Model-Free Wide-Area Monitoring of Power Grids via Cutset Voltages

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***Abstract*— The so-called cutset voltage has previously been proposed as a model-based circuit reduction technique, and as a method to calculate an aggregate measure of grid stress from wide-area synchrophasor measurements. Here we contribute to the theory of the cutset voltage by showing that it can be written as the difference between the mean voltage levels on either side of the chosen cut, plus an accompanying error term. We then show that this error vanishes under a simple graph-theoretic condition on the weighted degrees of nodes adjacent to the cutset. In this case, the model-based cutset voltage can be computed in a model-free manner by taking wide-area voltage measurements and averaging. We extend our results to the case of a voltage defined across an entire area of the network.**

I. Introduction

The maturity of phasor measurement unit (PMU) technology has opened the door for ubiquitous wide-area monitoring [1]–[3] of power systems, allowing the health of the grid to be assessed in real time. As more PMU units are deployed, the amount of available data also increases. New methods are now needed to extract meaning from this data, and to draw implications from the data to actionable grid control policies. For monitoring purposes, a key problem is the development of simple indicators which quantify the average stress level across the system.

One common stress measure for AC power grids is the difference in voltage phasor angles between adjacent buses. In the linearized DC Power Flow approximation, the real power p_{ij} which flows from bus i to bus j is given by

$$p_{ij} = \frac{1}{X_{ij}}(\theta_i - \theta_j)\,,$$

where X_{ij} is the effective line reactance of the branch joining i and j, and θ_i, θ_j are the voltage phasor angles. Large angle differences indicate large power flows, weak transmission lines, or both. The angular difference across a single line however can be a misleading indicator of the health of the total grid, as this angle difference is affected both negatively and positively by changes in the power injections at all other buses. Moreover, in meshed networks multiple transfer paths through the network may exist between two buses, with the direct path capturing only some of the interaction between the buses. It therefore becomes more useful to think in terms of areas (subsets of buses), and the power transfers between areas. As PMUs deliver only bus-by-bus measurements, techniques are needed to aggregate these measurements in a useful way for the monitoring of areas.

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One technique for aggregating this information, put forward by Dobson *et al.*, is a weighted-average difference of voltage differences between areas. The idea is to use the impedances of inter-area transmission lines to construct weights, and then take a weighted average of the voltage differences using these coefficients. The so-called cutset voltage was defined in [4], which summarizes the results and some extensions. The method has since been applied for line outage detection [5], [6] and voltage collapse monitoring for transmission corridors [7]. We review the relevant definitions in Section II-C.* Two disadvantages of the proposed procedure are (i) a relative lack of theoretical results and (ii) its reliance on effective branch impedances, which are sometimes known only imprecisely.

A. Contributions

Our goal here is to contribute to the theoretical foundations of the cutset voltage. After reviewing some circuit theory and recalling the definition of the cutset voltage in Section II, we proceed in Section III to derive a linear relationship between the cutset voltage and the difference between the (arithmetic) average voltage levels on either side of the defining cut. We show that these quantities are in fact equal, modulo a physically meaningful error term. We then study two special cases where this error term vanishes. First, we show that it vanishes if the voltage profiles on either side of the cut are flat; this result holds independent of the grid topology or impedances. Second, we show that the error term vanishes if the cutset induces an "almost equitable partition" [9] of the weighted graph capturing the topology and impedances of the grid. These partitions have also appeared in the context of model reduction, controllability, and disturbance rejection of consensus dynamics [10]–[12]. If either of these cases hold — or if the error term is otherwise known to be small — our result shows that the model-based cutset voltage can be computed in a model-free way by averaging voltage measurements. The analysis exploits tools from circuit and graph theory, along with model reduction ideas [10].

Here we present our results in terms of classic resistive circuits with DC voltages and currents, but the results apply to any set of "through" and "across" variables satisfying Kirchhoff's Current Law (KCL), Kirchhoff's Voltage Law (KVL), and Ohm's Law. In particular, the results apply to the DC Power Flow approximation $P = B\theta$ where P and θ are vectors of active power injections and voltage phases, and B is the susceptance matrix. The results can also be applied to the AC current balance relations $I = YV$, where

*See also [8] for a related averaging procedure.

I and V are vectors of complex nodal currents and voltages and Y is the complex admittance matrix.

