omega \in \mathbb{R}$$

- if this holds, then we have

$$H(\mathbf{j}\omega) = \frac{H_1(\mathbf{j}\omega)}{1 + H_1(\mathbf{j}\omega)H_2(\mathbf{j}\omega)}$$

Filtering via LTI systems

- to ***filter*** a signal means to emphasize or de-emphasize some subset of frequency content within the signal
- several common classes of filters are
  - (i) low-pass filters (attenuates/removes all “high”-frequency content)
  - (ii) high-pass filters (attenuates/removes all “low”-frequency content)
  - (iii) band-pass filter (attenuates/removes all content outside a band)
  - (iv) band-stop filter (attenuates/removes all content inside a band)

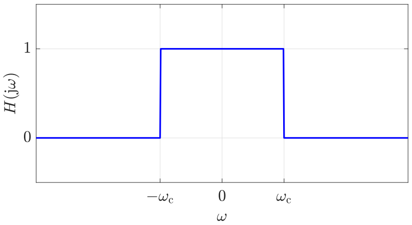
Filters can be conveniently designed using LTI systems through an appropriately chosen frequency response

- we will examine simple ideal and non-ideal filters

The ideal low-pass filter

For $\omega_c > 0$ consider the freq. response

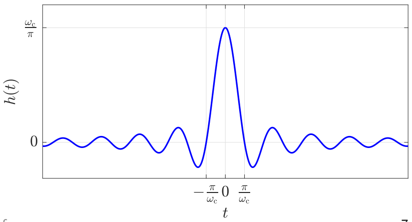
$$H(\mathbf{j}\omega) = \begin{cases} 1 & \text{if } -\omega_c \leq \omega < \omega_c \\ 0 & \text{otherwise} \end{cases}$$



This system perfectly passes frequencies $|\omega| < \omega_c$.

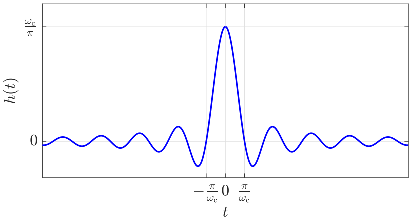
The impulse response is

$$h(t) = \frac{\sin(\omega_c t)}{\pi t}$$



The ideal low-pass filter

The filter is ***not causal***, since $h(t)$ is not zero for all $t < 0$. We therefore cannot use this ideal filter in any ***real-time*** application.



- we often instead prefer ***non-ideal*** but ***causal*** filters, because
  (i) they can be modelled using differential equations
  (ii) they can be easily implemented in hardware/software for real-time use

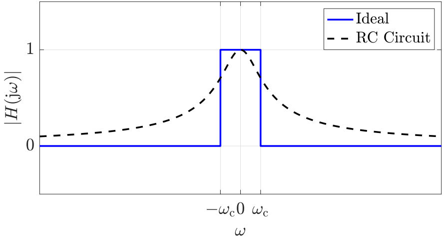
- another issue with the ideal LPF is that its impulse response ***oscillates***; when convolved in the time-domain, this tends to produce undesirable oscillations ("ripple") in the output signal

The RC circuit is a non-ideal LPF

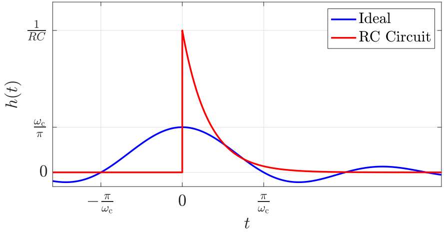
- the RC circuit is an example of a non-ideal but causal LPF

$$|H_{\mathrm{RC}}(\mathbf{j}\omega)| = \frac{1}{\sqrt{1+(RC\omega)^2}}$$

with $\frac{1}{RC} = \omega_{\mathrm{c}}$

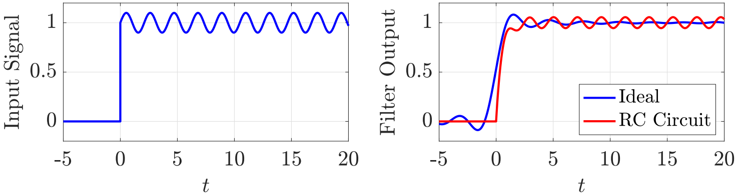


$$h_{\mathrm{RC}}(t) = \frac{1}{RC} e^{-t/RC} u(t)$$



Comparison of RC circuit and ideal LPF

- for $\omega_{\mathrm{c}} = 2$ rad/s, let's consider the input $x(t) = [1 + 0.1\sin(3t)]u(t)$

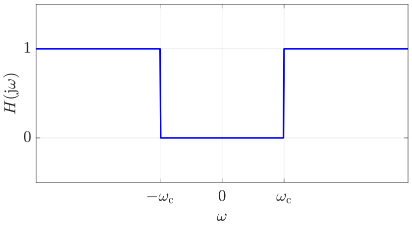


- the ideal filter
  (i) *anticipates* the input change (because it is not causal)
  (ii) perfectly removes the oscillatory component above ω_{c}
- the RC circuit
  (i) reacts only *after* the input changes (because it is causal)
  (ii) attenuates the oscillatory component, but cannot remove it

The ideal high-pass filter

For $\omega_c > 0$ consider the freq. response

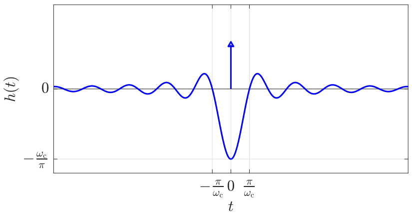
$$H(\mathbf{j}\omega) = \begin{cases} 1 & \text{if } |\omega| > \omega_c \\ 0 & \text{otherwise} \end{cases}$$



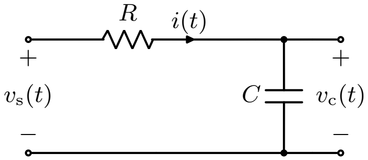
This system perfectly passes frequencies $|\omega| > \omega_c$.

The impulse response is

$$h(t) = \delta(t) - \frac{\sin(\omega_c t)}{\pi t}$$



A non-ideal HPF via the RC circuit



$$v_c(t) = \int_0^t \frac{1}{RC} e^{-(t-\tau)/RC} v_s(\tau)\,d\tau$$

- let's measure the ***resistor voltage*** instead of the capacitor voltage
- the voltage across the resistor is $v_r = v_s - v_c$, so

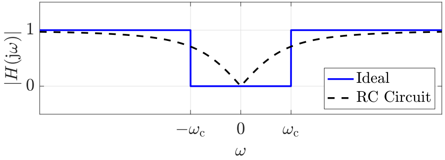
$$\begin{aligned} v_r(t) &= v_s(t) - \int_0^t \frac{1}{RC} e^{-(t-\tau)/RC} v_s(\tau)\,d\tau \\ &= \int_0^t \underbrace{\left[\delta(t-\tau) - \frac{1}{RC} e^{-(t-\tau)/RC}\right]}_{\tilde{h}(t-\tau)} v_s(\tau)\,d\tau \end{aligned}$$

A non-ideal HPF via the RC circuit

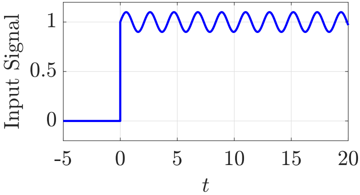
- this impulse response defines a non-ideal but causal HPF

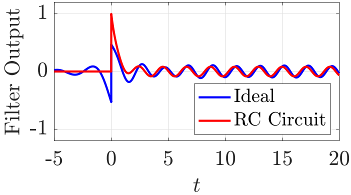
$$|\tilde{H}(\mathbf{j}\omega)| = 1 - \frac{1}{\sqrt{1+(RC\omega)^2}}$$

with $\frac{1}{RC} = \omega_c$



- for $\omega_c = 2$ rad/s, let's consider the input $x(t) = [1 + 0.1\sin(3t)]u(t)$





Relevant MATLAB commands

- MATLAB has very useful LTI system commands

```matlab
%% Define the system via its transfer function
s = tf('s');
H = 2/(s+2);

%% Plot the Bode plot, impulse response, step response
bode(H); impulse(H); step(H);

%% Simulate response to any input
h = 0.0001: T_max = 10;
t = 0:h:T_max;
u = sin(t).*exp(-t).*heaviside(t);
y = lsim(H,u,t);
```

Supplementary reading

- **O-W:** A. V. Oppenheim and S. Willsky, *Signals and Systems*, 2nd Ed.
- **BB:** B. Boulet, *Fundamentals of Signals and Systems*, Chp. 1.
- **BPL:** B. P. Lathi, *Signal Processing and Linear Systems*, Chp. 1, 8
- **K-S:** H. Kwakernaak and R. Sivan, *Modern Signals and Systems*, Chp. 2, Appendix A
- **EL-PV:** E. Lee and P. Varaiya, *Structure and Interpretation of Signals and Systems*, 2nd Ed.
- **ADL:** A. D. Lewis, *The Mathematical Theory of Signals and Systems*.

Topic	**O-W**	**BB**	**BPL**	**K-S**	**EL-PV**	**ADL**
Frequency response	6.2	8	4.4, 7.1, 7.2	7.5	8.1, 8.2	
Filters	6.3, 6.4	5, 8	4.5, 7.5–7.8	9.4	9	

Supplementary reading

- Oppenheim & Willsky, Chp. 3.2, 3.10, 6.2, 6.3, 6.7.1
- Boulet, Chp. 5, 8
- Lathi, Chp. 4, 7
- Kwakernaak & Sivan, Chp. 7.5

Personal Notes

Personal Notes

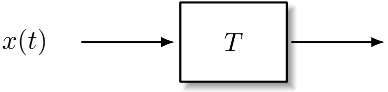
Personal Notes

8. Analysis of Continuous-Time Systems using the Laplace Transform

- response of a LTI system to a complex exponential input
- signals of exponential class and the Laplace transform
- response of a causal LTI system to a right-sided input
- analysis of LICC-ODEs via the Laplace transform
- steady-state interpretation of the freq. response for causal LTI systems

Warm-up: LTI systems with complex exponential inputs

- let T be a LTI system with impulse response h



$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau$$

- if the input x is a complex exponential $x(t) = e^{st}$ ***for s*** $\in \mathbb{C}$, then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\,\mathrm{d}\tau = \underbrace{\left[\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,\mathrm{d}\tau\right]}_{\text{“Transfer function” } H(s)} e^{st}$$

If the input is e^{st} then the output is ***also*** proportional to e^{st}, but ***scaled by the “transfer function”*** $H(s)$. Seems familiar!

Motivation for transfer function methods

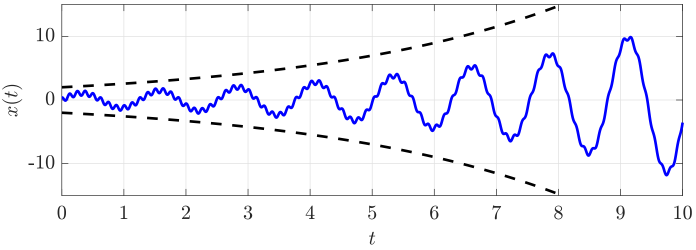
What is this new object $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,\mathrm{d}t$, and what use is it compared to the CTFT $H(\mathbf{j}\omega)$?

- **main idea:** we saw that the CTFT is not always well-defined; it turns out that $H(s)$ will exist in many situations where $H(\mathbf{j}\omega)$ does not
- while the theory here *can* be developed for two-sided signals and for non-causal LTI systems, we will study only
  - (i) CT signals that are ***right-sided*** (in this section, ***always*** from time 0)
  - (ii) CT systems that are LTI and ***causal***
- this setting describes ***physical systems*** (e.g., circuits, mechanical systems, etc.) . . . analyzing such systems is crucial in all engineering problems that involve "atoms" and not "bits"

Signals of exponential class

Definition 8.1. A right-sided CT signal x is of ***exponential class*** if $|x(t)| \leq Me^{\sigma t}$ for some constants $M > 0, \sigma \in \mathbb{R}$ and for all $t \geq 0$. The smallest possible value for σ that works is denoted by $\sigma^\star(x)$.

- **idea:** $\sigma^\star(x)$ is our sharpest bound on the "growth rate" of x.



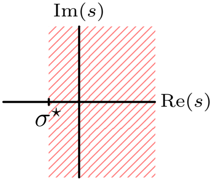
- **examples:** $e^{t^2}u(t)$ is not of exponential class

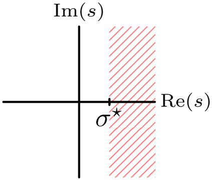
The right-sided Laplace transform

Definition 8.2 (Laplace Transform). Suppose x is of exponential class, and let $\mathcal{R}_x = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > \sigma^\star(x)\}$. The ***Laplace transform*** of x is the complex-valued function

$$X : \mathcal{R}_x \to \mathbb{C}, \qquad X(s) = \int_{0^-}^{\infty} x(t) e^{-st}\, \mathrm{d}t$$

- the lower limit "0^-" means we will include CT impulses at $t = 0$
- $X(s)$ is only defined for $s \in \mathcal{R}_x$, the ***region of convergence***





Existence of the Laplace transform

Let's check that this definition makes sense.

For any $s \in \mathbb{C}$, we can bound $X(s)$ as

$$|X(s)| \leq \left| \int_{0^-}^{\infty} x(t) e^{-st} \, \mathrm{d}t \right| \leq \int_{0^-}^{\infty} |x(t)| \cdot |e^{-st}| \, \mathrm{d}t$$

If $s = \sigma + \mathbf{j}\omega$, then $|e^{-st}| = |e^{-\sigma t} e^{-\mathbf{j}\omega t}| = e^{-\sigma t}$, so

$$|X(s)| \leq \int_{0^-}^{\infty} |x(t)| e^{-\sigma t} \, \mathrm{d}t \leq \int_{0^-}^{\infty} M e^{\sigma^* t} e^{-\sigma t} \, \mathrm{d}t.$$

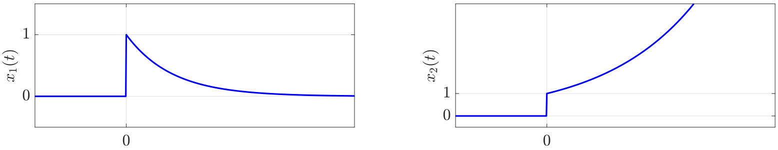
Since x is of exponential class, $|x(t)| \leq M e^{\sigma^* t}$, so

$$|X(s)| \leq \int_{0^-}^{\infty} M e^{-(\sigma - \sigma^*) t} \, \mathrm{d}t$$

For $\sigma > \sigma^*$ (i.e., for $s \in \mathcal{R}_x$), the exponential term is decaying, so the integral evaluates to a finite number. Thus $X(s)$ is well-defined for all $s \in \mathcal{R}_x$.

Example: LT of right-sided exponentials

- for $a > 0$ consider the signals $x_1(t) = e^{-at}u(t)$ and $x_2(t) = e^{at}u(t)$

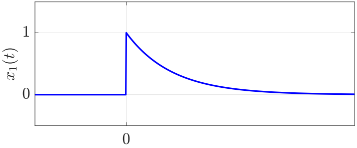


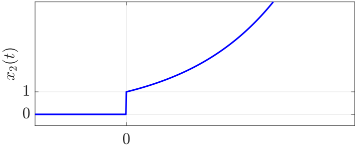
- the signal x_1 is exp. class with $\sigma^\star(x_1) = -a$, so $\mathcal{R}_{x_1} = \{s \mid \mathrm{Re}(s) > -a\}$.
- the LT of x_1 is

$$X_1(s) = \int_{0^-}^{\infty} e^{-at}u(t)e^{-st}\,\mathrm{d}t = \int_{0}^{\infty} e^{-(s+a)t}\,\mathrm{d}t = -\frac{1}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty}$$

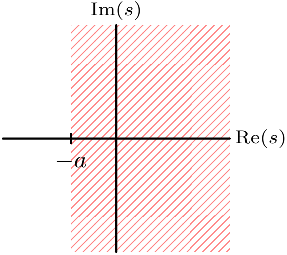
$$= \begin{cases} \frac{1}{s+a} & \text{if } s \in \mathcal{R}_{x_1} \\ \text{undefined} & \text{otherwise} \end{cases}$$

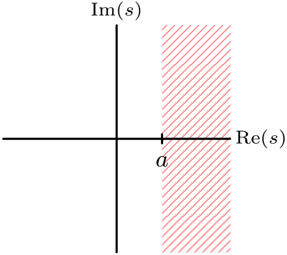
Example: LT of right-sided exponentials





$$X_1(s) = \frac{1}{s+a}, \text{ if } \mathrm{Re}(s) > -a \qquad X_2(s) = \frac{1}{s-a}, \text{ if } \mathrm{Re}(s) > a$$





Connection between LT and the CTFT

- **observation:** for the decaying exponential signal x_1, the region of convergence $\mathcal{R}_{x_1}$ *contains the imaginary axis* $\{s = \sigma + \mathbf{j}\omega \mid \sigma = 0\}$.
- it is therefore fine to substitute $s = \mathbf{j}\omega$, and we find that

$$X_1(s)\bigg|_{s=\mathbf{j}\omega} = \frac{1}{s+a}\bigg|_{s=\mathbf{j}\omega} = \frac{1}{\mathbf{j}\omega + a} = \text{CTFT of } x_1(t) = X_1(\mathbf{j}\omega)$$

General Fact: when $\mathcal{R}_x$ *contains the imaginary axis*, the CTFT $X(\mathbf{j}\omega)$ of $x(t)$ is obtained by substutitng $s = \mathbf{j}\omega$ into the LT $X(s)$

- when $\mathcal{R}_x$ ***does not*** contain the imaginary axis, the CTFT ***cannot*** be obtained from the LT; the CTFT may not even exist!

Important Laplace transforms

Name	$x(t)$	$X(s)$	$\mathcal{R}_x$
Impulse	$\delta(t)$	1	all s
Step	$u(t)$	$1/s$	$\mathrm{Re}(s) > 0$
Ramp	$tu(t)$	$1/s^2$	$\mathrm{Re}(s) > 0$
Monomial	$t^n u(t)$	$n!/s^{n+1}$	$\mathrm{Re}(s) > 0$
Sine	$\sin(\omega_0 t)u(t)$	$\omega_0/(s^2 + \omega_0^2)$	$\mathrm{Re}(s) > 0$
Cosine	$\cos(\omega_0 t)u(t)$	$s/(s^2 + \omega_0^2)$	$\mathrm{Re}(s) > 0$
Exponential	$e^{at}u(t)$	$1/(s-a)$	$\mathrm{Re}(s) > a$
Exp/Sin	$e^{at}\sin(\omega_0 t)u(t)$	$\omega_0/[(s-a)^2 + \omega_0^2]$	$\mathrm{Re}(s) > a$
Exp/Cos	$e^{at}\cos(\omega_0 t)u(t)$	$(s-a)/[(s-a)^2 + \omega_0^2]$	$\mathrm{Re}(s) > a$

Properties of the Laplace transform

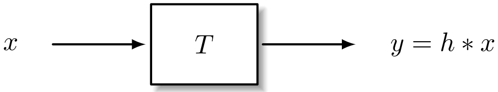
We let x be a right-sided CT signal with LT X.

Name	$x(t)$	$X(s)$	$\mathcal{R}_x$
Superposition	$\alpha x_1(t) + \beta x_2(t)$	$\alpha X_1(s) + \beta X_2(s)$	at least $\mathcal{R}_{x_1} \cap \mathcal{R}_{x_2}$
Time-shift by $t_0 \geq 0$	$x(t-t_0)u(t-t_0)$	$e^{-t_0 s} X(s)$	$\mathcal{R}_x$
Differentiation	$\dot{x}(t)$	$sX(s)$	at least $\mathcal{R}_x$
Integration up to $t \geq 0$	$\int_0^t x(\tau)\,\mathrm{d}\tau$	$\frac{1}{s}X(s)$	at least $\mathcal{R}_x \cap \mathbb{C}_{>0}$
Convolution	$(x_1 * x_2)(t)$	$X_1(s)X_2(s)$	at least $\mathcal{R}_{x_1} \cap \mathcal{R}_{x_2}$
Initial value theorem	$x(0^+)$	$\lim_{s\to\infty} sX(s)$	–
Final value theorem	$\lim_{t\to\infty} x(t)$	$\lim_{s\to 0} sX(s)$	–

Response of a causal LTI system and the LT

The Laplace transform can be used to compute the response of a causal LTI CT system to a general right-sided input.

- T causal LTI, impulse response h with LT $H(s)$ and RoC $\mathcal{R}_h$
- x right-sided with LT $X(s)$ and RoC $\mathcal{R}_x$



- by the convolution property of the LT, we have

$$Y(s) = H(s)X(s) \qquad \text{with RoC at least } \mathcal{R}_h \cap \mathcal{R}_x$$

Response of a causal LTI system and the LT

- Time-domain expression for output $y(t)$:

$$\text{Convolution:} \qquad y(t) = \int_{0^-}^{t} h(t-\tau)x(\tau)\,\mathrm{d}\tau, \qquad t \geq 0$$

- Laplace-domain expression for output $Y(s)$

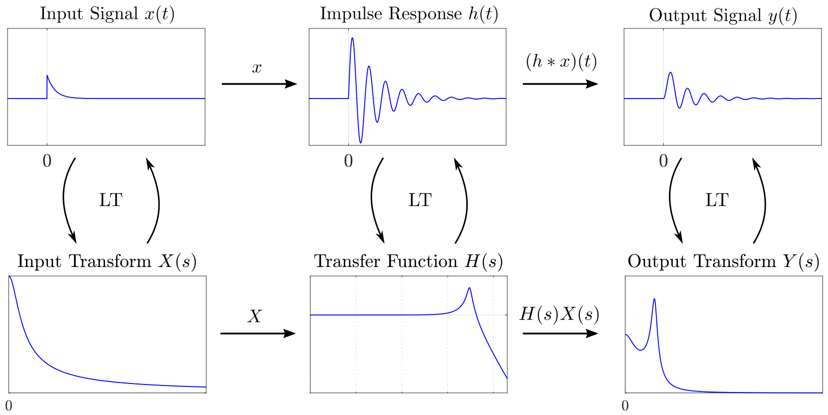
$$\text{Multiplication:} \qquad Y(s) = H(s)X(s), \qquad s \in \mathcal{R}_y \supseteq \mathcal{R}_h \cap \mathcal{R}_x$$

In the Laplace domain, the output $Y(s)$ is given by the ***product*** of the transfer function

$$H(s) = \int_{0^-}^{\infty} h(t)e^{-st}\,\mathrm{d}t$$

and the input $X(s)$. Much simpler than convolution!