B. Notation

For a set $\mathcal{S}$, $|\mathcal{S}|$ denotes its cardinality. If $\mathcal{T} \subset \mathcal{S}$, then $\mathcal{S} \setminus \mathcal{T} = \{x \in \mathcal{S} \;:\; x \notin \mathcal{T}\}$. For $A \in \mathbb{R}^{n\times n}$, A^{T} is its transpose. The $n \times n$ identity matrix is $\mathbb{I}_n$, $\mathbb{0}$ is a matrix of zeros of appropriate dimension, while $\mathbb{1}_n$ (resp. $\mathbb{0}_n$) are n-vectors of all ones and all zeros, respectively. For a matrix or vector X, $\operatorname{im} X$ denotes its image.

II. Review of Circuits and the Cutset Voltage

A. Graphs, Circuit Laws, and Associated Matrices

Throughout the paper we consider a resistive circuit described by a connected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ where $\mathcal{N}$ is the set of nodes (or buses) and $\mathcal{E}$ is the set of directed edges (or branches, or lines). For simplicity we set $|\mathcal{N}| = n$ and $|\mathcal{E}| = \ell$. To each edge $e \sim (i,j) \in \mathcal{E}$ we associate a branch conductance $g_e = g_{ij} > 0$, quantifying the (inverse) resistance of the edge. The *incidence matrix* $A \in \mathbb{R}^{n\times \ell}$ of $\mathcal{G}$ is defined component-wise as $A_{ke} = 1$ if node k is the source node of edge $e \sim (i,j)$ and as $A_{ke} = -1$ if node k is the sink node of edge e, with all other elements being zero. It follows that for $x \in \mathbb{R}^n$, $A^{\mathsf{T}}x \in \mathbb{R}^{\ell}$ is the vector with entries $x_i - x_j$ for $(i,j) \in \mathcal{E}$. It is sometimes convenient to consider the branches as undirected as well, and in this case we write $\{i,j\} \in \mathcal{E}$.

To each node $i \in \mathcal{N}$ we associate a current injection $I_i \in \mathbb{R}$ and a nodal voltage $V_i \in \mathbb{R}$, while to each edge $e \sim (i,j) \in \mathcal{E}$ we associate an edge current $i_e \in \mathbb{R}$ and a voltage difference $v_e \in \mathbb{R}$. Let $I = (I_1, \ldots, I_n)$, $V = (V_1, \ldots, V_n)$, $i = (i_1, \ldots, i_\ell)$ and $v = (v_1, \ldots, v_\ell)$ be the associated vectors obtained by stacking the components. Then KCL, KVL, and Ohm's Law are written as [13]

$$I = Ai\,, \tag{1a}$$
$$v = A^{\mathsf{T}}V\,, \tag{1b}$$
$$i = \Lambda v\,, \tag{1c}$$

where $\Lambda = \operatorname{diag}(\{g_{ij}\}) \in \mathbb{R}^{\ell\times\ell}$ is the diagonal matrix of branch conductances. Eliminating the branch variables i and v from (1), we obtain the nodal equations

$$I = GV\,, \tag{2}$$

where $G = G^{\mathsf{T}} = A\Lambda A^{\mathsf{T}}$ is the *conductance (Laplacian) matrix* of the graph $\mathcal{G}$, with elements $G_{ij} = -g_{ij}$ for $\{i,j\} \in \mathcal{E}$, $G_{ii} = \sum_{\{i,j\}\in\mathcal{E}} g_{ij}$, and zero otherwise.[†] In particular, it holds that 0 is a simple eigenvalue of G with $G\mathbb{1}_n = \mathbb{0}_n$.

B. Specialization to a Two-Area Circuit

Consider now a cutset[‡] of the circuit, which induces a partitioning of the nodes into two areas $\mathcal{N} = \mathcal{N}_a \cup \mathcal{N}_b$ representing the nodes on either side of the cutset, with $\mathcal{N}_a \cap \mathcal{N}_b = \emptyset$. For notational simplicity, we sometimes denote in short the cardinality of $\mathcal{N}_a$ by a, and that of $\mathcal{N}_b$ by b. The cutset also induces a partitioning of the edge set as $\mathcal{E} = \mathcal{E}^{aa} \cup \mathcal{E}^{ab} \cup \mathcal{E}^{bb}$, where $\mathcal{E}^{aa}$ (resp. $\mathcal{E}^{bb}$) are the edges interior to $\mathcal{N}_a$ (resp. $\mathcal{N}_b$) and $\mathcal{E}^{ab}$ are the edges of the cutset between $\mathcal{N}_a$ and $\mathcal{N}_b$. Without loss of generality, we assume that each directed edge $(i,j) \in \mathcal{E}^{ab}$ originates from $\mathcal{N}_a$, so that edges are oriented from $\mathcal{N}_a$ to $\mathcal{N}_b$. With this, the incidence matrix $A \in \mathbb{R}^{n\times\ell}$ may be partitioned as