Response of a LTI system and the LT



Example: RC circuit

$$h(t) = \tfrac{1}{RC} e^{-t/RC} u(t), \qquad x(t) = \frac{1}{\Delta}\left[u(t) - u(t-\Delta)\right]$$

Let's compute the output using the LT method

- The two LTs are

$$H(s) = \frac{1}{RCs+1}, \qquad X(s) = \frac{1}{\Delta}\frac{1}{s} - \frac{1}{\Delta}\frac{e^{-\Delta s}}{s} = \frac{1}{\Delta}\frac{1}{s}(1 - e^{-\Delta s})$$

- Therefore

$$Y(s) = \frac{1}{\Delta}\frac{1}{s(RCs+1)}(1 - e^{-\Delta s})$$

- Using partial fractions, note that

$$\frac{1}{s(RCs+1)} = \frac{A}{s} + \frac{B}{RCs+1} = \frac{(ARC+B)s + A}{s(RCs+1)}$$

so $A = 1$ and $B = -RC$

Example: RC circuit

- Therefore we have

$$Y(s) = \frac{1}{\Delta}\left(\frac{1}{s} - \frac{RC}{RCs+1}\right)(1 - e^{-\Delta s})$$
$$= \frac{1}{\Delta}\left(\frac{1}{s} - \frac{1}{s+\frac{1}{RC}}\right) - \frac{1}{\Delta}\left(\frac{1}{s} - \frac{1}{s+\frac{1}{RC}}\right)e^{-\Delta s}$$

- Going back to the time-domain, we have

$$y(t) = \frac{1}{\Delta}(1 - e^{-t/RC})u(t) - \frac{1}{\Delta}(1 - e^{-(t-\Delta)/RC})u(t-\Delta)$$

which is the same solution we obtained via convolution

Note: The LT method can also be used to include ***non-zero initial conditions*** the response calculations. We don't care about this in ECE216, but in other courses you might.

Analysis of LICC-ODEs via Laplace Transform

- **recall:** the LICC-ODE $Q(D)y(t) = P(D)x(t)$ defines a causal LTI system which produces a unique right-sided output $y(t)$ for each right-sided input $x(t)$
- the quantities $Q(D)$ and $P(D)$ are the linear differential operators

$$Q(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$$

$$P(D) = b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0$$

and $D\xi(t) = \frac{\mathrm{d}\xi(t)}{\mathrm{d}t}$ is the derivative operation

We can use the Laplace transform to compute the impulse response of this causal LTI system.

Analysis of LICC-ODEs via Laplace Transform

The key idea is to take the LT of both sides of the differential equation $Q(D)y(t) = P(D)x(t)$, then rearrange.

- using the LT derivative rule

$$(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)Y(s)$$
$$= (b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0)X(s)$$

- rearranging

$$\frac{Y(s)}{X(s)} = H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{P(s)}{Q(s)}$$

The transfer function of an LTI system defined by a LICC-ODE is very easy to find; this is often the preferred way to compute outputs.

Analysis of LICC-ODEs via Laplace Transform

- now we have the ***rational*** function

$$H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{P(s)}{Q(s)},$$

  and you can compute the impulse response $h(t)$ by:

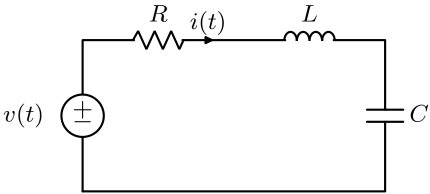
  (i) factoring the denominator
  (ii) performing a partial fraction expansion on $H(s)$
  (iii) computing the inverse Laplace transform term-by-term

- if $m \leq n$, you will obtain an expression of the form

$$h(t) = b_0\delta(t) + \left[\sum_{k=1}^{n} b_k t^{m_k} e^{p_k t} \sin(\omega_k t + \phi_k)\right] u(t)$$

  for appropriate constants $b_k, p_k, m_k, \omega_k, \phi_k$ . . . often complicated, but occasionally useful

Example: the series RLC circuit



$$\frac{\mathrm{d}^2 i(t)}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{LC}i(t) = \frac{1}{L}\frac{\mathrm{d}v(t)}{\mathrm{d}t}$$

- Following our procedure we obtain

$$H(s) = \frac{I(s)}{V(s)} = \frac{\frac{1}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \underbrace{\frac{\frac{1}{L}s}{s^2 + 2\zeta\omega_{\mathrm{n}}s + \omega_{\mathrm{n}}^2}}_{\text{standard form}}$$

- Equating the coefficients, you can easily find that

$$\omega_{\mathrm{n}} = \text{"natural frequency"} = \sqrt{\frac{1}{LC}}, \qquad \zeta = \text{"damping ratio"} = \frac{R}{2}\sqrt{\frac{C}{L}}.$$

Steady-state interpretation of the frequency response

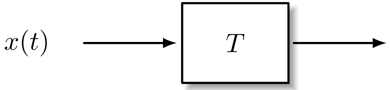
- in Chap. 7, we understood ***frequency response*** $H(\mathbf{j}\omega)$ as telling us about the response of a LTI system to *two-sided* complex exp. inputs

$$x(t) = e^{\mathbf{j}\omega_0 t} \quad \Longrightarrow \quad y(t) = H(\mathbf{j}\omega_0)e^{\mathbf{j}\omega_0 t} = |H(\mathbf{j}\omega_0)|e^{\mathbf{j}(\angle H(\mathbf{j}\omega_0))}e^{\mathbf{j}\omega_0 t}$$

- in this chapter though we have been looking at ***right-sided*** inputs and ***causal*** systems – how should our interpretation change?

- consider applying a *right-sided* complex exponential input

$$x(t) = e^{\mathbf{j}\omega_0 t}u(t)$$



$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau$$

Steady-state interpretation of the frequency response

- **main assumption:** h has finite action, so T is ***BIBO stable***
- the system responds with the output signal

$$
\begin{aligned}
y(t) &= \left[\int_{-\infty}^{\infty} h(\tau) e^{\mathbf{j}\omega_0 (t-\tau)} u(t-\tau)\,\mathrm{d}\tau\right] u(t) \\
&= \left[\int_{-\infty}^{t} h(\tau) e^{-\mathbf{j}\omega_0 \tau}\,\mathrm{d}\tau\right] \underbrace{e^{\mathbf{j}\omega_0 t} u(t)}_{x(t)} && \text{(prop. of unit step)} \\
&= \underbrace{\left[\int_{-\infty}^{\infty} h(\tau) e^{-\mathbf{j}\omega_0 \tau}\,\mathrm{d}\tau\right]}_{\text{CTFT of } h} x(t) - \underbrace{\left[\int_{t}^{\infty} h(\tau) e^{-\mathbf{j}\omega_0 \tau}\,\mathrm{d}\tau\right]}_{\triangleq F(t)} x(t) && \text{(rewriting)} \\
&= H(\mathbf{j}\omega_0) x(t) - F(t) x(t)
\end{aligned}
$$

The first term looks familiar from our CTFT analysis of LTI systems. But what is this new term $F(t)$?

Steady-state interpretation of the frequency response

- since h has finite action, we can bound $|F(t)|$ as

$$|F(t)| = \left|\int_t^\infty h(\tau)e^{-\mathbf{j}\omega_0\tau}\,\mathrm{d}\tau\right| \leq \int_t^\infty |h(\tau)|\,\mathrm{d}\tau$$

$$= \underbrace{\int_{-\infty}^\infty |h(\tau)|\,\mathrm{d}\tau}_{\text{some number } M} - \underbrace{\int_{-\infty}^t |h(\tau)|\,\mathrm{d}\tau}_{m(t)}$$

where $m(t) \geq 0$ satisfies $\lim_{t\to\infty} m(t) = M$. So

$$\lim_{t\to\infty} |F(t)| \leq M - \lim_{t\to\infty} m(t) = 0 \qquad \Longrightarrow \qquad \lim_{t\to\infty} F(t) = 0.$$

The new term $F(t)$ vanishes as $t \to \infty$. We think of this term as capturing the ***transient*** response.

Steady-state interpretation of the frequency response

- if we define the ***steady-state response*** as

$$y_{\mathrm{ss}}(t) \triangleq H(\mathbf{j}\omega_0)x(t) = |H(\mathbf{j}\omega_0)|e^{\mathbf{j}(\angle H(\mathbf{j}\omega_0))}e^{\mathbf{j}\omega_0 t}u(t)$$

  then

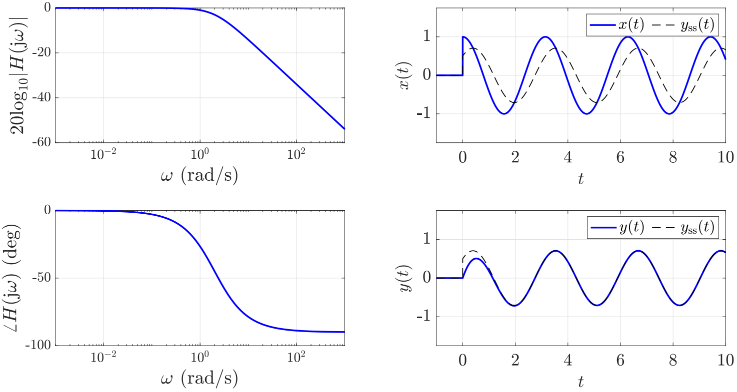
$$\lim_{t\to\infty}[y(t) - y_{\mathrm{ss}}(t)] = \lim_{t\to\infty} F(t)x(t) = 0!$$

- the output $y(t)$ converges to $y_{\mathrm{ss}}(t)$!

For a causal and BIBO stable system, the frequency response $H(\mathbf{j}\omega_0)$ quantifies the ***steady-state*** amplitude scaling and phase shift experienced by ***right-sided*** complex exponential signal.

Example: steady-state response of RC circuit

- $RC = 1/2$, input is $x(t) = \cos(\omega_0 t)u(t)$ with $\omega_0 = 2$ rad/s



Steady-state interpretation of the frequency response

Theorem 8.1. For a causal and BIBO stable CT LTI system

(i) the response to a sinusoidal input is sinusoidal in steady-state;

(ii) the s.s. response has the same frequency as the input;

(iii) the s.s response amplitude is the input amplitude times $|H(\mathbf{j}\omega_0)|$;

(iv) the s.s response phase is the input phase plus a shift of $\angle H(\mathbf{j}\omega_0)$.

- The frequency response $H(\mathbf{j}\omega)$ of a BIBO stable physical system can be ***measured*** by applying a sinusoidal inputs, waiting, and recording the steady-state output
- Bode plots are the standard way to visualize $H(\mathbf{j}\omega)$

Relevant MATLAB commands

- working with transfer functions for simulations

```
s = tf('s');
H_1 = 1/(s^2 + 0.1*s+1);%Define two TFs
H_2 = 1/(5*s+1);
bode(H_1*H_2);          %Plot's Bode diagram
step(H_1 + H_2);        %Response with unit step input
```

- MATLAB also has symbolic tools for doing LTs

```
syms s;
syms t a real;
x = exp(-a*t)*heaviside(t); %Define a right-sided signal
X = laplace(x);             %Compute the LT
x_recovered = ilaplace(X);  %Compute the inverse LT
```

Supplementary reading

- **O-W:** A. V. Oppenheim and S. Willsky, *Signals and Systems*, 2nd Ed.
- **BB:** B. Boulet, *Fundamentals of Signals and Systems*.
- **BPL:** B. P. Lathi, *Signal Processing and Linear Systems*.
- **K-S:** H. Kwakernaak and R. Sivan, *Modern Signals and Systems*.
- **EL-PV:** E. Lee and P. Varaiya, *Structure and Interpretation of Signals and Systems*.
- **ADL:** A. D. Lewis, *The Mathematical Theory of Signals and Systems*.

Topic	**O-W**	**BB**	**BPL**	**K-S**	**EL-PV**	**ADL**
Laplace transform	9.1–9.6, 9.9	6	6.1, 6.2	8.1–8.5	13.3–13.7	V4 9.1
Transfer functions	9.7	7	6.3–6.5	8.6–8.7	13.3–13.7	

Personal Notes

Personal Notes

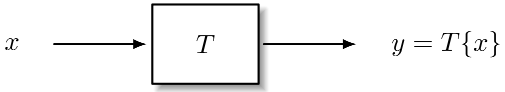
Personal Notes

9. Fundamentals of Discrete-Time Systems

- definition of a DT system and examples
- linearity, causality, time-invariance
- memory, invertibility, stability
- linear time-invariant (LTI) systems
- impulse response of a LTI system
- response of a LTI system and convolution
- LTI system properties and the impulse response
- finite impulse response (FIR) systems
- difference equations and DT LTI systems
- frequency-domain analysis of DT LTI systems

What is a DT system

- a ***DT system*** is some operation T that takes a DT input and produces exactly one DT output

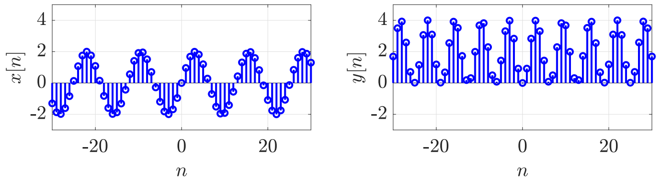


- **notation:** $T\{x\}[n]$ is the *value* of the output signal at time $n \in \mathbb{Z}$.
- while CT systems often arise from physics, DT systems tend to come from engineering processes or phenomenological models, such as
  (i) discretization of CT systems,
  (ii) digital filtering or signal processing,
  (iii) economic/financial models, ...

Pointwise definition of a DT system

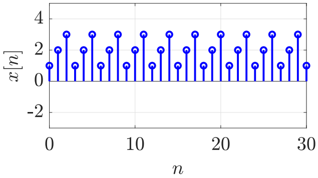
One way to define a DT system is ***pointwise***: give a formula for $y[n] = T\{x\}[n]$ in terms of (potentially) *all* the input signal values $\{x[n]\}_{n\in\mathbb{Z}}$.

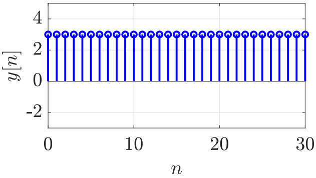
- **example:** the system T_{sq} defined by $y[n] = (x[n])^2$ produces an output which is the squared value of the input at each $n \in \mathbb{Z}$



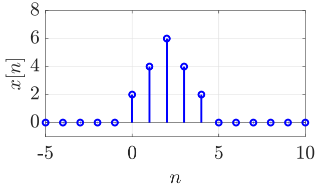
Pointwise definition of a DT system

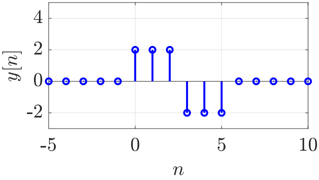
- **example:** the system defined by $y[n] = \max\{x[n], x[n-1], x[n-2]\}$





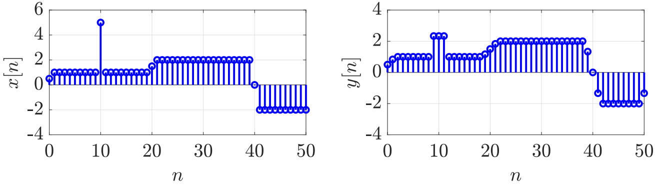
- **example:** the system T_{diff} defined by $y[n] = x[n] - x[n-1]$



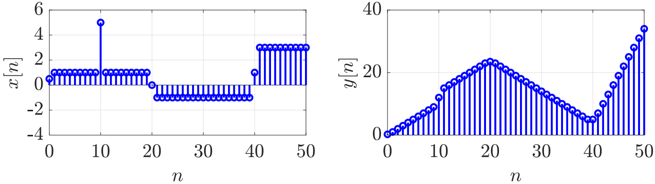


Pointwise definition of a DT system

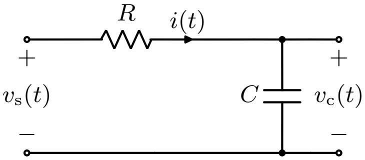
- **example:** the three-point moving average filter
  $y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$



- **example:** the summing system T_{sum} defined by $y[n] = \sum_{k=-\infty}^{n} x[k]$



Example: discretized RC circuit



$$RC\frac{\mathrm{d}v_{\mathrm{c}}}{\mathrm{d}t} + v_{\mathrm{c}}(t) = v_{\mathrm{s}}(t)$$

- suppose we ***sample*** the values of the voltages every T_{s} seconds, and approximate the derivative with a ***first difference***

$$RC\frac{v_{\mathrm{c}}(nT_{\mathrm{s}}) - v_{\mathrm{c}}((n-1)T_{\mathrm{s}})}{T_{\mathrm{s}}} + v_{\mathrm{c}}((n-1)T_{\mathrm{s}}) = v_{\mathrm{s}}((n-1)T_{\mathrm{s}})$$

- rearranging, we have the ***recursion***

$$v_{\mathrm{c}}[n] = (1 - \tfrac{T_{\mathrm{s}}}{RC})v_{\mathrm{c}}[n-1] + \tfrac{T_{\mathrm{s}}}{RC}v_{\mathrm{s}}[n-1]$$

Example: a bank account

Discrete-time systems are often useful for modelling financial or economic processes that are indexed at discrete times (e.g., quarterly)

- consider a ***bank account*** where
  - (i) $s[n]$ = savings balance at end of month n (we start at month $n = 0$)
  - (ii) $x[n]$ = earnings deposited at end of month n
- let $r > 0$ be the monthly interest rate, $s[0] = x[0]$ the initial balance
- a simple model describing the balance evolution is

$$s[n] = \underbrace{(1+r)s[n-1]}_{\text{old balance plus interest}} + \underbrace{x[n]}_{\text{new deposit}}$$

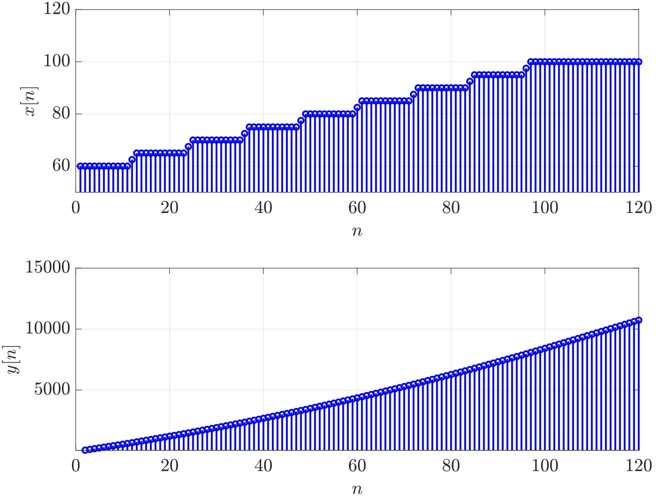
- this is again a ***recursive*** definition of a DT system

Example: a bank account

- in this simple case, we can actually convert the recursive definition into a ***pointwise*** definition by iterating:

$$
\begin{aligned}
s[0] &= x[0] \\
s[1] &= (1+r)s[0] + x[1] = (1+r)x[0] + x[1] \\
s[2] &= (1+r)s[1] + x[2] = (1+r)^2 x[0] + (1+r)x[1] + x[2] \\
&\vdots \\
s[n] &= \sum_{k=0}^{n} (1+r)^{n-k} x[k]
\end{aligned}
$$

Example: a bank account



Properties of DT systems: linearity and causality

Definition 9.1. A DT system T is ***linear*** if for any two input signals $x, \tilde{x}$ and any two constants $\alpha, \tilde{\alpha} \in \mathbb{C}$ it holds that $T\{\alpha x + \tilde{\alpha}\tilde{x}\} = \alpha T\{x\} + \tilde{\alpha} T\{\tilde{x}\}$.

- the summing system $y[n] = \sum_{k=-\infty}^{n} x[k]$ is linear
- the squaring system $y[n] = (x[n])^2$ is not linear

Definition 9.2. A DT system T is ***causal*** if for all $n \in \mathbb{Z}$, the output value $y[n]$ depends only on the present and previous input values $\{\ldots, x[n-3], x[n-2], x[n-1], x[n]\}$.

- the 1-step delay system $y[n] = x[n-1]$ is causal
- the system $y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$ is not causal

Properties of DT systems: time-invariance

- ***notation:*** let $x_k[n] = x[n-k]$ be short form notation for a time-shifted signal

Definition 9.3. A DT system T is ***time-invariant*** if for any input signal x with output signal $y = T\{x\}$, it holds that $y_k = T\{x_k\}$ for all possible time shifts $k \in \mathbb{Z}$.

- ***physical meaning:*** an experiment on the system *tomorrow* will produce the same results as an experiment on the system *today*.
- the squaring system $y[n] = (x[n])^2$ is time-invariant
- the summing system $y[n] = \sum_{l=-\infty}^{n} x[l]$ is time-invariant
- the system $y[n] = \sin(n/10)x[n]$ is not time-invariant

Example: time-invariance of reversal system

- consider the ***time-reversal*** system T_{tr} defined by

$$y[n] = x[-n]$$

- let x be an input with correspond. output

$$y[n] = T_{\text{tr}}\{x\}[n] = x[-n]$$

- if we simply shift the obtained output, we obtain

$$y_k[n] = y[n-k] = x[-(n-k)] = x[-n+k]$$

- if we shift the input signal as $x_k[n] = x[n-k]$, we compute the output

$$T_{\text{tr}}\{x_k\}[n] = x_k[-n] = x[-n-k]$$

These two signals are not equal, so the system is ***not*** time-invariant.