$$A = \begin{bmatrix} A_a^{aa} & A_a^{ab} & \mathbb{0} \\ \mathbb{0} & A_b^{ab} & A_b^{bb} \end{bmatrix}\,, \tag{3}$$

where the lower (resp. upper) indices indicate the number of rows (resp. columns) of the submatrix. For example, $A_a^{aa} \in \mathbb{R}^{|\mathcal{N}_a|\times|\mathcal{E}^{aa}|}$ describes the interior connections between nodes in $\mathcal{N}_a$, while A_a^{ab} and A_b^{ab} describe inter-area connections for the source and sink ends, respectively. We will use a similar labeling convention for vectors in what follows. With this partitioning and notation, we have that

$$I = \begin{bmatrix} I_a \\ I_b \end{bmatrix}, \quad V = \begin{bmatrix} V_a \\ V_b \end{bmatrix}, \quad i = \begin{bmatrix} i_{aa} \\ i_{ab} \\ i_{bb} \end{bmatrix}, \quad v = \begin{bmatrix} v_{aa} \\ v_{ab} \\ v_{bb} \end{bmatrix},$$

along with $\Lambda = \operatorname{blkdiag}(\Lambda_{aa}, \Lambda_{ab}, \Lambda_{bb})$, where all subvectors and submatrices are defined in the obvious way by areas and edge types. The conductance matrix takes the form

$$G = \begin{bmatrix} G_{aa} & G_{ab} \\ G_{ba} & G_{bb} \end{bmatrix}\,, \tag{4}$$

where the various submatrices are given in Lemma A.2, and the nodal equations (2) become

$$\begin{bmatrix} I_a \\ I_b \end{bmatrix} = \begin{bmatrix} G_{aa} & G_{ab} \\ G_{ba} & G_{bb} \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix}. \tag{5}$$

C. Definition of the Cutset Voltage

With the previous notations, the cutset voltage from [4] becomes extremely simple to define. The net current i_{cut} which flows across the cutset from $\mathcal{N}_a$ to $\mathcal{N}_b$ is given by

$$i_{\mathrm{cut}} = \mathbb{1}_{ab}^{\mathsf{T}} i_{ab} = \sum\nolimits_{e\sim(i,j)\in\mathcal{E}^{ab}} i_e\,. \tag{6}$$

where $\mathbb{1}_{ab}$ stands for $\mathbb{1}_{|\mathcal{E}^{ab}|}$. We define the *cutset conductance* g_{cut} by

$$g_{\mathrm{cut}} \triangleq \mathbb{1}_{ab}^{\mathsf{T}}\Lambda_{ab}\mathbb{1}_{ab} = \sum\nolimits_{(i,j)\in\mathcal{E}^{ab}} g_{ij}\,, \tag{7}$$

and finally define the *cutset voltage* as the ratio

$$v_{\mathrm{cut}} \triangleq i_{\mathrm{cut}}/g_{\mathrm{cut}}\,. \tag{8}$$

Note that *by definition*, the cutset voltage satisfies the Ohm's Law-type relationship (8). Ostensibly then, v_{cut} will change in a manner consistent with the intuition of system operators: as the net current flow from $\mathcal{N}_a$ to $\mathcal{N}_b$ increases, or as the total "parallel" conductance between the areas decreases, the cutset voltage increases. The cutset voltage is a model-based indicator, in that it requires values for the branch conductances. These branch conductances (or in the AC case, susceptances) are sometimes poorly known however, which

[†]For notational simplicity we ignore shunt elements, but they can be included with relatively few modifications, see [4, Appendix B].

[‡]A *cutset* of a connected graph $\mathcal{G}$ is a set of edges $\mathcal{E}_{\mathrm{cut}} \subset \mathcal{E}$ such that the graph $(\mathcal{N}, \mathcal{E}\setminus\mathcal{E}_{\mathrm{cut}})$ obtained by removing the cutset edges is disconnected.

will tend to introduce error into the calculation of $v_{\rm cut}$; we will return to this point in Section III.