Properties of DT systems: memory and invertibility

Definition 9.4. A DT system T is ***memoryless*** if for all times $n \in \mathbb{Z}$, the output value $y[n]$ depends only on the input value $x[n]$.

- the squaring system $y[n] = (x[n])^2$ is memoryless
- the summing system $y[n] = \sum_{k=-\infty}^{n} x[k]$ is not memoryless

Definition 9.5. A DT system T is ***invertible*** if there exists another DT system T_{inv} such that $T_{\text{inv}}\{T\{x\}\} = T\{T_{\text{inv}}\{x\}\} = x$ for all inputs x.

- the 1-step delay system $y[n] = x[n-1]$ is invertible, with inverse $y[n] = x[n+1]$.
- the squaring system $y[n] = (x[n])^2$ is not invertible

Properties of DT systems: stability (linear systems)

- **recall:** a DT signal x has ***finite amplitude*** or is ***bounded*** if $\|x\|_\infty = \max_{n\in\mathbb{Z}} |x[n]|$ is finite, and if so, we write $x \in \ell_\infty$

Definition 9.6. A linear DT system T is ***Bounded-Input Bounded-Output (BIBO) stable*** if there is a constant $K \geq 0$ such that $\|y\|_\infty \leq K\|x\|_\infty$ for all bounded inputs x and outputs $y = T\{x\}$.

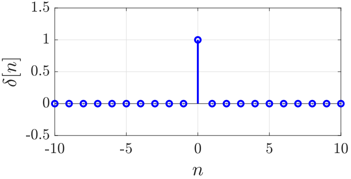
"Bounded inputs produce bounded outputs"

- the 1-step delay system $y[n] = x[n-1]$ is BIBO stable
- the summing system $y[n] = \sum_{k=-\infty}^{n} x[k]$ is not BIBO stable
- the difference system $y[n] = x[n] - x[n-1]$ is BIBO stable

Linear Time-Invariant (LTI) systems

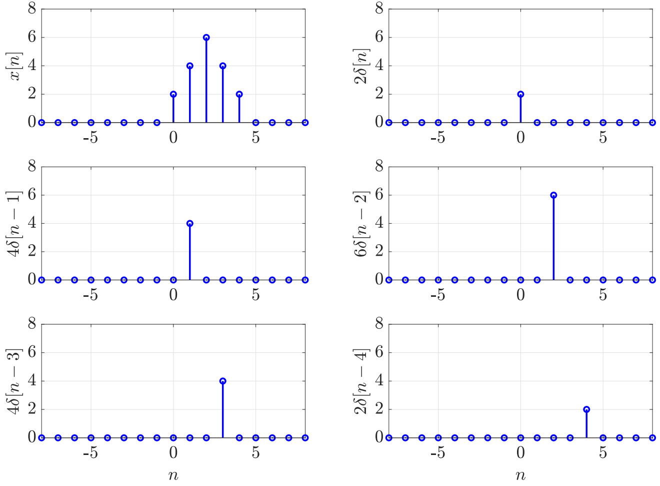
- we now focus on systems which are ***both*** linear and time-invariant
- remember that, roughly speaking
  1. linearity: "the superposition principle holds"
  2. time-invariance: "the system will be the same tomorrow as it is today"
- **recall:** the DT unit impulse and sifting formula

$$\delta[n] = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases}$$



Sifting formula: $$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \quad \text{or} \quad x = \sum_{k=-\infty}^{\infty} x[k]\delta_k$$

Visualization of sifting formula



The impulse response of a LTI system

Definition 9.7. The ***impulse response*** h of a DT LTI system T is the response $T\{\delta\}$ to a unit impulse δ input applied at $n = 0$, i.e., $h = T\{\delta\}$.

- **example:** the system $y[n] = \sum_{k=-\infty}^{n} x[k]$ has impulse response

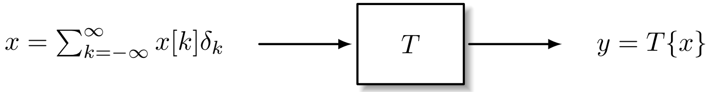
$$h[n] = \sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n \geq 0 \end{cases} = u[n]$$

- **example:** the system $y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$ has impulse response

$$h[n] = \frac{1}{3}(\delta[n-1] + \delta[n] + \delta[n+1])$$

Response of a DT LTI system

- let's apply the input x to a DT LTI system



- using linearity of T, we calculate that

$$y = T\{x\} = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta_k\right\} = \sum_{k=-\infty}^{\infty} x[k]T\{\delta_k\}$$

- since T is ***time-invariant***, the response $T\{\delta_k\}$ to an impulse at time k is equal to the response $h = T\{\delta_0\}$ shifted by k time units, i.e.,

$$T\{\delta_k\} = h_k, \qquad \text{or explicitly} \qquad T\{\delta_k\}[n] = h[n-k]$$

Response of a DT LTI system

- putting everything together, we have

$$y[n] = T\{x\}[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k]$$

This is the DT ***convolution*** of the signals h and x, and is denoted by $y = h * x$

- compare convolution in continuous and discrete time:

CT Convolution

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,\mathrm{d}\tau$$

DT Convolution

$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k]$$

Examples of DT convolution

- **example:** consider the impulse response $h[n] = \delta[n - n_0]$, a shifted impulse function. We compute via convolution that

$$y[n] = \sum_{k=-\infty}^{\infty} \delta[n - k - n_0]x[k] = x[n - n_0]$$

so this impulse response corresponds to a n_0-step delay system

- **example:** consider the impulse response $h[n] = \delta[n] - \delta[n - 1]$. We compute that

$$y[n] = \sum_{k=-\infty}^{\infty} \delta[n - k]x[k] - \delta[n - 1 - k]x[k] = x[n] - x[n - 1]$$

so this is actually the impulse response of T_{diff}!

Properties of DT convolution

- as with CT signals, we can think of convolution as an operation which takes two signals v, w and returns another signal $v * w$ defined as

$$(v * w)[n] = \sum_{k=-\infty}^{\infty} v[n-k]w[k].$$

For any DT signals v, w, x and any constants α, β the following hold:

(i) ***superposition:*** $x * (\alpha v + \beta w) = \alpha(x * v) + \beta(x * w)$

(ii) ***commutative:*** $v * w = w * v$

(iii) ***time-invariance:*** $v * w_k = (v * w)_k$

(iv) ***identity element:*** $\delta * x = x$

- as with CT signals, convolution is not always well-defined, but Proposition 6.1 applies also to DT convolution with no changes

Example: DT convolution and polynomial multiplication

- convolution pops up in some other unexpected places
- consider the two polynomials P and Q defined by

$$P(s) = \sum_{k=0}^{n} a[k]s^k, \qquad Q(s) = \sum_{\ell=0}^{m} b[\ell]s^\ell.$$

  with coefficients $a[0], \ldots, a[n]$ and $b[0], \ldots, b[m]$

- if we pad both sets of coefficients with zeros on either side, i.e.,

$$\ldots, 0, \underbrace{a[-2]}_{=0}, \underbrace{a[-1]}_{=0}, a[0], \ldots, a[n], \underbrace{a[n+1]}_{=0}, 0, \ldots$$

  we can think of a and b as DT signals defined for all $k \in \mathbb{Z}$, and we can then write P and Q as

$$P(s) = \sum_{k=-\infty}^{\infty} a[k]s^k, \qquad Q(s) = \sum_{\ell=-\infty}^{\infty} b[\ell]s^\ell.$$

Example: DT convolution and polynomial multiplication

- multiplying these two poly. we get a degree $n+m$ poly.

$$P(s)Q(s) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a[k]b[\ell]s^{k+\ell} \qquad \text{(double sum)}$$

$$= \sum_{r=-\infty}^{\infty} \underbrace{\sum_{\ell=-\infty}^{\infty} a[r-\ell]b[\ell]\, s^{r}}_{\triangleq c[r]} \qquad (r = k+\ell)$$

- therefore, we get coefficients $c[r] = (a * b)[r]$

The coefficients of a product of polynomials are given by the DT convolution of the original coefficients!

LTI system properties and the impulse response

- **summary:** the output of any DT LTI system is given by the convolution of the impulse response with the input signal

$$y = h * x \qquad\qquad y[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k]$$

- h is very useful for ***understanding properties*** of LTI systems

Theorem 9.1. A DT LTI system T with impulse response $h = T\{\delta\}$ is

(i) causal if and only if $h[n] = 0$ for all $n < 0$;

(ii) memoryless if and only if $h[n] = \alpha\delta[n]$ for some $\alpha \in \mathbb{C}$;

(iii) BIBO stable iff h has finite action, i.e., $h \in \ell_1$, in which case

$$\|y\|_\infty \leq \|h\|_1 \|x\|_\infty, \quad \text{for all } x \in \ell_\infty \text{ with } y = T\{x\}.$$

Proof of Theorem 9.1

Recall that our convolution formula for the output is

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[n-k]x[k]. \tag{3}$$

(i): A system is causal if $y[n]$ depends only on $\{\ldots, x[n-2], x[n-1], x[n]\}$. If $h[n] = 0$ for all $n < 0$, then (3) simplifies to

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k] = h[0]x[n] + h[1]x[n-1] + \cdots$$

which shows that T is causal. Conversely, if T is causal, then all terms $k = -\infty$ to $k = -1$ in (3) must be zero for ***any*** input x, so it must be that $h[n] = 0$ for all $n < 0$.

(ii): A system is memoryless if $y[n]$ depends only on $x[n]$, and not on the value of x at any other time. If $h[n] = 0$ for all $n \neq 0$, then (3) simplifies to $y[n] = h[0]x[n]$ so we conclude that T is memoryless. Conversely, if T is memoryless, then all terms in (3) must be zero except for $k = 0$ any for ***any*** x, so it must be that $h[n] = 0$ for all $n \neq 0$.

Proof of Theorem 9.1

(iii): Suppose that $h \in \ell_1$ and let $x \in \ell_\infty$. Then

$$|y[n]| \leq \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \leq \sum_{k=-\infty}^{\infty} |h[k]| \cdot |x[n-k]| \leq \|x\|_\infty \sum_{k=-\infty}^{\infty} |h[k]|$$

so $\|y\|_\infty \leq \|h\|_1 \|x\|_\infty$. Since x was arbitrary, T is BIBO stable with $K = \|h\|_1$.

To show the converse, we show that if h is does not have finite ation, then T is not BIBO stable. To show that T is not BIBO stable, we need only construct *one bounded input* which produces an *unbounded* output. Consider the candidate input

$$x[n] = \begin{cases} h[-n]^*/|h[-n]| & \text{if } h[-n] \neq 0 \\ 0 & \text{if } h[-n] = 0 \end{cases}$$

which is bounded since $|x[n]| = 1$ or $|x[n]| = 0$ for all $n \in \mathbb{Z}$. We compute that

$$y[0] = \sum_{k=-\infty}^{\infty} h[-k]x[k] = \sum_{k=-\infty}^{\infty} h[-k]\frac{h[-k]^*}{|h[-k]|} = \sum_{k=-\infty}^{\infty} \frac{|h[-k]|^2}{|h[-k]|} = \sum_{k=-\infty}^{\infty} |h[k]|.$$

which equals $+\infty$ since $h \notin \ell_1$, so y is not bounded, and hence T is not BIBO stable.

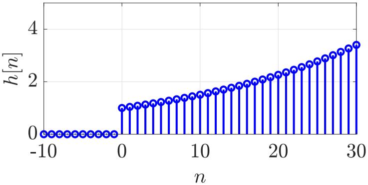
Example: bank account model

- the equation for the bank account model is

$$s[n] = \sum_{k=0}^{n} (1+r)^{n-k} x[k], \qquad n \geq 0.$$

- the impulse response is computed to be

$$h[n] = \sum_{k=0}^{n} (1+r)^{n-k} \delta[k] = (1+r)^n u[n]$$



- note that
  - (i) the system is causal, since $h[n] = 0$ for all $n < 0$;
  - (ii) the system is not memoryless, since, e.g., $h[1] = 1 + r \neq 0$
  - (iii) the system is ***not*** BIBO stable, since

$$\sum_{k=-\infty}^{\infty} (1+r)^k u[k] = \sum_{k=0}^{\infty} (1+r)^k = +\infty$$

Example: bank account model

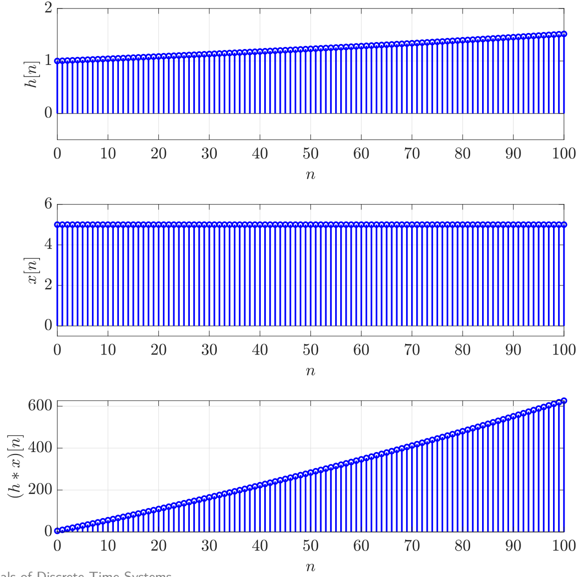
- consider the input $x[n] = 5u[n]$, corresponding to a \$5,000/month deposit
- for $n < 0$, the response $(h * x)[n]$ will be zero
- we compute the response $y = h * x$ for $n \geq 0$ as

$$
\begin{aligned}
y[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k] &= \sum_{k=-\infty}^{\infty} (1+r)^{n-k} u[n-k] \cdot (5u[k]) \\
&= 5 \sum_{k=0}^{n} (1+r)^{n-k} \\
&= 5 \frac{1-(1+r)^{n+1}}{1-(1+r)} = 5 \frac{(1+r)^{n+1}-1}{r}
\end{aligned}
$$

- therefore, we have the final answer

$$
y[n] = (h * x)[n] = \begin{cases} 0 & \text{if } n < 0 \\ 5 \frac{(1+r)^{n+1}-1}{r} & \text{if } n \geq 0 \end{cases}
$$

Example: bank account model



Finite impulse response (FIR) systems

- a causal ***finite impulse response (FIR)*** system is a DT LTI system of the form

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = b_0 x[n] + b_1 x[n-1] \cdots + b_M x[n-M]$$

where $\{b_k\}_{k=0}^{M}$ are constants and M is the ***order*** of the system

> "The current output value is a finite linear combination of the past and present input values"

- FIR systems are commonly used in signal processing and filtering; one can also consider non-causal versions by incorporating terms proportional to $x[n+1]$, $x[n+2]$, etc.

Finite impulse response (FIR) systems

- we compute the impulse response of an FIR system to be

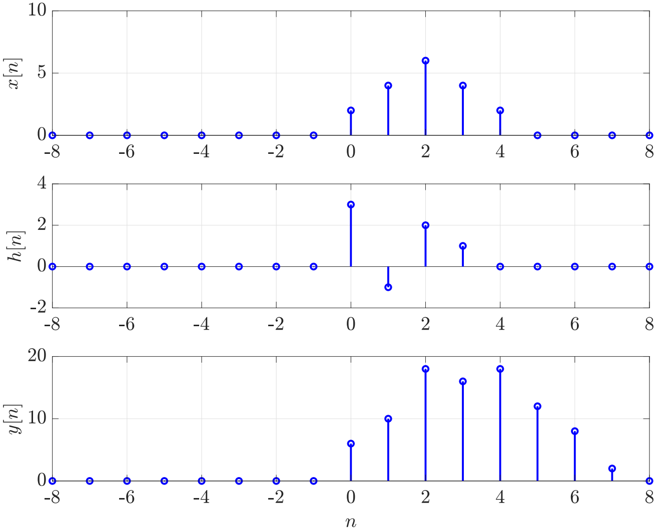
$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n & \text{if } 0 \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

- the impulse response is non-zero only for a ***finite*** number of time steps, hence the terminology FIR system

- note that this LTI system is
  (i) causal, since $h[n] = 0$ for all $n < 0$
  (ii) BIBO stable, since

$$\sum_{k=-\infty}^{\infty} |h[n]| = \sum_{k=0}^{M} |b_k| = \text{ some number } < \infty.$$

Example: FIR system



Difference equations and DT systems

- DT systems sometimes arise from ***recursive*** relationships called ***difference equations***; we saw this when we discretized the RC circuit, and in the bank account model
- sometimes it is easy to transform the recursive definition into a pointwise definition, sometimes it is more difficult
- **fact:** under mild technical conditions, difference equations with inputs always lead to DT systems, even if you can't solve the recursion explicitly!

We will now study a special case of this general fact, and argue that ***linear, inhomogeneous, constant-coefficient difference equations*** can define causal DT LTI systems.

LICC-DEs and DT systems

- we consider an n^{th} order, linear, inhomogeneous constant-coefficient difference equation (***LICC-DE***)

  $$a_0 y[k] + a_1 y[k-1] + \cdots + a_n y[k-n] = b_0 x[k] + \cdots + b_m x[k-m]$$

  which must hold for all times $k \in \mathbb{Z}$. In this equation

  (i) x is a given signal,

  (ii) y is the unknown to be solved for,

  (iii) n, m are nonnegative integers, and

  (iv) the coefficients $a_0, \ldots, a_n, b_0, \ldots, b_m$ are real constants.

- without loss of generality, we will assume that $a_0 = 1$
- **example:** bank model $y[k] = (1+r)y[k-1] + x[k]$
- **examaple:** RC circuit $y[k] = \left(1 - \frac{T_s}{RC}\right) y[k-1] + \frac{T_s}{RC} x[k-1]$

LICC-DEs and DT systems

- we will use the short-form notation $D^\ell y[k] = y[k-\ell]$, and set

$$Q(D) = 1 + a_1 D + a_2 D^2 + \cdots + a_n D^n$$
$$P(D) = b_0 + b_1 D + b_2 D^2 + \cdots + b_m D^m$$

- we can now write our difference equation

$$y[k] + a_1 y[k-1] + \cdots + a_n y[k-n] = b_0 x[k] + \cdots + b_m x[k-m]$$

in the short form

$$Q(D)y[k] = P(D)x[k] \qquad \text{(LICC-DE)}$$

When does (LICC-DE) define a (causal, LTI) DT system?

LICC-DEs define LTI DT systems

- as with LICC-ODEs for CT systems, we will consider only right-sided inputs and outputs

Theorem 9.2. For each right-sided input $x[k]$, the LICC-DE $Q(D)y[k] = P(D)x[k]$ possesses exactly one right-sided solution $y[k]$, and therefore defines a DT system $y = T\{x\}$. Moreover

- the system T is linear, time-invariant, and causal;
- the system T is BIBO stable if and only if all roots of

$$Q(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

in the complex variable $z \in \mathbb{C}$ have magnitude less than one.

Example: discretized RC circuit

- the RC circuit discretized with sampling period $T_{\rm s}$ is

$$v_{\rm c}[n] - (1 - \tfrac{T_{\rm s}}{RC})v_{\rm c}[n-1] = \tfrac{T_{\rm s}}{RC}v_{\rm s}[n-1]$$

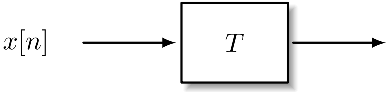
- this is a difference equation with $n = m = 1$, $a_1 = -(1 - \frac{T_{\rm s}}{RC})$, $b_0 = 0$, and $b_1 = \frac{T_{\rm s}}{RC}$.
- the polynomial is $Q(z) = z - (1 - \frac{T_{\rm s}}{RC}) = 0$, which has one root.
- the root has magnitude less than one if and only if

$$-1 < 1 - \tfrac{T_{\rm s}}{RC} < 1 \qquad \text{or} \qquad T_{\rm s} < RC.$$

The original CT model is BIBO stable. The DT model is ***also*** BIBO stable when sampling period is *sufficiently small*!

Warm-up: LTI systems with complex exponential inputs

- let T be a LTI system with impulse response h



$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- if the input x is a complex exponential $x[n] = e^{\mathbf{j}\omega_0 n}$, then

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{\mathbf{j}\omega_0(n-k)} = \underbrace{\left[\sum_{k=-\infty}^{\infty} h[k]e^{-\mathbf{j}\omega_0 k}\right]}_{\text{DTFT of } h \text{ at } \omega = \omega_0} e^{\mathbf{j}\omega_0 n}$$

If the input is a complex exp., then the output is ***also*** a complex exp. with the ***same frequency***, but ***scaled by the DTFT of*** h!

Frequency response of a LTI system

The DTFT of the impulse response is known as the system's ***frequency response***, and is denoted by $H(e^{\mathbf{j}\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-\mathbf{j}\omega n}$.

- write $H(e^{\mathbf{j}\omega}) \in \mathbb{C}$ in polar form as $H(e^{\mathbf{j}\omega}) = |H(e^{\mathbf{j}\omega})|e^{\mathbf{j}\angle H(e^{\mathbf{j}\omega})}$

$$x[n] = e^{\mathbf{j}\omega_0 n} \quad \Longrightarrow \quad y[n] = H(e^{\mathbf{j}\omega_0})e^{\mathbf{j}\omega_0 n} = |H(e^{\mathbf{j}\omega_0})|e^{\mathbf{j}(\angle H(e^{\mathbf{j}\omega_0}))}e^{\mathbf{j}\omega_0 n}$$

(i) ***"eigenfunction" property:*** if $x[n] = e^{\mathbf{j}\omega_0 n}$, then $y[n] \propto e^{\mathbf{j}\omega_0 n}$

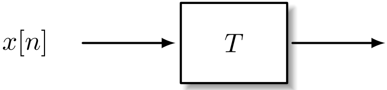
(ii) ***amplitude scaling:*** output amplitude is scaled by $|H(e^{\mathbf{j}\omega_0})|$

(iii) ***phase shifting:*** output phase is shifted by $\angle H(e^{\mathbf{j}\omega_0})$

A LTI system amplitude-scales and phase-shifts any complex exp. input!

Response of a LTI system and the DTFT

These ideas generalize to more general input signals



$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k]$$

- **recall:** convolution in time $\iff$ multiplication in frequency

$$y[n] = (h * x)[n] \qquad \iff \qquad Y(e^{\mathbf{j}\omega}) = H(e^{\mathbf{j}\omega})X(e^{\mathbf{j}\omega})$$

In the frequency domain, the output spectrum $Y(e^{\mathbf{j}\omega})$ is given by the ***product*** of the frequency response $H(e^{\mathbf{j}\omega})$ and the input spectrum $X(e^{\mathbf{j}\omega})$. Much simpler than convolution!

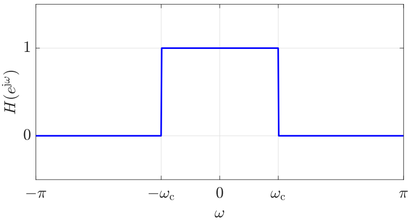
Proof of convolution $\iff$ multiplication

$$
\begin{aligned}
Y(e^{\mathbf{j}\omega}) &= \sum_{n=-\infty}^{\infty} y[n]e^{-\mathbf{j}\omega n} && \text{(def. of DTFT for } y\text{)} \\
&= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} h[n-k]x[k]\right] e^{-\mathbf{j}\omega n} && \text{(convolution)} \\
&= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h[n-k]x[k]e^{-\mathbf{j}\omega n} && \text{(change order of summation)} \\
&= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} h[n-k]e^{-\mathbf{j}\omega n} && \text{(group terms)} \\
&= \sum_{k=-\infty}^{\infty} x[k] \left[\sum_{\ell=-\infty}^{\infty} h[\ell]e^{-\mathbf{j}\omega(k+\ell)}\right] && (\ell = n-k) \\
&= \left[\sum_{\ell=-\infty}^{\infty} h[\ell]e^{-\mathbf{j}\omega\ell}\right] \left[\sum_{k=-\infty}^{\infty} x[k]e^{-\mathbf{j}\omega k}\right] && \text{(rearrange)} \\
&= H(e^{\mathbf{j}\omega})X(e^{\mathbf{j}\omega}) && \text{(def. of DTFT)}
\end{aligned}
$$

The ideal discrete-time low-pass filter

For $0 < \omega_c < \pi$ consider

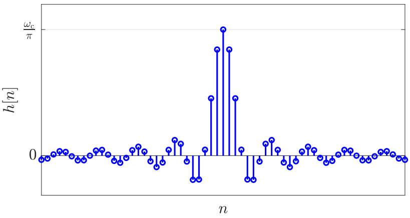
$$H(e^{\mathbf{j}\omega}) = \begin{cases} 1 & \text{if } -\omega_c \leq \omega < \omega_c \\ 0 & \text{otherwise} \end{cases}$$



This system perfectly passes frequencies $|\omega| < \omega_c$.

The impulse response is

$$h[n] = \begin{cases} \frac{\sin(\omega_c n)}{\pi n}, & n \neq 0 \\ \frac{\omega_c}{\pi}, & n = 0. \end{cases}$$

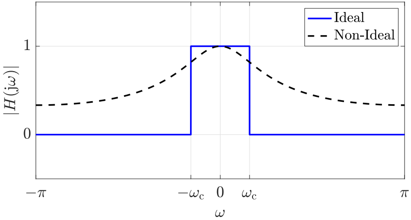


Some non-ideal discrete-time low-pass filters

- the filter is ***not causal*** since $h[n] \neq 0$ for all $n < 0$. We can come up with some causal alternatives.

Basic causal low-pass filter

$$L(e^{\mathbf{j}\omega}) = \frac{1 - \omega_c}{1 - \omega_c e^{-\mathbf{j}\omega}}$$



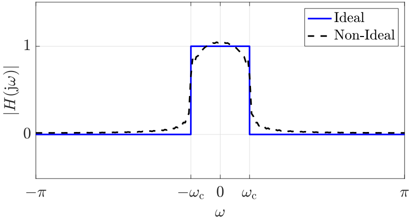
A better option is to

(i) take the ideal $h[n]$

(ii) time shift it to the right

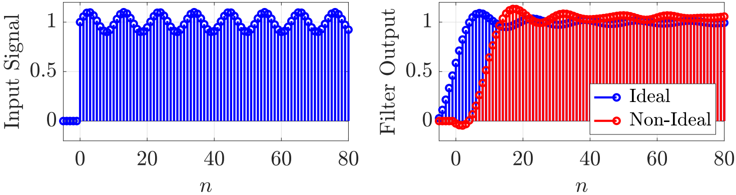
(iii) multiply by $u[n]$

(iv) compute its DTFT



Comparison of ideal and non-ideal filters

- for $\omega_c = 0.4$ rad/sample, let's consider the input $x[n] = (1 + 0.1\sin(0.6n))u[n]$

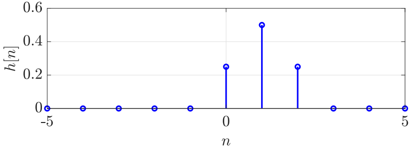


- the ideal filter *anticipates* the input change
- the non-ideal filter reacts only *after* the input change (and in this case, initially reacts in the "wrong" direction)
- both filters do a good job of removing the oscillatory component

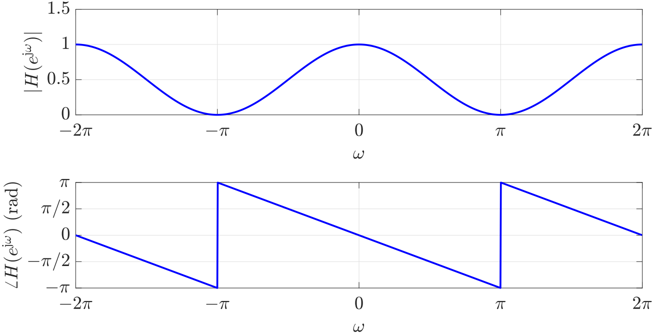
Example: three-point moving average as a FIR LPF

Consider the impulse response

$$h[n] = \frac{1}{4}\delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2]$$



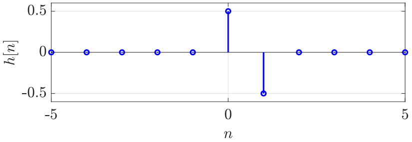
- in Chapter 4 we computed that $H(e^{\mathbf{j}\omega}) = \frac{1}{4}e^{-\mathbf{j}\omega}(2 + 2\cos(\omega))$



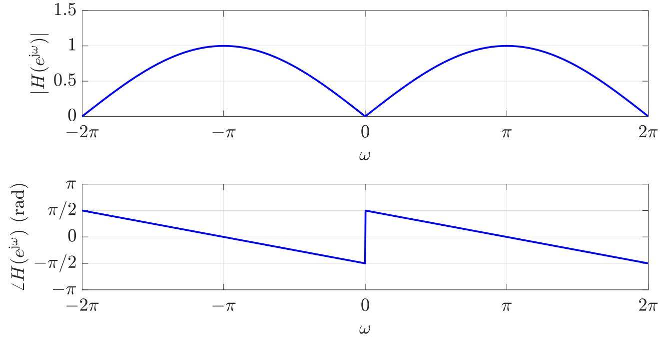
Example: difference system as a FIR HPF

Consider the impulse response

$$h[n] = \frac{1}{2}(x[n] - x[n-1])$$



- **try it yourself:** the frequency resp. is $H(e^{\mathrm{j}\omega}) = e^{-\mathrm{j}(\frac{\omega}{2} - \frac{\pi}{2})} \sin(\omega/2)$



Laplace transform analysis of DT LTI systems

- in Chap. 8 we used the Laplace transform to analyze ***causal CT*** LTI systems with right-sided (from time 0) inputs
- there is a directly analogous tool called the ***z-transform*** for ***causal DT*** LTI systems with right-sided inputs
- for a DT signal x right-sided from time 0, its z-transform is defined as

$$X : \mathcal{R}_x \to \mathbb{C}, \qquad X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

where $\mathcal{R}_x$ is a region of convergence

> The z-transform can actually be understood using what we know about the Laplace transform and sampling.

Laplace transform analysis of DT LTI systems

- given a right-sided DT signal x, choose any sampling period $T_s > 0$ and construct the fictitious continuous-time signal

$$x_{\mathrm{ct}}(t) = \sum_{n=0}^{\infty} x[n]\delta(t - nT_{\mathrm{s}})$$

- the Laplace transform of x_{ct} is

$$\begin{aligned} X_{\mathrm{ct}}(s) &= \int_0^{\infty} \left(\sum_{n=0}^{\infty} x[n]\delta(t - nT_{\mathrm{s}}) \right) e^{-st}\,\mathrm{d}t \\ &= \sum_{n=0}^{\infty} x[n] \int_0^{\infty} \delta(t - nT_{\mathrm{s}}) e^{-st}\,\mathrm{d}t \\ &= \sum_{n=0}^{\infty} x[n] (\underbrace{e^{sT_{\mathrm{s}}}}_{\text{“}z\text{”}})^{-n} = X(z) \end{aligned}$$

From this observation, one can begin to translate all our CT system results over to DT systems.

Laplace transform analysis of DT LTI systems

The following results can all be established.

- the transfer function $H(z)$ of a causal DT LTI system is $H(z) = \sum_{n=0}^{\infty} h[n]z^{-n}$, the z-transform of the impulse response
- if the RoC of $H(z)$ contains the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$, then the freq. response $H(e^{\mathbf{j}\omega})$ can be obtained by substituting $z = e^{\mathbf{j}\omega}$ into $H(z)$
- the output of a causal DT LTI system can be computed in the z-domain via $Y(z) = H(z)X(z)$
- the z-transform can be used to easily determine transfer functions of systems defined by LICC-DEs

Relevant MATLAB commands

- computing a DT convolution

```
%% bank model example
%% define time
N_max = 100;
n = -N_max:1:N_max;

%% define impulse response and input
r = 0.05/12;
h = (1+r).^n.*heaviside(n); h(n==0) = 1;
x = 5*heaviside(n); x(n==0) = 5;

%% compute convolution
y = conv(x,h,'same');
```

Supplementary reading

- **O-W:** A. V. Oppenheim and S. Willsky, *Signals and Systems*, 2nd Ed.
- **BB:** B. Boulet, *Fundamentals of Signals and Systems*.
- **BPL:** B. P. Lathi, *Signal Processing and Linear Systems*.
- **K-S:** H. Kwakernaak and R. Sivan, *Modern Signals and Systems*.
- **EL-PV:** E. Lee and P. Varaiya, *Structure and Interpretation of Signals and Systems*, 2nd Ed.
- **ADL:** A. D. Lewis, *The Mathematical Theory of Signals and Systems*.

Topic	**O-W**	**BB**	**BPL**	**K-S**	**EL-PV**	**ADL**
Definition of a system	1.5	1	1.6	3.1, 3.2	1.2, 2.3	
System properties	1.6	1	1.7	3.1, 3.2, 3.6		
LTI systems, impulse response	2.1, 2.3	2	9.1–9.4	3.3, 3.4		
Convolution	2.1	2	9.3	3.5	9.1	
System interconnections	10.8	1		3.9		
Difference equations	2.4	3	9.5	4.1–4.6	13.7	

Personal Notes

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10. Appendix: More on the DT Fourier Series and DT Fourier Transform

- introduction
- more on the discrete-time Fourier series (DTFS)
- DTFS as a numerical approximation of the CTFS
- more on the DT Fourier transform (DTFT)
- DTFT as the CTFT of a sampled CT signal

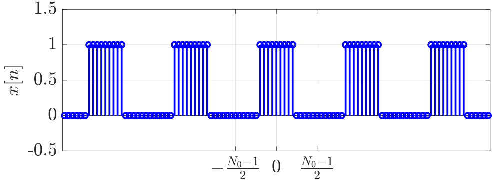
Introduction

The CTFS, DTFS, CTFT, and DTFT share many similarities and properties; once one understands the CTFS/CTFT, the corresponing DT methods are typically easy to pick up and use.

- this appendix contains additional information regarding the DT Fourier Series and DT Fourier Transform
- the material has been placed in this appendix not because it is unimportant, but to minimize monotonous repetition in the lectures
- refer to this appendix as needed for additional exposition

The discrete-time Fourier series (DTFS)

- let x be a DT periodic signal with fundamental period N_0



Theorem 10.1. The ***discrete-time Fourier series*** of x is

$$x[n] = \sum_{k=0}^{N_0-1} \alpha_k e^{\mathbf{j}k\omega_0 n}, \qquad \alpha_k = \frac{1}{N_0} \sum_{l=0}^{N_0-1} x[l] e^{-\mathbf{j}k\omega_0 l}.$$

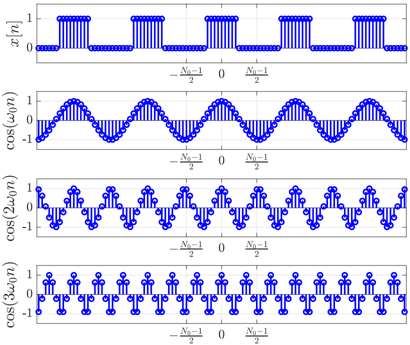
- we will now derive these formulas, mirroring the CT derivation

Derivation of the DTFS via approximation

- for some positive integer K, consider the ***order K approximation***

$$\hat{x}_K[n] = \sum_{k=0}^{K-1} \alpha_k e^{\mathbf{j}k\omega_0 n}, \qquad \omega_0 = \frac{2\pi}{N_0},$$

where we try to approximate x via a sum of DT complex exponentials



note: all the exponentials of different frequencies

$$0, \omega_0, 2\omega_0, 3\omega_0, \ldots, (N_0-1)\omega_0$$

are periodic and fit ***perfectly*** within our fundamental period N_0 of x

Derivation of the DTFS via approximation

- since $e^{\mathbf{j}k\omega_0 n}$ is *periodic in frequency*, the two signals $e^{\mathbf{j}k\omega_0 n}$ and $e^{\mathbf{j}(k+N_0)\omega_0 n}$ are actually the same signal:

$$e^{\mathbf{j}(k+N_0)\omega_0 n} = e^{\mathbf{j}k\omega_0 n}e^{\mathbf{j}N_0\omega_0 n} = e^{\mathbf{j}k\omega_0 n}e^{\mathbf{j}2\pi n} = e^{\mathbf{j}k\omega_0 n}.$$

Therefore, we restrict ourselves to $K \leq N_0$, since adding additional terms to the sum will not give us any higher-frequency exponentials.

- approximation error quantified via ***mean-square error***

$$J(\alpha_0, \ldots, \alpha_{K-1}) = \frac{1}{N_0}\sum_{n=0}^{N_0-1} |x[n] - \hat{x}_K[n]|^2.$$

Our goal: find the choice of constants $\{\alpha_k\}_{k=0}^{K-1}$ which ***minimizes*** J.

Orthogonality of complex exponentials

- DT exponentials satisfy the following orthogonality relationship

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1} e^{\mathbf{j}m\omega_0 n} e^{-\mathbf{j}\ell\omega_0 n} = \begin{cases} 1 & \text{if } m = \ell + kN_0,\ k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

For $m = \ell + kN_0$, we have $\frac{1}{N_0}\sum_{n=0}^{N_0-1} e^{\mathbf{j}(m-\ell)\omega_0 n} = \frac{1}{N_0}\sum_{n=0}^{N_0-1} e^{\mathbf{j}kN_0\frac{2\pi}{N_0}n} = 1$.
For the other case, we have

$$\begin{aligned}
\frac{1}{N_0}\sum_{n=0}^{N_0-1} e^{\mathbf{j}(m-\ell)\omega_0 n} &= \frac{1}{N_0}\sum_{n=0}^{N_0-1} (e^{\mathbf{j}(m-\ell)\omega_0})^n \\
&= \frac{1}{N_0}\frac{1 - e^{\mathbf{j}(m-\ell)\omega_0 N_0}}{1 - e^{\mathbf{j}(m-\ell)\omega_0}} \\
&= \frac{1}{N_0}\frac{1 - e^{\mathbf{j}(m-\ell)2\pi}}{1 - e^{\mathbf{j}(m-\ell)\omega_0}} = \frac{1}{N_0}\frac{1-1}{1 - e^{\mathbf{j}(m-\ell)\omega_0}} = 0
\end{aligned}$$

where we used the geometric series formula and $e^{\mathbf{j}2\pi n} = 1$ for all integers n.

Optimal selection of coefficients

Theorem 10.2 (Optimal Coefficients). The selection of coefficients $\{\alpha_k\}_{k=0}^{K-1}$ which minimizes the mean-squared error J is

$$\alpha_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-\mathbf{j}k\omega_0 n}$$

where $\omega_0 = 2\pi/N_0$.

Proof: The function of interest is

$$J = \frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n] - \hat{x}_K[n]|^2 = \frac{1}{N_0} \sum_{n=0}^{N_0-1} (x[n] - \hat{x}_K[n])^*(x[n] - \hat{x}_K[n])$$

where we used that $|z|^2 = z^* z$. Expanding out, we have

$$J = \frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2 - x[n]^* \hat{x}_K[n] - \hat{x}_K[n]^* x[n] + |\hat{x}_K[n]|^2$$

Optimal selection of coefficients

We compute that

$$|\hat{x}_K[n]|^2 = \left(\sum\nolimits_{\ell=0}^{K-1} \alpha_\ell e^{\mathbf{j}\ell\omega_0 n}\right)^* \left(\sum\nolimits_{m=0}^{K-1} \alpha_m e^{\mathbf{j}m\omega_0 n}\right)$$
$$= \sum_{\ell=0}^{K-1}\sum_{m=0}^{K-1} \alpha_\ell^* \alpha_m e^{\mathbf{j}(m-\ell)\omega_0 n}.$$

Substituting into J, we can write things out as

$$J = \frac{1}{N_0}\sum_{n=0}^{N_0-1}\left[|x[n]|^2 - x[n]^* \sum_{m=0}^{K-1}\alpha_m e^{\mathbf{j}m\omega_0 n} - x[n]\sum_{m=0}^{K-1}\alpha_m^* e^{-\mathbf{j}m\omega_0 n} + \sum_{\ell=0}^{K-1}\sum_{m=0}^{K-1}\alpha_\ell^*\alpha_m e^{\mathbf{j}(m-\ell)\omega_0 n}\right]$$

If we define $\beta_m = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-\mathbf{j}m\omega_0 n}$ then we can more simply write this as

$$J = \left[\frac{1}{N_0}\sum_{n=0}^{N_0-1}|x[n]|^2\right] - \sum_{m=0}^{K-1}(\alpha_m\beta_m^* + \alpha_m^*\beta_m) + \sum_{\ell=0}^{K-1}\sum_{m=0}^{K-1}\alpha_\ell^*\alpha_m \underbrace{\frac{1}{N_0}\sum_{n=0}^{N_0-1} e^{\mathbf{j}(m-\ell)\omega_0 n}}_{=1 \text{ iff } m=\ell}$$

Optimal selection of coefficients

We therefore have that

$$J = \left[\frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2 \right] - \sum_{m=0}^{K-1} (\alpha_m \beta_m^* + \alpha_m^* \beta_m) + \sum_{m=0}^{K-1} |\alpha_m|^2$$

$$= \left[\frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2 \right] + \sum_{m=0}^{K-1} (-\alpha_m \beta_m^* - \alpha_m^* \beta_m + |\alpha_m|^2)$$

If we add and subtract $|\beta_m|^2$ inside the sum, we can complete the square:

$$J = \left[\frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2 \right] + \sum_{m=0}^{K-1} (|\beta_m|^2 - \alpha_m \beta_m^* - \alpha_m^* \beta_m + |\alpha_m|^2) - \sum_{m=0}^{K-1} |\beta_m|^2$$

$$= \left[\frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2 \right] + \sum_{m=0}^{K-1} (\beta_m - \alpha_m)^* (\beta_m - \alpha_m) - \sum_{m=0}^{K-1} |\beta_m|^2$$

The first and third terms do not depend at all on α! Therefore, the best thing we can do to minimize J is to make the middle term zero. We therefore find that $\alpha_k = \beta_k$, which completes the proof. •

Summary of results

The order K approximation $\hat{x}_K$ of an N_0-periodic DT signal x is

$$\hat{x}_K[n] = \sum_{k=0}^{K-1} \alpha_k e^{\mathbf{j}k\omega_0 n}, \quad \alpha_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-\mathbf{j}k\omega_0 n}.$$

- roughly speaking, the magnitude of α_k tells us how strongly the frequency $\omega_k = k\omega_0$ is present in the overall signal x
- the $k = 0$ term in $\hat{x}_K[n]$ is constant; this is called the "dc" term
- we limit ourselves to $K \leq N_0$, since we know there are only N_0 distinct complex exponentials which are periodic with period N_0
- to compute α_k, you can sum over ***any*** interval of length N_0, i.e., if convenient you can instead use the sum $\sum_{n=\eta}^{\eta+N_0-1}$ for any $\eta \in \mathbb{Z}$.

Convergence of the approximation

- in *continuous-time*, we had our approximation $\hat{x}_K(t)$ of a periodic signal $x(t)$, and we saw that the ***energy*** in the error signal $\hat{x}_K(t) - x(t)$ tended to zero as $K \to \infty$; we needed an ***infinite*** linear combination of complex exponential functions to achieve zero error

- in discrete-time, we know that there is no point in considering $K > N_0$, since there are only a finite number of distinct periodic complex exponential functions with period N_0.

- a ***major*** consequence of this fact is that our approximation $\hat{x}_K[n]$ of a DT periodic signal becomes ***exact*** if $K = N_0$ (much simpler than CT signals!)