Finally, from (1b)–(1c) it holds for any $e \sim (i,j) \in \mathcal{E}$ that $i_e = g_e(V_i - V_j)$. It follows that (8) equals

$$v_{\rm cut} = \sum\nolimits_{(i,j)\in\mathcal{E}^{ab}} \frac{g_{ij}}{g_{\rm cut}}(V_i - V_j)\,. \tag{9}$$

This shows that the cutset voltage is a particular weighted average of voltage differences across lines in the cutset, with the weights given by $g_{ij}/g_{\rm cut}$.

*Remark 1 (***Cutset Selection***):* We have assumed in the previous calculations that a cutset has been chosen for monitoring. By its definition in (8), the cutset voltage $v_{\rm cut}$ is independent of intra-area current flows, and *only* attempts to measure an aggregate inter-area stress across the cut. In particular, the cutset voltage (8) is proportional to the *net* current flow across the cut. It follows that for $v_{\rm cut}$ to provide a useful indication of the stress across the cut, *cutsets must be selected such that current flows primarily in one direction across the cut.*[§] We assume that this is the case in what follows. In practice, the presence or planned presence of PMUs in a network will determine the cutsets (or in later extensions, areas) which can potentially be monitored, and a procedure for placing sensors and selecting optimal monitoring cuts remains as an open problem.

A large cutset voltage will always indicate a stressed cutset, and therefore a stressed overall grid. We caution against drawing the converse implication however, and erroneously concluding based on a small cutset voltage that the overall grid is lightly stressed. Drawing this implication requires the additional knowledge that *intra*-area current flows are small, or equivalently that voltages on either side of the cutset are fairly uniform. This is an additional criteria which the cutset voltage does not attempt to capture. □

III. Main Results: Average Voltage Differences and the Cutset Voltage

The cutset voltage (8) is a definition based on intuition from circuit theory. In this section we will formally relate this definition to a decomposition of the circuit equations (5). To begin our analysis, observe that the cutset conductance (7) is the parallel combination of the conductances of the cutset lines. This combination procedure can be thought of as an approximation, which assumes that the cutset lines are *actually* in parallel. For example, this is true if all nodes within area $\mathcal{N}_a$ (resp. $\mathcal{N}_b$) are at the same potential. In this case, the reduction procedure is exact, and the cutset voltage $v_{\rm cut}$ equals the (uniform) voltage difference along every edge $(i,j) \in \mathcal{E}^{ab}$. In other words, the network effectively contracts down to two nodes with multiple parallel connections. With these ideas in mind, we decompose the vector of nodal voltages as

$$\begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} \mu_a \mathbb{1}_a \\ \mu_b \mathbb{1}_b \end{bmatrix} + \begin{bmatrix} \widetilde{V}_a \\ \widetilde{V}_b \end{bmatrix}, \tag{10}$$

where $\mu_a, \mu_b > 0$ are scalars and $\widetilde{V}_a$, $\widetilde{V}_b$ are vectors which satisfy $\mathbb{1}_a^{\mathsf{T}}\widetilde{V}_a = \mathbb{1}_b^{\mathsf{T}}\widetilde{V}_b = 0$. That is, in each area, we break the subvector of nodal voltages into two pieces: a uniform profile at voltage μ and a vector $\widetilde{V}$ in the subspace orthogonal to to the image of $\mathbb{1}$. This decomposition is unique, in that given V_a and V_b we can determine μ and $\widetilde{V}$ uniquely via

$$\mu_a = \frac{\mathbb{1}_a^{\mathsf{T}} V_a}{|\mathcal{N}_a|}\,, \quad \widetilde{V}_a = \Pi_a V_a \tag{11a}$$

$$\mu_b = \frac{\mathbb{1}_b^{\mathsf{T}} V_b}{|\mathcal{N}_b|}\,, \quad \widetilde{V}_b = \Pi_b V_b \tag{11b}$$

where

$$\Pi_a = \mathbb{I}_a - \frac{1}{|\mathcal{N}_a|}\mathbb{1}_a\mathbb{1}_a^{\mathsf{T}}\,, \quad \Pi_b = \mathbb{I}_b - \frac{1}{|\mathcal{N}_b|}\mathbb{1}_b\mathbb{1}_b^{\mathsf{T}}\,, \tag{12}$$