Theorem 10.3. Let x be a periodic DT signal with period $N_0 \in \mathbb{Z}_{\geq 1}$, and let $\hat{x}_{N_0}$ be our approximation where we keep N_0 terms. Then $\hat{x}_{N_0}[n] = x[n]$ for all $n \in \mathbb{Z}$.

Convergence of the approximation

Proof: The optimal coefficients are given by

$$\alpha_k = \frac{1}{N_0} \sum_{l=0}^{N_0-1} x[l] e^{-\mathbf{j}k\omega_0 l}$$

With $K = N_0$, our MSE approximation $\hat{x}_K$ is

$$\hat{x}_{N_0}[n] = \sum_{k=0}^{N_0-1} \alpha_k e^{\mathbf{j}k\omega_0 n} = \sum_{k=0}^{N_0-1} \left[\frac{1}{N_0} \sum_{l=0}^{N_0-1} x[l] e^{-\mathbf{j}k\omega_0 l} \right] e^{\mathbf{j}k\omega_0 n}$$

$$= \frac{1}{N_0} \sum_{l=0}^{N_0-1} x[l] \left[\sum_{k=0}^{N_0-1} e^{\mathbf{j}k\omega_0 (n-l)} \right]$$

We previously determined though that

$$\frac{1}{N_0} \sum_{k=0}^{N_0-1} e^{\mathbf{j}k\omega_0 (n-l)} = \begin{cases} 1 & \text{if } n = l \\ 0 & \text{if } n \neq l \end{cases}$$

and therefore

$$\hat{x}_{N_0}[n] = x[n] \qquad \text{(approximation is exact at all times)}$$

Matrix-vector notation for DTFS

- our formula for the DTFS coefficients α_k is

$$\alpha_k = \frac{1}{N_0}\sum_{l=0}^{N_0-1} x[l]e^{-\mathbf{j}k\omega_0 l}, \qquad k \in \{0,\ldots,N_0-1\}.$$

- we can write these equations together in matrix-vector notation as

$$\underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N_0-1} \end{bmatrix}}_{:=\boldsymbol{\alpha}} = \frac{1}{N_0}\underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-\mathbf{j}(1)\omega_0(1)} & e^{-\mathbf{j}(1)\omega_0(2)} & \cdots & e^{-\mathbf{j}(1)\omega_0(N_0-1)} \\ 1 & e^{-\mathbf{j}(2)\omega_0(1)} & e^{-\mathbf{j}(2)\omega_0(2)} & \cdots & e^{-\mathbf{j}(2)\omega_0(N_0-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-\mathbf{j}(N_0-1)\omega_0(1)} & e^{-\mathbf{j}(2)\omega_0(N_0-1)} & \cdots & e^{-\mathbf{j}(N_0-1)\omega_0(N_0-1)} \end{bmatrix}}_{:=\boldsymbol{H}}\underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N_0-1] \end{bmatrix}}_{:=\boldsymbol{x}}$$

which gives us the very simple compact formula $\boldsymbol{\alpha} = \frac{1}{N_0}\boldsymbol{H}\boldsymbol{x}$

- $\boldsymbol{H}$ is symmetric, has complex entries, and has elements $\boldsymbol{H}_{kn} = e^{-\mathbf{j}k\omega_0 n}$
- $\boldsymbol{H}$ is known as the ***discrete Fourier transform*** matrix; as you can see it is quite structured, which makes it very amenable to fast algorithms for computing $\boldsymbol{\alpha}$.

Matrix-vector notation for DTFS

- our formula for $\hat{x}_{N_0}$ in terms of α_k is

$$\hat{x}_{N_0}[n] = \sum_{k=0}^{N_0-1} \alpha_k e^{\mathbf{j}k\omega_0 n}, \qquad n \in \{0, \ldots, N_0 - 1\}.$$

- we can write these equations together in matrix-vector notation as

$$\underbrace{\begin{bmatrix} \hat{x}_{N_0}[0] \\ \hat{x}_{N_0}[1] \\ \hat{x}_{N_0}[2] \\ \vdots \\ \hat{x}_{N_0}[N_0 - 1] \end{bmatrix}}_{:=\hat{\boldsymbol{x}}_{\boldsymbol{N_0}}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{\mathbf{j}\omega_0} & e^{\mathbf{j}\omega_0 2} & \cdots & e^{\mathbf{j}\omega_0 (N_0-1)} \\ 1 & e^{\mathbf{j}\omega_0 2} & e^{\mathbf{j}\omega_0 4} & \cdots & e^{\mathbf{j}\omega_0 2(N_0-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{\mathbf{j}\omega_0 (N_0-1)} & e^{\mathbf{j}\omega_0 (N_0-1)2} & \cdots & e^{\mathbf{j}\omega_0 (N_0-1)(N_0-1)} \end{bmatrix}}_{:=\boldsymbol{G}} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N_0-1} \end{bmatrix}}_{\boldsymbol{\alpha}}$$

which gives us the very simple compact formula $\hat{\boldsymbol{x}}_{\boldsymbol{N_0}} = \boldsymbol{G}\boldsymbol{\alpha}$

- **note:** the matrix $\boldsymbol{G}$ is symmetric and has complex elements $\boldsymbol{G}_{nk} = e^{\mathbf{j}k\omega_0 n}$

Matrix-vector notation for DTFS

- combining the formulas $\boldsymbol{\alpha} = \frac{1}{N_0}\boldsymbol{H}\boldsymbol{x}$ and $\hat{\boldsymbol{x}}_{\boldsymbol{N_0}} = \frac{1}{N_0}\boldsymbol{G}\boldsymbol{\alpha}$ we find that

$$\hat{\boldsymbol{x}}_{\boldsymbol{N_0}} = \frac{1}{N_0}\boldsymbol{G}\boldsymbol{H}\boldsymbol{x}$$

- what is the product $\frac{1}{N_0}\boldsymbol{G}\boldsymbol{H}$? We can compute its elements:

$$\begin{aligned}
\frac{1}{N_0}(\boldsymbol{G}\boldsymbol{H})_{nm} = \sum_{k=0}^{N_0-1} G_{nk}H_{km} &= \frac{1}{N_0}\sum_{k=0}^{N_0-1} e^{\mathbf{j}k\omega_0 n}e^{-\mathbf{j}k\omega_0 m} \\
&= \frac{1}{N_0}\sum_{k=0}^{N_0-1} e^{\mathbf{j}k\omega_0(n-m)} \\
&= \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

- therefore, $\frac{1}{N_0}\boldsymbol{G}\boldsymbol{H}$ is the ***identity matrix***, and we conclude that $\hat{\boldsymbol{x}}_{\boldsymbol{N_0}} = \boldsymbol{x}$. This is again the statement that the approximation is perfect!

Numerically approximating the CTFS coefficients

Recall our result for ***continuous-time*** Fourier series:

Let x_{ct} be a periodic CT signal with fundamental period T_0. Then

$$\hat{x}_\infty(t) = \lim_{K \to \infty} \hat{x}_K(t) = \sum\nolimits_{k=-\infty}^{\infty} \alpha_{\mathrm{ct},k} e^{\mathbf{j}k\omega_0 t}$$

is called the ***continuous-time Fourier series (CTFS)*** of x, where

$$\alpha_{\mathrm{ct},k} = \frac{1}{T_0} \int_0^{T_0} x_{\mathrm{ct}}(t) e^{-\mathbf{j}k\omega_0 t}\, \mathrm{d}t, \qquad \omega_0 = \frac{2\pi}{T_0}.$$

- the integral for computing $\alpha_{\mathrm{ct},k}$ can be difficult (or impossible) to compute analytically; how could we numerically approximate it? Here is one approach.

Numerically approximating the CTFS coefficients

- first, we ***sample*** the signal x_{ct} with sampling period T_{s}:

$$x[n] = x_{\text{ct}}(t)\Big|_{t=nT_{\text{s}}} = x_{\text{ct}}(nT_{\text{s}})$$

- to ensure that $x[n]$ is periodic, we need to sample an ***integer*** number of times per period T_0. If we want N_0 samples per period, we choose T_{s} such that $N_0 = T_0/T_{\text{s}}$, i.e., $T_{\text{s}} = T_0/N_0$

- we can ***approximate*** the value of $\alpha_{\text{ct},k}$ as

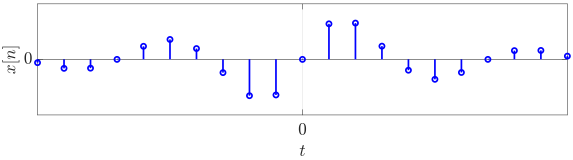
$$\begin{aligned}
\alpha_{\text{ct},k} = \frac{1}{T_0}\int_0^{T_0} x_{\text{ct}}(t)e^{-\mathbf{j}k\omega_0 t}\,\mathrm{d}t &\approx \frac{1}{T_0}\sum_{n=0}^{N_0-1} x_{\text{ct}}(nT_{\text{s}})\cdot T_{\text{s}}\cdot e^{-\mathbf{j}k\omega_0(nT_{\text{s}})} \\
&= \frac{T_{\text{s}}}{T_0}\sum_{n=0}^{N_0-1} x[n]e^{-\mathbf{j}k\frac{2\pi}{T_0}(nT_{\text{s}})} \\
&= \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-\mathbf{j}k\frac{2\pi}{N_0}n} = \alpha_k \quad \text{(DTFS coeff.!!)}
\end{aligned}$$

Numerically approximating the CTFS coefficients

- the DTFS coefficients $\{\alpha_k\}_{k=0}^{N_0-1}$ of the sampled signal $x[n]$ can be used to ***numerically approximate*** the CTFS coefficients $\{\alpha_{\mathrm{ct},k}\}_{k=-\infty}^{\infty}$ of the continuous-time signal $x_{\mathrm{ct}}(t)$
- the quality of the approximation depends on the number of samples per period N_0; as N_0 increases, the approximation will become better, because we will be using more DTFS coefficients
- since the DTFS coefficients can be computed very efficiently via numerical linear algebra (specifically, using the FFT algorithm), this leads to efficient methods for approximating the CTFS coefficients

Derivation of the DT Fourier Transform

- suppose we have a general DT signal x, e.g.,



Key idea: an *aperiodic* signal is a *periodic* signal with infinite period . . .

Steps we will take:

(i) window the aperiodic signal to $[-N, N]$, then periodize it

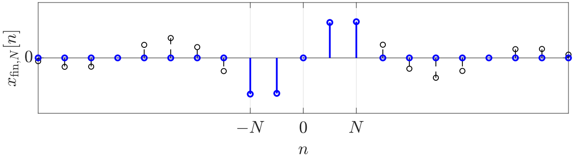
(ii) compute the DTFS of the periodized signal

(iii) take the limit as $N \to \infty$

Derivation of the DT Fourier Transform

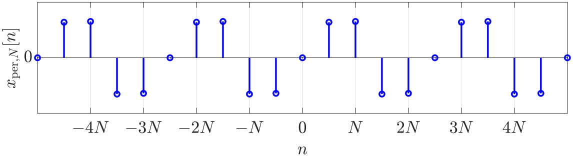
- we begin by ***windowing*** x to obtain a finite-duration signal

$$x_{\text{fin},N}[n] = x[n] \cdot (u[n+N] - u[t-(N+1)])$$



- we can now ***periodize*** $x_{\text{fin},N}$ to obtain the $2N+1$-periodic signal

$$x_{\text{per},N}[n] = \sum_{m=-\infty}^{\infty} x_{\text{fin},N}(n - m(2N+1))$$



Derivation of the DT Fourier Transform

- since $x_{\mathrm{per},N}$ is $2N+1$-periodic, we can ***represent it via the DTFS***

$$x_{\mathrm{per},N}[n] = \sum\nolimits_{k=-N}^{N} \alpha_k e^{\mathbf{j}k\omega_0 n}$$

where $\omega_0 = (2\pi)/(2N+1)$ is the fundamental ang. frequency

- the corresponding DTFS coefficients α_k are given by

$$\alpha_k = \frac{1}{2N+1} \sum_{n=-N}^{N} x_{\mathrm{per},N}[n] e^{-\mathbf{j}k\omega_0 n}$$

- however, over the range $[-N, N]$, we have $x_{\mathrm{per},N}[n] = x[n]$, so

$$\alpha_k = \frac{1}{2N+1} \sum_{n=-N}^{N} x[n] e^{-\mathbf{j}k\omega_0 n}$$

Derivation of the DT Fourier Transform

- as some simplifying notation, if we define the function

$$X : \mathbb{R} \to \mathbb{C}, \qquad X(e^{\mathbf{j}\omega}) = \sum_{n=-N}^{N} x[n] e^{-\mathbf{j}\omega n}$$

then the DTFS coefficients are simply ***samples*** of X

$$\alpha_k = \frac{1}{2N+1} X(e^{\mathbf{j}k\omega_0}), \qquad k \in \{-N, \ldots, N\}$$

- plugging this back into the DTFS, we find that

$$x_{\text{per},N}[n] = \sum_{k=-N}^{N} \frac{1}{2N+1} X(e^{\mathbf{j}k\omega_0}) e^{\mathbf{j}k\omega_0 n}$$

and substituting $\omega_0 = \frac{2\pi}{2N+1}$ we obtain

$$x_{\text{per},N}[n] = \frac{1}{2\pi} \sum_{k=-N}^{N} \frac{2\pi}{2N+1} X(e^{\mathbf{j}k\frac{2\pi}{2N+1}}) e^{\mathbf{j}k\frac{2\pi}{2N+1}n}$$

Derivation of the DT Fourier Transform

Our Fourier Series: $$x_{\text{per},N}[n] = \frac{1}{2\pi}\sum_{k=-N}^{N}\frac{2\pi}{2N+1}X(e^{\mathbf{j}k\frac{2\pi}{2N+1}})e^{\mathbf{j}k\frac{2\pi}{2N+1}n}$$

Recall that the integral $\int_a^b f(\omega)\,\mathrm{d}\omega$ is defined as the limit of a *Riemann sum*, where one splits the interval $[a,b]$ into M intervals of width $\Delta\omega = (b-a)/M$ and considers

$$\int_a^b f(\omega)\,\mathrm{d}\omega = \lim_{M\to\infty}\sum_{k=1}^{M}\Delta\omega\cdot f(k\cdot\Delta\omega) = \lim_{M\to\infty}\sum_{k=1}^{M}\tfrac{b-a}{M}f(k\cdot\tfrac{b-a}{M})$$

Comparing, we may set

$$a=-\pi,\quad b=\pi,\quad M=2N+1,\quad f(\omega)=X(e^{\mathbf{j}\omega})e^{\mathbf{j}\omega n}$$

Derivation of the DT Fourier Transform

- as $N \to \infty$, the DTFS sum becomes the ***integral***

$$\lim_{N\to\infty} x_{\mathrm{per},N}[n] = x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{\mathbf{j}\omega})e^{\mathbf{j}\omega n}\,\mathrm{d}\omega$$

- similarly, as $N \to \infty$ the function X becomes

$$X(e^{\mathbf{j}\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-\mathbf{j}\omega n}$$

These last two formulas extend the DTFS to aperiodic signals!

The DT Fourier Transform (DTFT)

Definition 10.1. The ***discrete-time Fourier transform (DTFT)*** of a DT signal x is the function $X : \mathbb{R} \to \mathbb{C}$ defined pointwise by

$$X(e^{\mathbf{j}\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-\mathbf{j}\omega n}.$$

We call X the ***Fourier transform*** or ***spectrum*** of x.

- we think of X (when it exists) as providing a ***frequency domain*** representation of the signal x
- note that X is a function of the ***continuous*** frequency variable $\omega \in \mathbb{R}$

Existence of the DTFT

Theorem 10.4. If x has finite action, i.e., $x \in \ell_1$, then the CTFT $X(e^{\mathbf{j}\omega})$ is well-defined.

Proof: Since x has finite action, we can bound the spectrum as

$$|X(e^{\mathbf{j}\omega})| = \left| \sum_{n=-\infty}^{\infty} x[n]e^{-\mathbf{j}\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |e^{-\mathbf{j}\omega n}||x[n]|$$

$$= \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

so $X(e^{\mathbf{j}\omega})$ is well-defined for all $\omega \in \mathbb{R}$.

- there are lots of DT signals which do not satisfy this condition, but nonetheless have well-defined DTFTs
- other signals (e.g., $x[n] = 2^n u[n]$) simply ***do not*** have a DTFT

The inverse discrete-time Fourier transform

Definition 10.2. The ***inverse discrete-time Fourier transform (inverse DTFT)*** of a DT spectrum X is the DT signal $x : \mathbb{Z} \to \mathbb{C}$ defined pointwise by

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{\mathbf{j}\omega}) e^{\mathbf{j}\omega n} \, \mathrm{d}\omega.$$

- the integral can be taken over an interval of length 2π; this is because the spectrum X is always 2π-periodic in ω
- as the name suggests, the inverse DTFT is the inverse of the DTFT, meaning that if we start with x, apply the DTFT, and then apply the inverse DTFT, we recover the original signal x

Proof of inverse relationship

Let x be a DT signal and assume that the DTFT $X(e^{\mathbf{j}\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-\mathbf{j}\omega n}$ is well-defined. We compute the inverse DTFT of X to be

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{\mathbf{j}\omega})e^{\mathbf{j}\omega n}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{k=-\infty}^{\infty} x[k]e^{-\mathbf{j}\omega k}\right]e^{\mathbf{j}\omega n}\,\mathrm{d}\omega$$

$$= \sum_{k=-\infty}^{\infty} x[k]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathbf{j}\omega(n-k)}\,\mathrm{d}\omega\right]$$

Using orthogonality of CT complex exponentials, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathbf{j}\omega(n-k)}\,\mathrm{d}\omega = \delta[n-k] = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

Therefore, the above simplifies to

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n]$$

so we recover the original signal x.

The DTFT of a complex exponential signal

- consider the DT complex exponential $z[n] = e^{\mathbf{j}\nu n}$ where $-\pi \leq \nu \leq \pi$
- if we try to directly compute the DTFT of $z[n] = e^{\mathbf{j}\nu n}$ we have that

$$Z(e^{\mathbf{j}\omega}) = \sum_{n=-\infty}^{\infty} e^{\mathbf{j}\nu n} e^{-\mathbf{j}\omega n} = \sum_{n=-\infty}^{\infty} e^{\mathbf{j}(\nu-\omega)n}$$

  and it is not clear how to further evaluate this sum

- as an alternative, let's instead write out the inverse DTFT equation

$$e^{\mathbf{j}\nu n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} Z(e^{\mathbf{j}\omega}) e^{\mathbf{j}\omega n} \, \mathrm{d}\omega$$

  and try to guess what $Z(e^{\mathbf{j}\omega})$ must equal for this to be true.

- we can now guess $Z(e^{\mathbf{j}\omega}) = 2\pi\delta(\omega - \nu)$ for $-\pi \leq \omega \leq \pi$, where $\delta(\omega - \nu)$ is the *continuous-time* unit impulse function from Chapter 2.

The DTFT of a complex exponential signal

We can calculate that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} Z(e^{\mathbf{j}\omega})e^{\mathbf{j}\omega n}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega-\nu)e^{\mathbf{j}\omega n}\,\mathrm{d}\omega = e^{\mathbf{j}\omega\nu}$$

so our guess worked!

For $-\pi \leq \nu \leq \pi$, the DTFT of the DT signal $x[n] = e^{\mathbf{j}\nu n}$ is

$$X(e^{\mathbf{j}\omega}) = 2\pi\delta(\omega-\nu), \qquad -\pi \leq \omega \leq \pi.$$

The DTFT of a periodic signal

We can also apply the DTFT to periodic signals

- let x be a periodic DT signal with fundamental period N_0, and let $\omega_0 = 2\pi/N_0$ be the fundamental angular frequency
- we represent x using the discrete-time Fourier ***series***

$$x[n] = \sum_{k=0}^{N_0-1} \alpha_k e^{\mathbf{j}k\omega_0 n}.$$

- going term by term, the DTFT spectrum is (the periodic extension of)

$$X(e^{\mathbf{j}\omega}) = 2\pi \sum_{k=0}^{N_0-1} \alpha_k \delta(\omega - k\omega_0)$$

The DTFT of a periodic signal is ***finite sum*** of impulse functions located at the first N_0 multiples of the fundamental frequency ω_0!

Energy and the DTFT

- signals with finite energy are often the nicest case to consider

Theorem 10.5. If x has finite energy, i.e., $x \in \ell_2$, then

(i) X has finite energy, i.e., $X \in \mathsf{L}_2^{\text{per}}$, and

(ii) the signal and its spectrum satisfy ***Parseval's relation***

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{\mathbf{j}\omega})|^2 \, \mathrm{d}\omega$$

A beautiful and surprising relationship between the energy of the signal and the energy of its spectrum.