are the projection matrices onto the subspaces orthogonal to $\mathbb{1}_a$ and $\mathbb{1}_b$, respectively. In particular, μ_a and μ_b are the (arithmetic) average nodal voltages in areas $\mathcal{N}_a$ and $\mathcal{N}_b$. Substituting the decomposition (10) into the nodal equations (5), we obtain

$$\begin{aligned}
\begin{bmatrix} I_a \\ I_b \end{bmatrix} &= \begin{bmatrix} G_{aa} & G_{ab} \\ G_{ba} & G_{bb} \end{bmatrix} \left(\begin{bmatrix} \mu_a \mathbb{1}_a \\ \mu_b \mathbb{1}_b \end{bmatrix} + \begin{bmatrix} \widetilde{V}_a \\ \widetilde{V}_b \end{bmatrix} \right) \\
&= \begin{bmatrix} \mu_a G_{aa}\mathbb{1}_a + \mu_b G_{ab}\mathbb{1}_b \\ \mu_a G_{ba}\mathbb{1}_a + \mu_b G_{bb}\mathbb{1}_b \end{bmatrix} + \begin{bmatrix} G_{aa} & G_{ab} \\ G_{ba} & G_{bb} \end{bmatrix} \begin{bmatrix} \widetilde{V}_a \\ \widetilde{V}_b \end{bmatrix} \\
&= \begin{bmatrix} (\mu_b - \mu_a)G_{ab}\mathbb{1}_b \\ -(\mu_b - \mu_a)G_{ba}\mathbb{1}_a \end{bmatrix} + \begin{bmatrix} G_{aa} & G_{ab} \\ G_{ba} & G_{bb} \end{bmatrix} \begin{bmatrix} \widetilde{V}_a \\ \widetilde{V}_b \end{bmatrix}
\end{aligned} \tag{13}$$

where we have used the facts that $G_{aa}\mathbb{1}_a = -G_{ab}\mathbb{1}_b$ and that $G_{bb}\mathbb{1}_b = -G_{ba}\mathbb{1}_a$.[¶] Substituting KCL (1a) into the left-hand side of (13) and using the incidence matrix (3), we obtain

$$\begin{bmatrix} A_a^{aa} i_{aa} + A_a^{ab} i_{ab} \\ A_b^{ab} i_{ab} + A_b^{bb} i_{bb} \end{bmatrix} = \begin{bmatrix} (\mu_b - \mu_a)G_{ab}\mathbb{1}_b \\ -(\mu_b - \mu_a)G_{ba}\mathbb{1}_a \end{bmatrix} + G \begin{bmatrix} \widetilde{V}_a \\ \widetilde{V}_b \end{bmatrix}. \tag{14}$$

Left-multiplying (14) by $(\mathbb{1}_a^{\mathsf{T}}, \mathbb{1}_b^{\mathsf{T}})$, using Lemma A.1 (i) and (iii), and noting (using Lemma A.1 and A.2) that $\mathbb{1}_a^{\mathsf{T}} G_{ab}\mathbb{1}_b = \mathbb{1}_b^{\mathsf{T}} G_{ba}\mathbb{1}_a = -g_{\rm cut}$, the previous two equations reduce to

$$\begin{aligned}
i_{\rm cut} &= g_{\rm cut}(\mu_a - \mu_b) + \mathbb{1}_a^{\mathsf{T}}(G_{aa}\widetilde{V}_a + G_{ab}\widetilde{V}_b)\,, \\
-i_{\rm cut} &= -g_{\rm cut}(\mu_b - \mu_a) + \mathbb{1}_b^{\mathsf{T}}(G_{ba}\widetilde{V}_a + G_{bb}\widetilde{V}_b)\,.
\end{aligned}$$