Converting between time and frequency domain

- there are useful patterns to recognize when converting between the time and frequency-domain representations of a signal
- **Example:** consider the DT spectrum defined by

$$X(e^{\mathbf{j}\omega}) = \sum_{k \in \mathcal{K}} \alpha_k e^{-\mathbf{j}\omega k}.$$

where $\mathcal{K}$ is some index set. This is a sum of complex exponentials, and we know from our example of the n_0-step delay that this corresponds to delays in the time-domain signal. Therefore

$$x[n] = \sum_{k \in \mathcal{K}} \alpha_k \delta[n-k]$$

Converting between time and frequency domain

- **Example:** Consider the DT signal $x[n] = \alpha^n u[n]$ where $|\alpha| < 1$. We compute the DTFT of x to be

$$X(e^{\mathbf{j}\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-\mathbf{j}\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-\mathbf{j}\omega})^n = \frac{1}{1 - \alpha e^{-\mathbf{j}\omega}}.$$

  Now suppose instead we are given the spectrum

$$X(e^{\mathbf{j}\omega}) = \sum_{k \in \mathcal{K}} \frac{b_k}{1 - \alpha_k e^{-\mathbf{j}\omega}}.$$

  Using the previous result term by term, the corresponding DT signal x must be given by

$$x[n] = \sum_{k \in \mathcal{K}} b_k \alpha_k^n u[n].$$

Converting between time and frequency domain

- **Example:** Consider the DT spectrum

$$H(e^{\mathbf{j}\omega}) = \frac{1}{(1-\alpha e^{-\mathbf{j}\omega})(1-\beta e^{-\mathbf{j}\omega})}$$

where $|\alpha| < 1$, $|\beta| < 1$, and $\alpha \neq \beta$. We split H into two terms using the method of ***partial fraction expansion***. Begin by writing

$$H(e^{\mathbf{j}\omega}) = \frac{a}{1-\alpha e^{-\mathbf{j}\omega}} + \frac{b}{1-\beta e^{-\mathbf{j}\omega}} = \frac{(a+b)-(a\beta+b\alpha)e^{-\mathbf{j}\omega}}{(1-\alpha e^{-\mathbf{j}\omega})(1-\beta e^{-\mathbf{j}\omega})}$$

for some constants a, b that we must solve for. Equating the two expressions for H, we find that $a+b=1$ and $a\beta+b\alpha=0$, from which it follows that $a=\alpha/(\alpha-\beta)$ and $b=\beta/(\beta-\alpha)$. Therefore

$$h[n] = \frac{\alpha}{\alpha-\beta}\alpha^n u[n] + \frac{\beta}{\beta-\alpha}\beta^n u[n].$$

Converting between time and frequency domain

- **Example:** Consider the DT spectra

$$H_1(e^{\mathbf{j}\omega}) = 1 + 2e^{-\mathbf{j}n_0\omega}, \qquad H_2(e^{\mathbf{j}\omega}) = \frac{1}{1-\alpha e^{-\mathbf{j}\omega}}.$$

Now multiply the two spectra to obtain

$$\begin{aligned} H(e^{\mathbf{j}\omega}) = H_1(e^{\mathbf{j}\omega})H_2(e^{\mathbf{j}\omega}) &= \frac{1+2e^{-\mathbf{j}n_0\omega}}{1-\alpha e^{-\mathbf{j}\omega}} \\ &= \frac{1}{1-\alpha e^{-\mathbf{j}\omega}} + 2\frac{1}{1-\alpha e^{-\mathbf{j}\omega}}e^{-\mathbf{j}n_0\omega} \end{aligned}$$

The first term we studied in the previous example. The second term is the same thing scaled by 2, and then multiplied by $e^{-\mathbf{j}n_0\omega}$ which we know arises from an n_0-step delay. Therefore

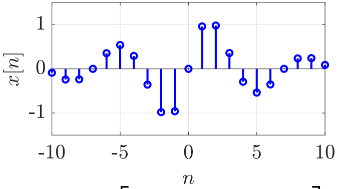
$$h[n] = \alpha^n u[n] + 2\alpha^{n-n_0}u[n-n_0].$$

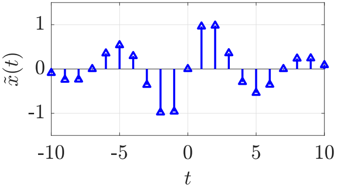
DTFT as the CTFT of a sampled CT signal

The DTFT of a DT signal x can also be obtained by applying the ***CTFT*** to the fictitious *continuous-time* signal

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-n)$$

which has impulses at all $n \in \mathbb{Z}$, weighted by the value $x[n]$.





$$\tilde{X}(\mathbf{j}\omega) = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x[n]\delta(t-n) \right] e^{-\mathbf{j}\omega t}\,\mathrm{d}t = \sum_{n=-\infty}^{\infty} x[n] e^{-\mathbf{j}\omega n} = X(e^{\mathbf{j}\omega})$$

11. Bonus: Vector Space Concepts in Signals & Systems

- introduction
- position, length, and angle in $\mathbb{R}^N$
- position, length, and angle in $\mathbb{C}^N$
- projections onto subspaces and change of basis
- change-of-basis interpretation of DTFS
- filtering as projection onto subspaces
- from filtering to (circular) convolution
- circulant matrices

Introduction

- this section contains material for students who wish to begin building a richer mathematical conception of signals and systems
- underlying the CTFS/DTFS/CTFT/DTFT, there are some fundamental ***unifying*** mathematical ideas from linear algebra
- understanding the deeper mathematical concepts helps to build intuition, and is particularly useful for more complex signal processing tasks such as image and video processing
- the key ideas are ***geometric***:
  - (i) signals are vectors in a ***vector space***
  - (ii) Fourier analysis is ***projection*** onto the basis of complex exp. signals
  - (iii) systems are ***transformations*** of these vectors

Definition of a vector space

Definition 11.1 (Vector space). A ***vector space*** over the set of ***scalars*** Λ is a set Σ of ***vectors*** equipped with the following two operations:

1. vector addition, which is a map $+ : \Sigma \times \Sigma \to \Sigma$ taking two vectors $x, y \in \Sigma$ and producing a new vector $x, y \in \Sigma$ s.t.
   - *commutativity:* $x + y = y + x$;
   - *associativity:* $x + (y + z) = (x + y) + z$;
   - *zero vector:* there exists an element $\mathbb{0} \in \Sigma$ such that $x + \mathbb{0} = x$;
   - *additive inverse:* for each $x \in \Sigma$ there is some $y \in \Sigma$ s.t. $x + y = \mathbb{0}$;
2. scalar multiplication, which is a map $\cdot : \Lambda \times \Sigma \to \Sigma$ taking a scalar $\alpha \in \Lambda$ and a vector $x \in \Sigma$ and producing a new vector $\alpha x \in \Sigma$ s.t.
   - *vector distributivity:* $\alpha(x + y) = \alpha x + \alpha y$;
   - *scalar distributivity:* $(\alpha_1 + \alpha_2)x = \alpha_1 x + \alpha_2 x$;
   - *multiplicative identity element:* $1x = x$.

The vector space $\mathbb{R}^n$ over $\mathbb{R}$

Put simply: a vector space is a set of objects which are **closed** under addition and scalar multiplication.

- real Euclidean space $\Sigma = \mathbb{R}^N$ is a vector space over the scalars $\Lambda = \mathbb{R}$
- we often notate vectors in $\mathbb{R}^N$ as n-tuples

$$x = (x[0], x[1], x[2], \ldots, x[N-1]), \qquad \text{each } x[n] \in \mathbb{R}$$

- addition and scalar multiplication defined as

$$x + y = (x[0] + y[0], x[1] + y[1], \ldots, x[N-1] + y[N-1])$$
$$\alpha x = (\alpha x[0], \alpha x[1], \alpha x[2], \ldots, \alpha x[N-1]), \qquad \alpha \in \mathbb{R}$$

- the zero element is

$$\mathbb{0} = (0, 0, \ldots, 0)$$

Norms on vector spaces

A vector space is just a set of objects we can add and scale. If we want to talk about ***length*** or ***size***, we need to add another ingredient called a norm.

Definition 11.2 (Norm). A ***norm*** on a vector space Σ over Λ is a map $\|\cdot\| : \Sigma \to \mathbb{R}$ satisfying

(i) *homogeneity:* $\|\alpha x\| = |\alpha|\,\|x\|$ for all $\alpha \in \Lambda$ and all $x \in \Sigma$;

(ii) *non-negativity:* $\|x\| \geq 0$ for all $x \in \Sigma$;

(iii) *non-degeneracy:* $\|x\| = 0$ if and only if $x = \mathbb{0}$, and

(iv) *triangle inequality:* $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \Sigma$.

- a norm allows us to measure the ***size*** of a vector $\|x\|$ and the ***distance*** between two vectors $\|x - y\|$

Example: the vector space $\mathbb{R}^N$ over $\mathbb{R}$

- on the space $\mathbb{R}^N$ we usually use the ***Euclidean*** norm

$$\|x\| = \sqrt{\sum_{n=0}^{N-1} (x[n])^2} \quad \text{or} \quad \|x\|^2 = \sum_{n=0}^{N-1} (x[n])^2$$

  which measures the length from the origin to the "tip" of the vector x.

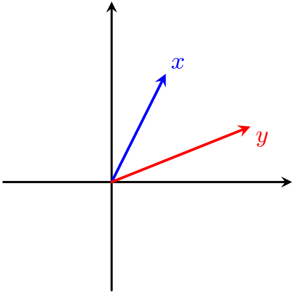
- the distance between two vectors x and y is then

$$\|x - y\| = \sqrt{\sum_{n=0}^{N-1} (x[n] - y[n])^2}.$$

- there are many ***other*** norms you can use on $\mathbb{R}^N$, but we won't need them in this course.

Defining angles between vectors on $\mathbb{R}^N$

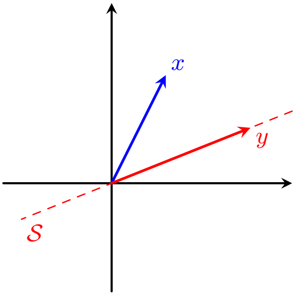
- the next ingredient we need is to define ***angles*** between vectors
  - consider two vectors $x, y \in \mathbb{R}^N$



Defining angles between vectors on $\mathbb{R}^N$

- the next ingredient we need is to define ***angles*** between vectors
  - consider two vectors $x, y \in \mathbb{R}^N$
  - consider the subspace

$$\mathcal{S} = \{z \in \mathbb{R}^N \mid z = \alpha y,\ \alpha \in \mathbb{R}\}$$



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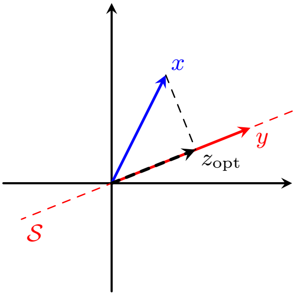
  - consider the subspace

$$\mathcal{S} = \{z \in \mathbb{R}^N \mid z = \alpha y,\ \alpha \in \mathbb{R}\}$$

  - find the closest point to x on $\mathcal{S}$

$$\min_{z \in \mathcal{S}} \|x - z\| = \min_{\alpha \in \mathbb{R}} \|x - \alpha y\|$$

and call the point $z_{\text{opt}} = \alpha_{\text{opt}} y$



Defining angles between vectors on $\mathbb{R}^N$

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  - consider the subspace

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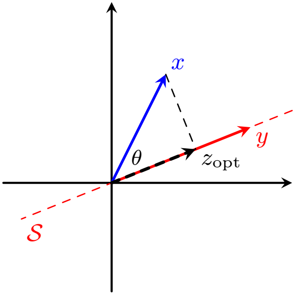
  - find the closest point to x on $\mathcal{S}$

$$\min_{z \in \mathcal{S}} \|x - z\| = \min_{\alpha \in \mathbb{R}} \|x - \alpha y\|$$

    and call the point $z_{\text{opt}} = \alpha_{\text{opt}} y$

  - ***define*** angle θ between x and y:

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\|z_{\text{opt}}\|}{\|x\|} = \frac{\|\alpha_{\text{opt}} y\|}{\|x\|}$$



Defining angles between vectors on $\mathbb{R}^N$

- to compute α_{opt}, we need to minimize the function

$$f(\alpha) = \|x - \alpha y\|^2 = \sum_{n=0}^{N-1} (x[n] - \alpha y[n])^2$$

- the minimum occurs when the derivative is zero:

$$f'(\alpha_{\text{opt}}) = 2 \sum_{n=0}^{N-1} (x[n] - \alpha y[n])(-y[n]) = 0$$

- solving, we find that

$$\alpha_{\text{opt}} = \frac{\sum_{n=0}^{N-1} x[n]y[n]}{\sum_{n=0}^{N-1} y[n]y[n]}$$

- we call the quantity in the numerator the ***inner product on*** $\mathbb{R}^N$

$$\langle x, y \rangle = \sum_{n=0}^{N-1} x[n]y[n]$$

The inner product on $\mathbb{R}^N$

$$\langle x, y\rangle = \sum_{n=0}^{N-1} x[n]y[n]$$

Observations:

(i) the inner product is *symmetric*: $\langle x, y\rangle = \langle y, x\rangle$

(ii) the inner product of a vector with itself is the squared norm:

$$\langle x, x\rangle = \sum_{n=0}^{N-1} x[n]x[n] = \sum_{n=0}^{N-1} (x[n])^2 = \|x\|^2.$$

(iii) the inner product distributes over linear combinations

$$\langle x, \alpha_1 y + \alpha_2 z\rangle = \alpha_1\langle x, y\rangle + \alpha_2\langle x, z\rangle$$

Defining angles between vectors on $\mathbb{R}^N$

- with this notation, we get a nice formulas for α_{opt} and z_{opt}

$$\alpha_{\text{opt}} = \frac{\langle x, y\rangle}{\langle y, y\rangle}, \qquad z_{\text{opt}} = \alpha_{\text{opt}} y = \frac{\langle x, y\rangle}{\langle y, y\rangle} y.$$

- the angle between x and y becomes

$$\cos\theta = \frac{\|z_{\text{opt}}\|}{\|x\|} = \frac{1}{\|x\|}\frac{|\langle x, y\rangle|}{\|y\|^2}\|y\| = \frac{|\langle x, y\rangle|}{\|x\|\|y\|}$$

How do we know that the RHS is between -1 and 1, so that we can actually solve $\cos(\theta) =$ RHS? We will answer this shortly. For the moment, let's just check some simple cases.

Defining angles between vectors on $\mathbb{R}^N$

- ***Colinear vectors:*** suppose that x is just a scaled version of y, i.e., $x = \gamma y$ for some $\gamma \geq 0$. Then the angle between the two vectors is

$$\cos(\theta) = \frac{|\langle x, y\rangle|}{\|x\|\|y\|} = \frac{|\langle \gamma y, y\rangle|}{\|\gamma y\|\|y\|} = \frac{|\gamma|\langle y, y\rangle}{|\gamma|\|y\|^2} = \frac{|\gamma|\|y\|^2}{|\gamma|\|y\|^2} = 1$$

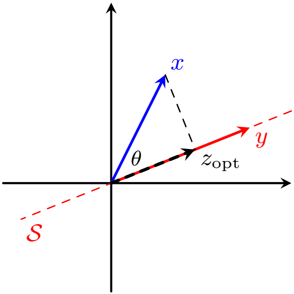
  which means that $\theta = 0$; this case checks out!

- ***Orthogonal vectors:*** note that if $\langle x, y\rangle = 0$, then $\cos(\theta) = 0$ and hence $\theta = 90^\circ$. Therefore, vectors forming a right angle have zero inner product. From our picture, the vectors y and $x - \alpha_{\text{opt}} y$ should form a right angle; let's check. We have

$$\begin{aligned} \langle x - \alpha_{\text{opt}} y, y\rangle &= \langle x, y\rangle - \alpha_{\text{opt}}\langle y, y\rangle \\ &= \langle x, y\rangle - \frac{\langle x, y\rangle}{\langle y, y\rangle}\langle y, y\rangle = 0 \end{aligned}$$

  so they do indeed form a right angle!

Defining angles between vectors on $\mathbb{R}^N$



- we think of z_{opt} as the ***best approximation*** to the vector x among all vectors in the subspace $\mathcal{S}$.
- it is "best" in the sense that the distance from x to $\mathcal{S}$ is minimized
- we call z_{opt} the ***projection*** onto the subspace spanned by y;

Inner products on vector spaces

We now generalize the idea of an inner product.

Definition 11.3 (Inner product). An ***inner product*** on a vector space Σ over Λ is a map $\langle \cdot, \cdot \rangle : \Sigma \times \Sigma \to \Lambda$ satisfying

(i) *conjugate symmetry:* $\langle x, y \rangle = \langle y, x \rangle^*$,

(ii) *linearity:* $\langle \alpha_1 x + \alpha_2 y, z \rangle = \alpha_1 \langle x, z \rangle + \alpha_2 \langle y, z \rangle$,

(iii) *non-negativity:* $\langle x, x \rangle \geq 0$ for all $x \in \Sigma$, and

(iv) *non-degeneracy:* $\langle x, x \rangle = 0$ if and only if $x = \mathbb{0}$.

- Two vectors $x, y \in \Sigma$ are said to be ***orthogonal*** if $\langle x, y \rangle = 0$
- Any inner product $\langle \cdot, \cdot \rangle$ also defines a ***norm*** $\|x\| = \sqrt{\langle x, x \rangle}$

The Cauchy-Schwarz Inequality

- whenever we have a vector space Σ with an inner product $\langle \cdot, \cdot \rangle$ we have a powerful relationship called the ***Cauchy-Schwarz Inequality***

$$|\langle x, y \rangle| \leq \|x\| \, \|y\|, \qquad \text{for any } x, y \in \Sigma$$

- dividing by $\|x\|\|y\|$, it therefore holds that

$$0 \leq \frac{|\langle x, y \rangle|}{\|x\| \, \|y\|} \leq 1 \qquad (\text{assuming } x, y \neq \mathbb{0}).$$

- therefore, the formula

$$\cos \theta = \frac{|\langle x, y \rangle|}{\|x\|\|y\|}$$

always yields a solution $-90^\circ \leq \theta \leq 90^\circ$.

The vector space $\mathbb{C}^N$ over $\mathbb{C}$

Many of our ideas for $\mathbb{R}^N$ carry over to $\mathbb{C}^N$ with only minor changes.

- complex Euclidean space $\Sigma = \mathbb{C}^N$ is a vector space over the scalars $\Lambda = \mathbb{C}$
- we often notate these vectors as n-tuples

$$x = (x[0], x[1], x[2], \ldots, x[N-1]), \qquad \text{each } x[n] \in \mathbb{C}$$

- addition and scalar multiplication defined as

$$x + y = (x[0] + y[0], x[1] + y[1], \ldots, x[N-1] + y[N-1])$$
$$\alpha x = (\alpha x[0], \alpha x[1], \alpha x[2], \ldots, \alpha x[N-1]), \qquad \alpha \in \mathbb{C}$$

- the zero element is defined as

$$\mathbb{0} = (0 + 0\mathbf{j}, 0 + 0\mathbf{j}, \ldots, 0 + 0\mathbf{j})$$

The norm on $\mathbb{C}^N$ over $\mathbb{C}$

- on the space $\mathbb{C}^N$ we usually use the ***Euclidean*** norm

$$\|x\| = \sqrt{\sum_{n=0}^{N-1} |x[n]|^2} = \sqrt{\sum_{n=0}^{N-1} x[n]^* x[n]}$$

- inside the sum is the ***magnitude squared*** $|x[n]|^2 = x[n]^* x[n]$ of x.
- the distance between two vectors $x, y \in \mathbb{C}^N$ is

$$\begin{aligned}\|x - y\| &= \sqrt{\sum_{n=0}^{N-1} |x[n] - y[n]|^2} \\ &= \sqrt{\sum_{n=0}^{N-1} (x[n] - y[n])^* (x[n] - y[n])}\end{aligned}$$

The inner product on $\mathbb{C}^N$ over $\mathbb{C}$

The inner product on $\mathbb{C}^N$ is defined as

$$\langle x, y \rangle = \sum_{n=0}^{N-1} x[n] y[n]^*$$

Note the ***complex conjugation*** of $y[n]$ in this formula.

(i) the inner product is *conjugate symmetric*: $\langle x, y \rangle = \langle y, x \rangle^*$

(ii) the inner product of a vector with itself is the squared norm:

$$\langle x, x \rangle = \sum_{n=0}^{N-1} x[n] x[n]^* = \sum_{n=0}^{N-1} |x[n]|^2 = \|x\|^2.$$

(iii) ***in the first argument***, the inner product distributes over linear combinations

$$\langle \alpha_1 x + \alpha_2 y, z \rangle = \alpha_1 \langle x, z \rangle + \alpha_2 \langle y, z \rangle$$

(iv) ***in the second argument***, we get the slightly modified conjugate formula

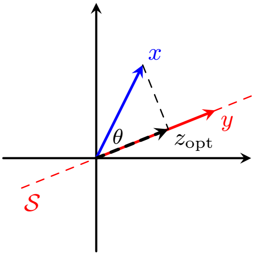
$$\langle z, \alpha_1 x + \alpha_2 y \rangle = \alpha_1^* \langle z, x \rangle + \alpha_2^* \langle z, y \rangle$$

Defining angles between vectors on $\mathbb{C}^N$

Our method of defining angles on $\mathbb{R}^N$ carries over to $\mathbb{C}^N$.

$$\alpha_{\text{opt}} = \frac{\langle x, y\rangle}{\langle y, y\rangle}, \qquad z_{\text{opt}} = \alpha_{\text{opt}} y.$$

$$\cos\theta = \frac{|\langle x, y\rangle|}{\|x\|\|y\|}$$



We now prove that this formula for α_{opt} minimizes $\|x - \alpha y\|$.

Proof of angle formula for $\mathbb{C}^N$

We again consider the function $f(\alpha) = \|x - \alpha y\|^2$. Write this using the inner product:

$$
\begin{aligned}
f(\alpha) &= \langle x - \alpha y, x - \alpha y \rangle \\
&= \langle x, x - \alpha y \rangle - \alpha \langle y, x - \alpha y \rangle && \text{(using linearity in 1st arg)} \\
&= \langle x - \alpha y, x \rangle^* - \alpha \langle x - \alpha y, y \rangle^* && \text{(using conjugate symm.)} \\
&= \|x\|^2 - \alpha^* \langle x, y \rangle - \alpha \langle y, x \rangle + \alpha \alpha^* \|y\|^2 && \text{(using lin. \& conjugate sym.)}
\end{aligned}
$$

If we define $\beta = \langle x, y \rangle / \|y\|^2$ then we can write this as

$$
\begin{aligned}
f(\alpha) &= \|x\|^2 - \alpha^* \beta \|y\|^2 - \alpha \beta^* \|y\|^2 + \alpha \alpha^* \|y\|^2 \\
&= \|x\|^2 + (|\alpha|^2 - \alpha^* \beta - \beta^* \alpha) \|y\|^2
\end{aligned}
$$

If we add and subtract $|\beta|^2 \|y\|^2$, we can complete the square to obtain

$$
\begin{aligned}
f(\alpha) &= \|x\|^2 + (|\alpha|^2 + |\beta|^2 - \alpha^* \beta - \beta^* \alpha) \|y\|^2 - |\beta|^2 \|y\|^2 \\
&= \|x\|^2 - |\beta|^2 \|y\|^2 + |\alpha - \beta|^2 \|y\|^2
\end{aligned}
$$

The first and second terms are independent of α. To minimize $f(\alpha)$, we therefore select $\alpha = \beta$, which shows the desired formula. •

The vector space $\ell_2([0, N_0 - 1])$ over $\mathbb{C}$

Consider the set $\ell_2([0, N_0 - 1])$ of all complex-valued DT signals which are periodic with period N_0 and have finite energy, i.e.,

$$\ell_2([0, N_0 - 1]) = \left\{ x : \mathbb{Z} \to \mathbb{C} \;\middle|\; x \text{ is } N_0\text{-periodic and } \sum_{k=0}^{N_0-1} |x[k]|^2 < \infty \right\}.$$

Indeed $\ell_2([0, N_0 - 1])$ is a vector space over $\mathbb{C}$:

(i) the sum of two N_0-periodic signals is again N_0-periodic $\Longrightarrow$ the set is closed under vector addition

(ii) an N_0-periodic signal scaled by a constant α is again N_0-periodic $\Longrightarrow$ the set is closed under scalar multiplication

(iii) the zero element $\mathbb{0}$ is the constant zero signal

The vector space $\ell_2([0, N_0 - 1])$ over $\mathbb{C}$

- **recall:** we established that there is a ***one-to-one correspondence*** between periodic signals x_{per} and finite-duration signals x_{fin}.
- therefore, each $x \in \ell_2([0, N_0 - 1])$ is equivalent to a vector

$$\boldsymbol{x} = (x[0], x[1], \ldots, x[N_0 - 1]) \in \mathbb{C}^{N_0}$$

- it follows that the two vector spaces $\ell_2([N_0 - 1])$ and $\mathbb{C}^{N_0}$ are ***equivalent*** (isomorphic); anything you learn about one can be used on the other.
- **aside:** as a consequence of the above, the finite-energy stipulation in the definition of $\ell_2([0, N_0 - 1])$ is redundant at the moment, but is more important when $N_0 \to \infty$ ...

Projections onto subspaces

- to define the angle between a vector x and the ***subspace*** $\mathcal{S} = \{\alpha y \mid \alpha \in \mathbb{R}\}$, we computed the ***projection*** of x onto $\mathcal{S}$
- $z_{\text{opt}} \in \mathcal{S}$ was the ***best approximation*** of x among all vectors $z \in \mathcal{S}$

How do we extend this idea to more general subspaces? We consider a particular kind of subspace which is very useful.

Definition 11.4 (Orthogonal set of vectors). A set of vectors $\{y_0, \ldots, y_{K-1}\}$ in $\mathbb{C}^N$ is ***orthogonal*** if $\langle y_i, y_j \rangle = 0$ for all $i \neq j$.

- given an orthogonal set of vectors $\{y_0, \ldots, y_{K-1}\}$, define

$$\mathcal{S} = \left\{ z \in \mathbb{C}^N \mid z = \sum_{k=0}^{K-1} \alpha_k y_k, \;\; \alpha_k \in \mathbb{C} \text{ for all } 0 \leq k \leq K-1 \right\}.$$

Projections onto subspaces

$$\mathcal{S} = \left\{ z \in \mathbb{C}^N \mid z = \sum_{k=0}^{K-1} \alpha_k y_k, \;\; \alpha_k \in \mathbb{C} \text{ for all } 0 \leq k \leq K-1 \right\}.$$

$\mathcal{S}$ is the set of all ***linear combinations*** (i.e., the ***span***) of $\{y_0, \ldots, y_{K-1}\}$

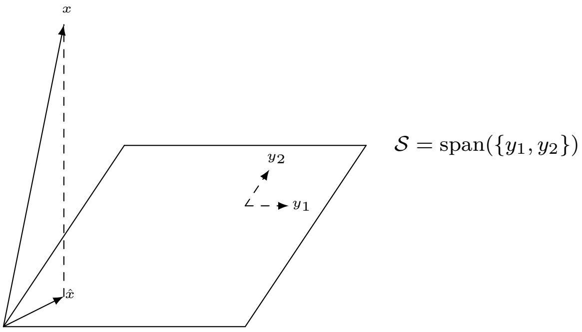
- given a vector $x \in \mathbb{C}^N$, we wish to compute the best approximation $\hat{x}_K$ of x contained within the subspace $\mathcal{S}$, i.e., we solve

$$\min_{\hat{x}_K \in \mathcal{S}} \|x - \hat{x}_K\|^2 = \min_{\alpha_1, \ldots, \alpha_{K-1} \in \mathbb{C}} \left\| x - \sum_{k=0}^{K-1} \alpha_k y_k \right\|^2$$

- we now prove that the optimal choice of coefficients is given by

$$\alpha_{k,\text{opt}} = \frac{\langle x, y_k \rangle}{\|y_k\|^2} \quad \Longrightarrow \quad \hat{x}_K = \sum_{k=0}^{K-1} \frac{\langle x, y_k \rangle}{\|y_k\|^2} y_k.$$

Projections onto subspaces



Projection of x onto $\mathcal{S}$ finds the point $\hat{x} \in \mathcal{S}$ which is ***closest*** to the original vector x

Proof of projection formula

We want to minimize the function

$$f(\alpha_0, \alpha_1, \ldots) = \left\| x - \sum_{k=0}^{K-1} \alpha_k y_k \right\|^2$$

$$= \left\langle x - \sum_{k=0}^{K-1} \alpha_k y_k , x - \sum_{k=0}^{K-1} \alpha_k y_k \right\rangle$$

$$= \langle x, x \rangle - \sum_{k=0}^{K-1} \alpha_k^* \langle x, y_k \rangle - \sum_{k=0}^{K-1} \alpha_k \langle y_k, x \rangle + \sum_{k=0}^{K-1} \sum_{j=0}^{K-1} \alpha_k^* \alpha_j \langle y_j, y_k \rangle$$

Since the vectors $\{y_0, \ldots, y_{K-1}\}$ form an orthogonal set, this simplifies to

$$f(\alpha_0, \alpha_1, \ldots) = \langle x, x \rangle - \sum_{k=0}^{K-1} \alpha_k^* \langle x, y_k \rangle - \sum_{k=0}^{K-1} \alpha_k \langle y_k, x \rangle + \sum_{k=0}^{K-1} \alpha_k^* \alpha_k \langle y_k, y_k \rangle$$

$$= \sum_{k=0}^{K-1} \left[\frac{1}{K} \|x\|^2 - \alpha_k^* \langle x, y_k \rangle - \alpha_k \langle y_k, x \rangle + \alpha_k^* \alpha_k \langle y_k, y_k \rangle \right]$$

Each term in this sum looks ***exactly*** like the calculation we did for projections in $\mathbb{C}^N$.

We therefore just go term by term and copy the result, seting $\alpha_k = \langle x, y_k \rangle / \|y_k\|^2$.

Projections onto subspaces

- the ***best approximation*** of x contained within $\mathcal{S}$ is given by

$$\hat{x}_K = \sum_{k=0}^{K-1} \underbrace{\frac{\langle x, y_k \rangle}{\|y_k\|^2} y_k}_{\text{projection of } x \text{ onto } y_k}$$

Comments:

- if one begins with a merely *linearly independent* set of vectors $\{y_0, \ldots, y_{K-1}\}$, one can obtain an *orthogonal* set through the Gram-Schmidt procedure, as covered in a linear algebra course.

- in $\mathbb{C}^N$, a very natural orthogonal set are the basis vectors

$$\mathbb{e}_k[n] = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{otherwise} \end{cases} \qquad 0 \leq k \leq N-1$$

i.e., the vectors $\mathbb{e}_0 = (1, 0, \ldots, 0), \mathbb{e}_1 = (0, 1, 0, \ldots, 0)$, and so on.

Representation in a basis

What happens when $K = N$? The subspace $\mathcal{S}$ consists of all linear combinations of a set of N orthogonal vectors $\{y_0, \ldots, y_{N-1}\}$; such a set forms a ***basis*** for $\mathbb{C}^N$. Therefore $\mathcal{S}$ is the ***entire vector space*** $\mathbb{C}^N$.

- in this case, our best approximation $\hat{x}_K$ is ***exact***, i.e., $\hat{x}_K = x$
- our approximation formula

$$\alpha_k = \frac{\langle x, y_k \rangle}{\|y_k\|^2}, \qquad \hat{x}_K = x = \sum_{k=0}^{N-1} \alpha_k y_k \tag{4}$$

  is the ***representation of x in the basis $\{y_0, \ldots, y_{N-1}\}$***
- the coefficient $\alpha_k = \frac{\langle x, y_k \rangle}{\|y_k\|^2}$ can be thought of as the amount that y_k contributes towards the overall vector x.

DTFS is just a representation of x in a new basis

Recall: Let x be a periodic DT signal with fundamental period N_0, and let $\omega_0 = 2\pi/N_0$. The decomposition

$$x[n] = \sum_{k=0}^{N_0-1} \alpha_k e^{\mathbf{j}k\omega_0 n}, \qquad n \in \{0, \ldots, N_0 - 1\}$$

with coefficients α_k given by

$$\alpha_k = \frac{1}{N_0} \sum_{l=0}^{N_0-1} x[l] e^{-\mathbf{j}k\omega_0 l}, \qquad k \in \{0, \ldots, N_0 - 1\}.$$

is called the ***discrete-time Fourier series (DTFS)*** of the signal x.

We now show that these formulas are performing a change of basis!

DTFS is just a representation of x in a new basis

- we are in the vector space $\ell_2([0, N_0 - 1])$, with inner product

$$\langle x, y \rangle = \sum_{n=0}^{N_0-1} x[n]y[n]^*.$$

- consider the set of vectors $\{\phi_0, \ldots, \phi_{N_0-1}\}$ defined by

$$\phi_k[n] = e^{\mathbf{j}k\omega_0 n}, \qquad n \in \{0, \ldots, N_0 - 1\}.$$

- we know from our previous calculations for the DTFS that

$$\langle \phi_k, \phi_j \rangle = \sum_{n=0}^{N_0-1} \phi_k[n]\phi_j[n]^* = \begin{cases} N_0 & \text{if } k = j \\ 0 & \text{if } k \neq j. \end{cases}$$

so these N_0 vectors form an orthogonal basis for $\ell_2([0, N_0 - 1])$.

DTFS is just a representation of x in a new basis

- let $x \in \ell_2([0, N_0 - 1])$ be a DT periodic signal
- if we want to represent x in the basis defined by $\{\phi_0, \dots, \phi_{N_0-1}\}$, we need only apply the formula (4), which we repeat here:

$$\alpha_k = \frac{\langle x, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle}, \qquad x = \sum_{k=0}^{N_0-1} \alpha_k \phi_k.$$

- note that this means we represent the signal x as

$$x[n] = \sum_{k=0}^{N_0-1} \alpha_k e^{\mathbf{j} k \omega_0 n}, \qquad n \in \{0, \dots, N_0 - 1\}.$$

- we compute the coefficient α_k to be

$$\alpha_k = \frac{1}{\langle \phi_k, \phi_k \rangle} \langle x, \phi_k \rangle = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] \phi_k[n]^* = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-\mathbf{j} k \omega_0 n}$$

which is ***exactly*** our formula for the DTFS coefficients!!

Matrix-vector interpretation of DTFS basis change

- **recall:** we developed matrix-vector formulas for the DTFS:

$$\boldsymbol{x} = \boldsymbol{G}\boldsymbol{\alpha}, \quad \boldsymbol{\alpha} = \tfrac{1}{N_0}\boldsymbol{H}\boldsymbol{x}.$$

  where $\boldsymbol{H}_{kn} = e^{-\mathbf{j}k\omega_0 n}$ and $\boldsymbol{G}_{nk} = e^{\mathbf{j}k\omega_0 n}$ and

$$\begin{aligned}\boldsymbol{\alpha} &= (\alpha_0, \alpha_1, \ldots, \alpha_{N_0-1}) \\ \boldsymbol{x} &= (x[0], x[1], \ldots, x[N_0-1])\end{aligned}$$

- note that $\boldsymbol{H}_{kn} = \boldsymbol{G}^*_{nk}$. As matrices, we say that $\boldsymbol{H}$ and $\boldsymbol{G}$ are *Hermitian* conjugates and we write that $\boldsymbol{G} = \boldsymbol{H}^{\mathrm{H}}$.
- we have previously observed that $\frac{1}{N_0}\boldsymbol{G}\boldsymbol{H} = I$, and therefore $\boldsymbol{H}\boldsymbol{H}^{\mathrm{H}} = N_0 I$.

DTFS basis change and Parseval's relation

- Let's compute the norm of $\boldsymbol{\alpha}$

$$\begin{aligned}\|\boldsymbol{\alpha}\|^2 = \langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle = \tfrac{1}{N_0^2} \langle \boldsymbol{H}\boldsymbol{x}, \boldsymbol{H}\boldsymbol{x} \rangle &= \tfrac{1}{N_0^2} (\boldsymbol{H}\boldsymbol{x})^{\mathrm{H}} (\boldsymbol{H}\boldsymbol{x}) \\ &= \tfrac{1}{N_0^2} \boldsymbol{x}^{\mathrm{H}} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{H} \boldsymbol{x} \\ &= \tfrac{1}{N_0} \|\boldsymbol{x}\|^2\end{aligned}$$

- writing this out explicitly, we find that

$$\frac{1}{N_0}\|\boldsymbol{x}\|^2 = \|\boldsymbol{\alpha}\|^2 \quad \text{or} \quad \frac{1}{N_0}\sum_{k=0}^{N_0-1} |x[k]|^2 = \sum_{k=0}^{N_0-1} |\alpha_k|^2$$

- this is ***Parseval's relation*** for the DTFS!

DTFS basis change as a rotation of vectors

- Consider two signals $x, y \in \ell_2([0, N_0 - 1])$ with associated vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^N$, and let $\boldsymbol{\alpha} = \frac{1}{N_0}\boldsymbol{H}\boldsymbol{x}$ and $\boldsymbol{\beta} = \frac{1}{N_0}\boldsymbol{H}\boldsymbol{y}$ be the associated vectors of DTFS coefficients.
- The inner product between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is

$$\begin{aligned}\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = \frac{1}{N_0^2}\langle \boldsymbol{H}\boldsymbol{x}, \boldsymbol{H}\boldsymbol{y} \rangle = \frac{1}{N_0^2}(\boldsymbol{H}\boldsymbol{y})^{\mathrm{H}}(\boldsymbol{H}\boldsymbol{x}) &= \frac{1}{N_0^2}\boldsymbol{x}^{\mathrm{H}}\boldsymbol{H}^{\mathrm{H}}\boldsymbol{H}\boldsymbol{x} \\ &= \frac{1}{N_0}\boldsymbol{y}^{\mathrm{H}}\boldsymbol{x} = \frac{1}{N_0}\langle \boldsymbol{x}, \boldsymbol{y} \rangle\end{aligned}$$

- the angle $\theta_{\alpha\beta}$ between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is

$$\cos\theta_{\alpha\beta} = \frac{|\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle|}{\|\boldsymbol{\alpha}\|\|\boldsymbol{\beta}\|} = \frac{N_0\langle|\boldsymbol{x}, \boldsymbol{y}\rangle|}{\sqrt{N_0}\|\boldsymbol{x}\|\sqrt{N_0}\|\boldsymbol{y}\|} = \frac{|\langle \boldsymbol{x}, \boldsymbol{y} \rangle|}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|} = \cos\theta_{xy}$$

The DTFS transformation ***preserves angles between vectors***! It is a “rotation” of vectors in $\mathbb{C}^{N_0}$.

Vector space concepts for CTFS/CTFT/DTFT

- the case of a DT periodic signal fits nicely with your current linear algebra knowledge, because the vector space $\ell_2([0, N_0 - 1])$ is *finite dimensional*
- when dealing with CT signals, or aperiodic DT signals, the associated vector spaces are ***infinite dimensional***
- while the mathematics becomes more complicated, the key ideas remain unchanged; all these transformations are just methods of representing a signal in a new basis, consisting of complex exponential signals in either CT or DT

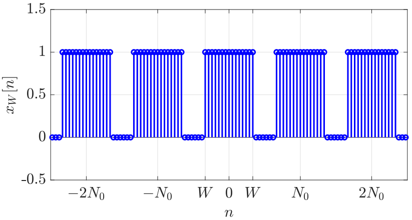
Filtering of signals as projection onto subspaces

- when we speak of "filtering" a signal, we usually mean that we modify the signal to accentuate certain characteristics
- the two most common types of filters are
  - ***high-pass*** filters, where we remove the low-frequency signal content
  - ***low-pass*** filters, where we remove the high-frequency signal content
- we can think of the action of such filters as projection onto subspaces generated by high-frequency and low-frequency collections of complex exponential signals, respectively
- we illustrate this by returning to a familiar example

Example: filtering the windowing signal

Let $N_0 \in \mathbb{Z}_{>0}$ be odd, and let $W \in \mathbb{Z}$ satisfy $0 \leq W \leq \frac{N_0-1}{2}$. Consider the signal

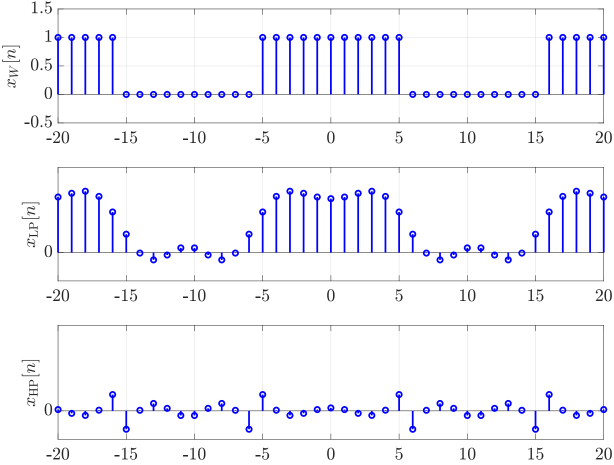
$$x_W[n] = \begin{cases} 1 & \text{if } |n| \leq W \\ 0 & \text{if } W < n \leq \frac{N_0-1}{2} \end{cases}$$



- we consider $N_0 = 21$ and $W = 5$
- consider the low-pass subspace $\mathcal{S}_{\text{LP}}$ and the high-pass subspace $\mathcal{S}_{\text{HP}}$

$$\mathcal{S}_{\text{LP}} = \text{span}\{e^{\mathbf{j}\omega_0(-4)n}, \ldots, e^{\mathbf{j}\omega_0(-1)n}, e^{\mathbf{j}\omega_0(0)n}, e^{\mathbf{j}\omega_0(1)n}, \ldots, e^{\mathbf{j}\omega_0(4)n}\}$$
$$\mathcal{S}_{\text{HP}} = \text{span}\{e^{\mathbf{j}\omega_0(-10)n}, \ldots, e^{\mathbf{j}\omega_0(-5)n}, e^{\mathbf{j}\omega_0(5)n}, \ldots, e^{\mathbf{j}\omega_0(10)n}\}$$

Example: filtering the windowing signal



From filtering to circular convolution

- recall that our DTFS representation of x is given by

$$x[n] = \sum_{k=0}^{N_0-1} \alpha_k e^{\mathbf{j}k\omega_0 n}, \qquad \alpha_k = \frac{1}{N_0} \sum_{l=0}^{N_0-1} x[l] e^{-\mathbf{j}k\omega_0 l}$$

- in these previous filtering examples, projection onto a subspace is equivalent to setting the remaining Fourier coefficients to ***zero***
- instead of setting them to *zero*, let us now generalize our idea of a "filter" as an operation which produces a new signal y by modifying each Fourier coefficient α_k of x via multiplication by a corresponding constant $\beta_k \in \mathbb{C}$
- in other words, the output y of the filter has Fourier coefficients $\gamma_k = \alpha_k \beta_k$

From filtering to circular convolution

- we can now represent y through the DTFS representation

$$y[n] = \sum_{k=0}^{N_0-1} \gamma_k e^{\mathbf{j}k\omega_0 n} = \sum_{k=0}^{N_0-1} \alpha_k \beta_k e^{\mathbf{j}k\omega_0 n}$$

- inserting the formula for α_k we have

$$y[n] = \sum_{k=0}^{N_0-1} \left[\frac{1}{N_0} \sum_{l=0}^{N_0-1} x[l] e^{-\mathbf{j}k\omega_0 l} \right] \beta_k e^{\mathbf{j}k\omega_0 n} = \sum_{l=0}^{N_0-1} x[l] \underbrace{\left[\sum_{k=0}^{N_0-1} \frac{\beta_k}{N_0} e^{\mathbf{j}k\omega_0 (n-l)} \right]}_{=h[n-l]}$$

- in brackets we have the DTFS representation of a signal with coefficients β_k / N_0; let's call this signal $h[n-l]$. Therefore

$$y[n] = \sum_{l=0}^{N_0-1} h[n-l] x[l]$$

- this is a convolution formula!

From filtering to circular convolution

Summary: If "filtering" means scaling the DTFS coefficients of the input x by constants $\beta_k \in \mathbb{C}$, then the filtered output is

$$y[n] = \sum_{l=0}^{N_0-1} h[n-l]x[l], \qquad n \in \{0, \dots, N_0 - 1\}$$

where h is the N_0-periodic DT signal with DTFS coefficients β_k / N_0.