These equations are in fact redundant, so we discard the second. Dividing the first by $g_{\rm cut}$ and using the definition of $v_{\rm cut}$, we find that

$$\begin{aligned}
\mu_a - \mu_b &= v_{\rm cut} - \mathbb{1}_a^{\mathsf{T}}(G_{aa}\widetilde{V}_a + G_{ab}\widetilde{V}_b)/g_{\rm cut} \\
&= v_{\rm cut} + (\mathbb{1}_b^{\mathsf{T}} G_{ba}\widetilde{V}_a - \mathbb{1}_a^{\mathsf{T}} G_{ab}\widetilde{V}_b)/g_{\rm cut} \\
&= v_{\rm cut} + (-g_a^{\mathsf{T}}\widetilde{V}_a + g_b^{\mathsf{T}}\widetilde{V}_b)/g_{\rm cut}\,,
\end{aligned}$$

where

$$g_a \triangleq -G_{ab}\mathbb{1}_b \in \mathbb{R}^a\,, \tag{15a}$$

$$g_b \triangleq -G_{ba}\mathbb{1}_a \in \mathbb{R}^b\,, \tag{15b}$$

[§]Large circulating current flows between the two areas would indicate a stressed grid, but lead to a small net current flow, and therefore a small cutset voltage, so the indicator is not useful in this case.

[¶]This follows from (4) and the fact that $G\mathbb{1}_n = \mathbb{0}_n$.

are the *weighted inter-area degree vectors* for areas $\mathcal{N}_a$ and $\mathcal{N}_b$. That is, $g_a \in \mathbb{R}^a$ is the vector of weighted nodal degrees for area $\mathcal{N}_a$ taking into account *only* edges contained in the cutset, and similarly for $g_b \in \mathbb{R}^b$. Using the inverse relations (11), we may go further and write

$$\begin{aligned}\mu_a - \mu_b &= v_{\text{cut}} + \frac{1}{g_{\text{cut}}}\begin{bmatrix}-g_a^{\mathsf{T}}\Pi_a & g_b^{\mathsf{T}}\Pi_b\end{bmatrix}\begin{bmatrix}V_a \\ V_b\end{bmatrix} \\ &= v_{\text{cut}} - \varepsilon_{\text{cut}}\,,\end{aligned}$$

where

$$\boxed{\varepsilon_{\text{cut}} \triangleq \frac{1}{g_{\text{cut}}}\begin{bmatrix}g_a^{\mathsf{T}}\Pi_a & -g_b^{\mathsf{T}}\Pi_b\end{bmatrix}\begin{bmatrix}V_a \\ V_b\end{bmatrix}\,,} \tag{16}$$

is the *cutset voltage error*. We have therefore proved the following result.

*Theorem 3.1 (***Cutset Voltage and Average Voltages***):* Consider the two-area circuit described by the nodal equations (5), and let μ_a and μ_b be the mean voltages in areas $\mathcal{N}_a$ and $\mathcal{N}_b$, respectively. Let the weighted inter-area degree vectors be as in (15), and the cutset voltage be as defined in (8). Then

$$\boxed{v_{\text{cut}} = \mu_a - \mu_b \,+\, \varepsilon_{\text{cut}}\,,} \tag{17}$$

where the cutset voltage error is as in (16).

From equation (17), we see that the cutset voltage equals the *difference in average nodal voltages* across the cutset, plus an error term ε_{cut} for which we have obtained an explicit formula. While the cutset voltage (8) is relatively straightforward to compute, it requires not only voltage measurements but also knowledge of the branch conductances g_{ij}. Exact knowledge of these parameters may prove problematic, especially in the DC power flow context where effective branch susceptances are calculated based on numerics. In contrast, (17) shows that the cutset voltage can be computed with only voltage measurements if the error term is somehow known to be sufficiently small. The next result characterizes some cases when the cutset voltage error term is in fact exactly zero. We first require a definition.

*Definition 1 (***Almost Equitable Partitions, [9]***):* For an integer $K \geq 2$, let $\pi \triangleq \{\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_K\}$ be a partition of a weighted undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$. For any node $i \in \mathcal{N}$ and any area $q \in \{1, \ldots, K\}$, let

$$g_{\text{parallel}}(i, \mathcal{N}_q) \triangleq \sum\nolimits_{j \in \mathcal{N}_q, \{i,j\} \in \mathcal{E}} g_{ij}\,,$$

be the total parallel conductance between node $i \in \mathcal{N}$ and area $\mathcal{N}_q$. We call π an *almost equitable partition* (AEP) of $\mathcal{G}$ if for each pair of distinct areas $p, q \in \{1, \ldots, K\}$, there exists a value $g_{pq} \in \mathbb{R}$ such that

$$g_{\text{parallel}}(i, \mathcal{N}_q) = g_{pq}\,,$$

for all nodes $i \in \mathcal{N}_p$.