- Two important differences with other convolution formulas so far:
  (i) x and h are both periodic with period N_0
  (ii) the sum runs only from 0 to $N_0 - 1$, not from $-\infty$ to ∞
- since x and h are N_0-periodic, we have $y[n + N_0] = y[n]$, so y is also N_0-periodic
- this is called ***circular convolution***, and is notated as

$$y = x \circledast h$$

Properties of circular convolution

- just like other convolution operations we have encountered, circular convolution satisfies several appealing properties

(i) ***commutative property:*** if x and h are two N_0-periodic DT signals, then $x \circledast h = h \circledast x$, or pointwise

$$\sum_{l=0}^{N_0-1} h[n-l]x[l] = \sum_{l=0}^{N_0-1} h[l]x[n-l]$$

(ii) ***linearity:*** if h, x, y are three N_0 periodic DT signals and $\alpha, \beta \in \mathbb{C}$ then

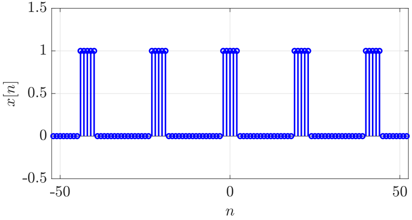
$$h \circledast (\alpha x + \beta y) = \alpha(h \circledast x) + \beta(h \circledast y)$$

(iii) ***time-invariance:*** if h, x are two N_0 periodic DT signals with $y = h \circledast x$ then for all $\Delta \in \mathbb{Z}$ we have that $y_\Delta = h \circledast x_\Delta$

Example 11.3: circular convolution

Consider the input signal

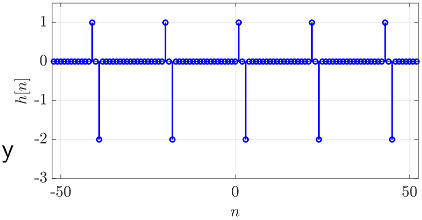
$$x[n] = \begin{cases} 1 & \text{if } |n| \leq 2 \\ 0 & \text{if } 2 < |n| \leq 10 \end{cases}$$



and the impulse response

$$h[n] = \delta[n-1] - 2\delta[n-3]$$

for $0 \leq n \leq 20$, repeated periodically



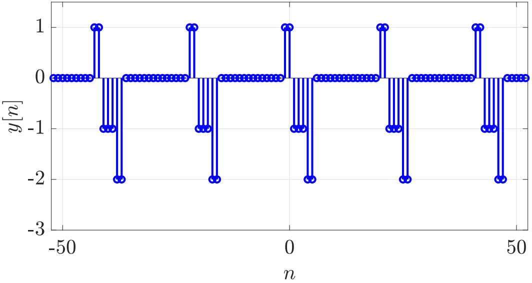
Example 11.3: circular convolution

- we can now compute for $0 \leq n \leq 20$ that

$$(h \circledast x)[n] = \sum_{l=0}^{N_0-1} h[n-l]x[l] = \sum_{l=0}^{20} x[l]\left(\delta[n-l-1] - 2\delta[n-l-3]\right)$$

$$= x[n-1] - 2x[n-3]$$

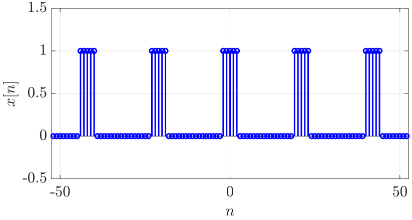
where x is as defined before



Example 11.4: circular convolution

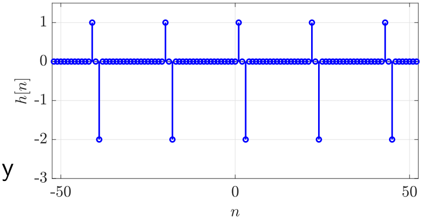
Consider the input signal

$$x[n] = \begin{cases} 1 & \text{if } |n| \leq 2 \\ 0 & \text{if } 2 < |n| \leq 10 \end{cases}$$



and the impulse response

$$h[n] = \delta[n-1] - 2\delta[n-3] + \delta[n-10]$$

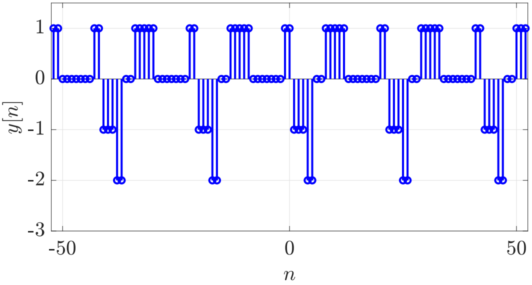


for $0 \leq n \leq 20$, repeated periodically

Example 11.4: circular convolution

- we can now similarly compute for $0 \le n \le 20$ that that

$$(h \circledast x)[n] = \sum_{l=0}^{N_0-1} h[n-l]x[l] = x[n-1] - 2x[n-3] + x[n-10].$$



Circulant matrices

- we now formalize circular convolution

$$(h \circledast x)[n] = y[n] = \sum_{l=0}^{N_0-1} h[n-l]x[l], \qquad n \in \{0, \ldots, N_0 - 1\}$$

  using matrix-vector notation

- we can write these equations all out explicitly as

$$\begin{aligned}
y[0] &= \sum\nolimits_{l=0}^{N_0-1} h[0-l]x[l] \\
y[1] &= \sum\nolimits_{l=0}^{N_0-1} h[1-l]x[l] \\
y[2] &= \sum\nolimits_{l=0}^{N_0-1} h[2-l]x[l] \\
&\vdots \\
y[N_0-1] &= \sum\nolimits_{l=0}^{N_0-1} h[N_0-1-l]x[l]
\end{aligned}$$

Circulant matrices

Writing this all in matrix-vector form, we have

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[N_0-1] \end{bmatrix} = \boldsymbol{C} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N_0-1] \end{bmatrix} \qquad \text{or} \qquad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$

where

$$\boldsymbol{C} = \begin{bmatrix} h[0] & h[-1] & h[-2] & \cdots & h[-(N_0-1)] \\ h[1] & h[0] & h[-1] & \cdots & h[1-(N_0-1)] \\ h[2] & h[1] & h[0] & \cdots & h[2-(N_0-1)] \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ h[N_0-1] & h[(N_0-1)-1] & h[(N_0-1)-2] & \cdots & h[(N_0-1)-(N_0-1)] \end{bmatrix}$$

However, we know that $h[n+N_0] = h[n]$ for all $n \in \mathbb{Z}$. We can use this to simplify $\boldsymbol{C}$!

Circulant matrices

$$
\boldsymbol{C} = \begin{bmatrix} h[0] & h[-1] & h[-2] & \cdots & h[-(N_0-1)] \\ h[1] & h[0] & h[-1] & \cdots & h[1-(N_0-1)] \\ h[2] & h[1] & h[0] & \cdots & h[2-(N_0-1)] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h[N_0-1] & h[(N_0-1)-1] & h[(N_0-1)-2] & \cdots & h[(N_0-1)-(N_0-1)] \end{bmatrix}
$$

$$
= \begin{bmatrix} h[0] & h[N_0-1] & h[N_0-2] & \cdots & h[1] \\ h[1] & h[0] & h[N_0-1] & \cdots & h[2] \\ h[2] & h[1] & h[0] & \cdots & h[3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h[N_0-1] & h[N_0-2] & h[N_0-3] & \cdots & h[0] \end{bmatrix}
$$

- note that $\boldsymbol{C}$ is constant along all diagonals; a ***Toeplitz*** matrix
- each diagonal corresponds to a value of the impulse response.

Circulant matrices

$$
\boldsymbol{C} = \begin{bmatrix} h[0] & h[N_0-1] & h[N_0-2] & \cdots & h[1] \\ h[1] & h[0] & h[N_0-1] & \cdots & h[2] \\ h[2] & h[1] & h[0] & \cdots & h[3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h[N_0-1] & h[N_0-2] & h[N_0-3] & \cdots & h[0] \end{bmatrix}
$$

- in addition to being Toeplitz, each row of $\boldsymbol{C}$ is just a one-element circular shift of the row above; such matrices are called ***circulant***
- there is a beautiful connection between this circulant matrix and the matrix-vector formulation of the DTFS developed in Chapter 6

Circulant matrices and the DTFS

- **Recall:** the matrix-vector formulas for the DTFS of a signal x

$$\boldsymbol{x} = \boldsymbol{G}\boldsymbol{\alpha}, \quad \boldsymbol{\alpha} = \tfrac{1}{N_0}\boldsymbol{H}\boldsymbol{x}.$$

  where $\boldsymbol{H}_{kn} = e^{-\mathbf{j}k\omega_0 n}$ and $\boldsymbol{G}_{nk} = e^{\mathbf{j}k\omega_0 n}$ and

$$\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{N_0-1})$$
$$\boldsymbol{x} = (x[0], x[1], \ldots, x[N_0 - 1])$$

- the matrices $\boldsymbol{H}$ and $\boldsymbol{G}$ satisfy $\boldsymbol{G} = \boldsymbol{H}^{\mathrm{H}}$ and $\boldsymbol{H}\boldsymbol{H}^{\mathrm{H}} = N_0 I$.

Let's see what happens if we try to compute the DTFS coefficients γ_k of the signal $y = h \circledast x$

Circulant matrices and the DTFS

- **recall:**
  (i) the result y of circular convolution was the signal having DTFS coefficients $\gamma_k = \alpha_k \beta_k$
  (ii) the impulse response h was correspondingly defined as

$$h[n] = \sum_{k=0}^{N_0-1} \frac{\beta_k}{N_0} e^{\mathbf{j}k\omega_0 n}$$

  so h has DTFS coefficients β_k / N_0
- using our vector formulas, the vector $\boldsymbol{\gamma}$ of DTFS coefficients of y is

$$\boldsymbol{\gamma} = \tfrac{1}{N_0} \boldsymbol{H} \boldsymbol{y} = \tfrac{1}{N_0} \boldsymbol{H} \boldsymbol{C} \boldsymbol{x}$$

- substituting $\boldsymbol{x} = \boldsymbol{G}\boldsymbol{\alpha} = \boldsymbol{H}^{\mathrm{H}} \boldsymbol{\alpha}$ we obtain

$$\boldsymbol{\gamma} = \tfrac{1}{N_0} \boldsymbol{H} \boldsymbol{C} \boldsymbol{H}^{\mathrm{H}} \boldsymbol{\alpha}$$

- it follows that the matrix $\boldsymbol{H}\boldsymbol{C}\boldsymbol{H}^{\mathrm{H}}$ must be ***diagonal*** with diagonal elements equal to $N_0 \beta_k$! The matrix $\boldsymbol{H}$ diagonalizes $\boldsymbol{C}$.

Circulant matrices and the DTFS

- we therefore have that

$$\mathrm{diag}(\beta_0, \beta_2, \ldots, \beta_{N_0-1}) = \frac{1}{N_0}\boldsymbol{H}\boldsymbol{C}\boldsymbol{H}^{\mathrm{H}}$$

  or equivalently $\boldsymbol{C} = \frac{1}{N_0}\boldsymbol{H}^{\mathrm{H}}\mathrm{diag}(\beta_0, \beta_2, \ldots, \beta_{N_0-1})\boldsymbol{H}$

- we can now write the output signal $\boldsymbol{y}$ as

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} = \underbrace{\left(\frac{1}{\sqrt{N_0}}\boldsymbol{H}\right)^{\mathrm{H}}}_{\text{unrotate}} \underbrace{\mathrm{diag}(\beta_0, \beta_2, \ldots, \beta_{N_0-1})}_{\text{scale}} \underbrace{\left(\frac{1}{\sqrt{N_0}}\boldsymbol{H}\right)}_{\text{rotate}} \boldsymbol{x}$$

- the operation of the LTI system defined by circular convolution can be interpreted as a rotation to a diagonal coordinate system, a scaling, and a rotation back to the original coordinate system

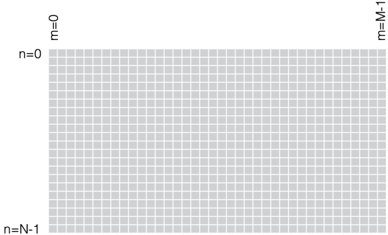
12. Bonus: Introduction to Image Processing

Introduction

- throughout the course we have focused on *one-dimensional* signals $x : \mathbb{R} \to \mathbb{C}$ in continuous-time or $x : \mathbb{Z} \to \mathbb{C}$ in discrete-time
- when we want to work with (digitized) ***images***, these are described by two-dimensional signals $x : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$
- it turns out that with the geometric methods of Chapter 6, this is not difficult; we just need to use a different basis and inner product, but all the ideas are the same
- we will look at how to implement simple filters on images to blur and accentuate edges; the discussion will be at a relatively high level

Images

- an image can be represented by an $N \times M$ matrix $\boldsymbol{X}$
- each entry $\boldsymbol{X}[n,m]$ is a pixel value



- in an 8-bit greyscale image, there are $2^8 = 256$ levels, with 0 being black and 255 being white, so $\boldsymbol{X}[n,m] \in \{0, 1, \ldots, 255\}$

Vector space and inner product for matrices

- the vector space of interest is the vector space of $N \times M$ matrices $\boldsymbol{X}$
- addition and scalar multiplication are defined element-wise by

$$(\boldsymbol{X}+\boldsymbol{Y})[n,m] = \boldsymbol{X}[n,m]+\boldsymbol{Y}[n,m], \qquad (\alpha\boldsymbol{X})[n,m] = \alpha\boldsymbol{X}[n,m].$$

- the inner product between two matrices is defined as

$$\langle \boldsymbol{X}, \boldsymbol{Y} \rangle = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \boldsymbol{X}[n,m]\boldsymbol{Y}[n,m]^*$$

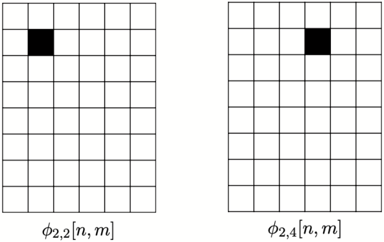
  and is called the ***Frobenius*** inner product

- this vector space has dimension NM; we need to define ***basis vectors*** for both time spatial domain and the associated two-dimensional frequency domain

Bases for two-dimensional signals

- the ***spatial*** basis vectors $\{\varphi_{k,l}\}$ for the space of $N \times M$ matrices are defined as

$$\varphi_{k,l}[n,m] = \delta[n-k]\delta[m-l] = \begin{cases} 1 & \text{if } (n,m) = (k,l) \\ 0 & \text{otherwise} \end{cases}$$



- these vectors are orthogonal (in fact, orthonormal), since

$$\langle \varphi_{k,l}, \varphi_{k',l'} \rangle = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \varphi_{k,l}[n,m]\varphi_{k',l'}[n,m]^*$$

$$= \sum_{n=0}^{N-1} \delta[n-k]\delta[n-k'] \sum_{m=0}^{M-1} \delta[m-l]\delta[m-l'] = \begin{cases} 1 & \text{if } (k,l) = (k',l') \\ 0 & \text{otherwise} \end{cases}$$

Bases for two-dimensional signals

- the projection of an image vector $\boldsymbol{X}$ onto the spatial basis vector $\varphi_{k,l}$ simply returns the value of the image in the (k,l) cell:

$$\begin{aligned}\frac{\langle \boldsymbol{X}, \varphi_{k,l}\rangle}{\|\varphi_{k,l}\|^2} = \langle \boldsymbol{X}, \varphi_{k,l}\rangle &= \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} \boldsymbol{X}[n,m]\varphi_{k,l}[n,m] \\ &= \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} \boldsymbol{X}[n,m]\delta[n-k]\delta[m-l] \\ &= \boldsymbol{X}[k,l]\end{aligned}$$

- put differently, each basis vector corresponds to one pixel location in the image, and projection onto that basis vector just picks out the value of that particular pixel

Bases for two-dimensional signals

- we also need a two-dimensional Fourier basis for the frequency domain
- let $\omega_0 = 2\pi/N$ and $\nu_0 = 2\pi/M$ be the fundamental frequencies for the two directions, and define the Fourier basis vectors

$$\phi_{k,l}[n,m] = e^{\mathbf{j}k\omega_0 n} e^{\mathbf{j}\nu_0 lm} = e^{\mathbf{j}(k\omega_0 n + l\nu_0 m)}$$

- these vectors are orthogonal since

$$\begin{aligned}
\langle \phi_{k,l}, \phi_{k',l'} \rangle &= \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} e^{\mathbf{j}k\omega_0 n} e^{\mathbf{j}l\nu_0 m} (e^{\mathbf{j}k'\omega_0 n} e^{\mathbf{j}l'\nu_0 m})^* \\
&= \sum_{n=0}^{N-1} e^{\mathbf{j}k\omega_0 n} e^{-\mathbf{j}k'\omega_0 n} \sum_{m=0}^{M-1} e^{\mathbf{j}l\nu_0 m} e^{-\mathbf{j}l'\nu_0 m} \\
&= \sum_{n=0}^{N-1} e^{\mathbf{j}(k-k')\omega_0 n} \sum_{m=0}^{M-1} e^{\mathbf{j}(l-l')\nu_0 m} \\
&= N\delta[k-k'] \cdot M\delta[l-l'] = \begin{cases} NM & \text{if } (k,l) = (k',l') \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Bases for two-dimensional signals

- the basis vectors $\phi_{k,l}[n,m] = e^{\mathbf{j}k\omega_0 n} e^{\mathbf{j}l\nu_0 m}$ are oscillating signals in ***two*** dimensions; they look like ***waves***

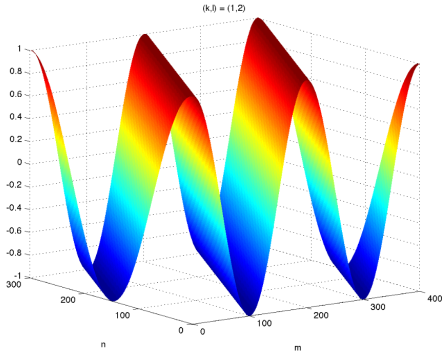


FIGURE 13.4. The real part of the basis vector $\phi_{1,2}$ where $N = 300$ and $M = 400$.

Bases for two-dimensional signals

- the values of k and l in $\phi_{k,l}[n,m] = e^{\mathbf{j}k\omega_0 n} e^{\mathbf{j}l\nu_0 m}$ define the direction and period of the wave

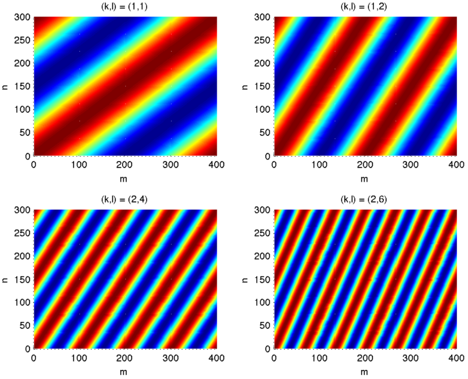


FIGURE 13.5. Four basis vectors $\phi_{k,l}$ for images of size $N = 300$ and $M = 400$: $\phi_{1,1}$, $\phi_{1,2}$, $\phi_{2,4}$, and $\phi_{2,6}$. In the plot the same coloring is used as in Fig. 13.4, i.e., red is close to one and blue is close to zero. The ratio k/l sets the angle of the wave front. Larger k and large l correspond to higher frequencies.

Filtering of two-dimensional signals

- we can now discuss low-pass and high-pass filtering images
  (i) a low-pass filter will smooth the image and blur edges
  (ii) a high-pass filter will select out the edges
- let $0 \leq K \leq N-1$ and $0 \leq L \leq M-1$ and define

$$\mathcal{S}_{\mathrm{LP}} = \mathrm{span}(\{\phi_{k,l}\}_{k=-K,\ldots,K,l=-L,\ldots,L})$$
$$\mathcal{S}_{\mathrm{HP}} = \mathrm{span}(\{\text{all basis vectors } \textbf{not} \text{ in } \mathcal{S}_{\mathrm{LP}}\})$$

- note that the choices of K and L control the dimensions of these subspaces
- to filter, we take an image, use the DTFT to represent the image in the Fourier basis, and then project onto these respective subspaces using the inner product

Example: filtering of two-dimensional signals

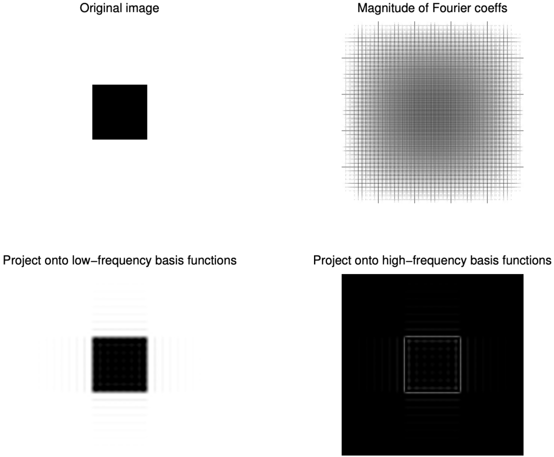


FIGURE 13.6. Example of of a two-dimensional 480×480 box signal that is black (grey-scale value 0) in the center box and white (grey-scale value 255) elsewhere. The magnitudes of the Fourier coefficients are shown, as are the projections onto the low-frequency basis functions ($0 \leq |k| \leq 23, 0 \leq |l| \leq 23$), and onto the balance of basis functions.

Example: filtering of two-dimensional signals

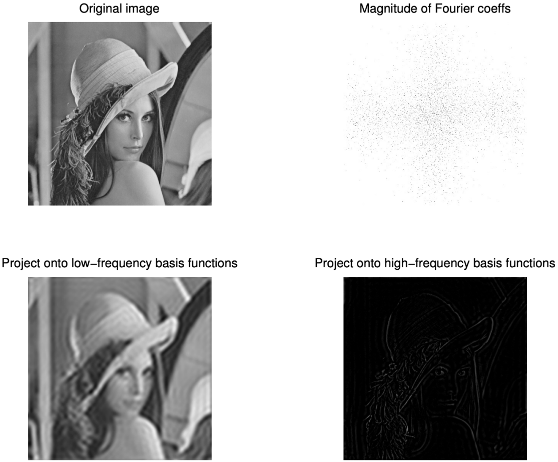


FIGURE 13.7. Example of of a two-dimensional 480×480 image. The magnitudes of the Fourier coefficients are show, as are the projections onto the low-frequency basis functions ($0 \le |k| \le 26, 0 \le |l| \le 26$), and onto the balance of basis functions.