In other words, a partition is almost equitable if *every* node in each area "sees" the same parallel conductance to the other areas. An example of such a partition for two-area network is shown in Figure 1. Notice that in our two-area network, $g_{\text{parallel}}(i, \mathcal{N}_b)$ is equal to the i^{th} element of g_a. Similar statement holds for $g_{\text{parallel}}(i, \mathcal{N}_a)$ and g_b, which yields the following technical lemma.

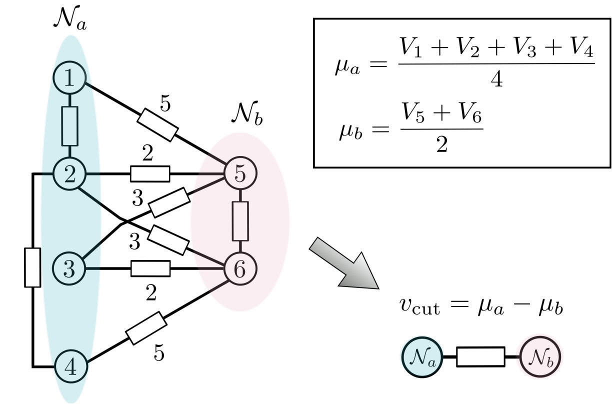

Fig. 1. An almost equitable partition $\pi = \{\{1, 2, 3, 4\}, \{5, 6\}\}$, with the resulting network reduction. In this case, $g_a = (5, 5, 5, 5)$, $g_b = (10, 10)$, and $g_{\text{cut}} = 20$.

*Lemma 3.2 (***AEPs and Inter-Area Degree Vectors***):* Consider the two-area network $(\{\mathcal{N}_a, \mathcal{N}_b\}, \mathcal{E})$ discussed previously, and let the weighted inter-area degree vectors g_a, g_b be as in (15). Then $\pi = \{\mathcal{N}_a, \mathcal{N}_b\}$ is an almost equitable partition of $\mathcal{G}$ if and only if $g_a \in \operatorname{im} \mathbb{1}_a$ and $g_b \in \operatorname{im} \mathbb{1}_b$.

We therefore have the following corollary of Theorem 3.1, which characterizes when the cutset error vanishes.

*Corollary 3.3 (***Zero Cutset Voltage Error***):* The cutset voltage error ε_{cut} vanishes and $v_{\text{cut}} = \mu_a - \mu_b$ if either

(i) *Uniform Area Voltages:* $V_a \in \operatorname{im} \mathbb{1}_a$ and $V_b \in \operatorname{im} \mathbb{1}_b$, or
(ii) *Almost Equitable Partitions:* $\pi = \{\mathcal{N}_a, \mathcal{N}_b\}$ is an almost equitable partition of $\mathcal{G}$.

Proof: In case (i), we compute using (12) that $\Pi_a V_a = \mathbb{0}_a$ and $\Pi_b V_b = \mathbb{0}_b$, so the result follows. Similarly, in case (ii), $g_a^{\mathsf{T}}\Pi_a = \mathbb{0}_a^{\mathsf{T}}$ and $g_b^{\mathsf{T}}\Pi_b = \mathbb{0}_b^{\mathsf{T}}$. ■

Corollary 3.3 (i) can be interpreted as the *strong intra-area coupling limit*, where $\Lambda_{aa}, \Lambda_{bb} \longrightarrow +\infty$. In this strong coupling limit, each area contracts down to a single node. It follows then that $\mu_a - \mu_b = v_{\text{cut}}$ exactly, independent of the exact conductance values or the interconnection pattern between the areas. Conversely, Corollary 3.3 (ii) holds independent of the nodal voltage values and current injections, and depends only on structural and weighting properties of the partition/cutset. While almost equitability is a specialized property that can not be expected to hold in test cases, the result provides insight into how the graph topology and weights influence the cutset voltage. If either condition holds, v_{cut} can easily be computed through centralized or distributed averaging of pure voltage measurements, with no knowledge of branch conductances required. While (9) says to "average the differences", (17) instead allows one to subtract two averages; inter-area computations are therefore largely replaced by intra-area ones.

*Remark 2 (***Extension to Bordering Buses):** From the expression (9), v_{cut} depends only on the voltages on the *borders* of the respective areas. Here, we use the coordinate change (10), where μ denotes the average voltage level of the *entire* area. When all buses are border buses (as in Figure 1) these two situations are equivalent, but in general they are not. Closing this theoretical gap requires refining our analysis by using more complicated block-matrix notation, and is deferred to an extended publication. □

IV. Extension to Voltage Across an Area

We now extend the previous arguments regarding the cutset voltage to the generalization of a "voltage across an area" [4, Section III]. The situation of interest is shown in Figure 2, where now a set of nodes $\mathcal{N}_m$ lies in the middle between the areas $\mathcal{N}_a$ and $\mathcal{N}_b$. We assume that $\mathcal{N}_m$ is a *nodal cutset* of $\mathcal{G}$, meaning that the graph $(\mathcal{V} \setminus \mathcal{N}_m, \mathcal{E}_{\mathrm{node-cut}})$ — obtained by removing the area $\mathcal{N}_m$ and any associated dangling edges — is disconnected. For simplicity, we also assume that the subgraph induced by the area $\mathcal{N}_m$ is connected; these assumptions can be relaxed without much difficulty.

For this problem setup, the incidence matrix takes the form

$$A = \begin{bmatrix} A_a^{aa} & A_a^{am} & \mathbb{0} \\ A_m^{am} & A_m^{mm} & A_m^{bm} \\ \mathbb{0} & A_b^{bm} & A_b^{bb} \end{bmatrix}, \tag{18}$$

with the accompanying nodal equations

$$\begin{bmatrix} I_a \\ I_m \\ I_b \end{bmatrix} = \begin{bmatrix} G_{aa} & G_{am} & \mathbb{0} \\ G_{ma} & G_{mm} & G_{mb} \\ \mathbb{0} & G_{bm} & G_{bb} \end{bmatrix} \begin{bmatrix} V_a \\ V_m \\ V_b \end{bmatrix}. \tag{19}$$

Applying Kron reduction [14] to the middle area $\mathcal{N}_m$, we obtain the reduced representation

$$\begin{bmatrix} I_a^{\mathrm{red}} \\ I_b^{\mathrm{red}} \end{bmatrix} = \begin{bmatrix} G_{aa}^{\mathrm{red}} & G_{ab}^{\mathrm{red}} \\ G_{ba}^{\mathrm{red}} & G_{bb}^{\mathrm{red}} \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix}, \tag{20}$$

where

$$G_{aa}^{\mathrm{red}} \triangleq G_{aa} - G_{am} G_{mm}^{-1} G_{ma} \tag{21a}$$

$$G_{bb}^{\mathrm{red}} \triangleq G_{bb} - G_{bm} G_{mm}^{-1} G_{mb} \tag{21b}$$

$$G_{ab}^{\mathrm{red}} = (G_{ba}^{\mathrm{red}})^{\mathsf{T}} \triangleq -G_{am} G_{mm}^{-1} G_{mb} \tag{21c}$$

$$I_a^{\mathrm{red}} \triangleq I_a - G_{am} G_{mm}^{-1} I_m \tag{21d}$$

$$I_b^{\mathrm{red}} \triangleq I_b - G_{bm} G_{mm}^{-1} I_m \,. \tag{21e}$$

Since by assumption there were no shunt conductances, G is an irreducible Laplacian matrix, and the matrix

$$G_{\mathrm{red}} \triangleq \begin{bmatrix} G_{aa}^{\mathrm{red}} & G_{ab}^{\mathrm{red}} \\ G_{ba}^{\mathrm{red}} & G_{bb}^{\mathrm{red}} \end{bmatrix}$$

inherits this property [14, Lemma II.1]. In particular, this implies that the current balance $\mathbb{1}_a^{\mathsf{T}} I_a^{\mathrm{red}} + \mathbb{1}_b^{\mathsf{T}} I_b^{\mathrm{red}} = 0$ holds for the reduced network. The reduced network described by (20) therefore has the form of the network considered in Section III, which is two areas $\mathcal{N}_a$ and $\mathcal{N}_b$ separated by a cutset of lines. The difference is that the lines of the cutset are now equivalent lines as determined by (21a)–(21c), and the current injections are now the equivalent current injections determined by (21d)–(21e). Let $\mathcal{G}_{\mathrm{red}} = (\mathcal{N}_{\mathrm{red}}, \mathcal{E}_{\mathrm{red}})$ denote the corresponding reduced graph, where‖

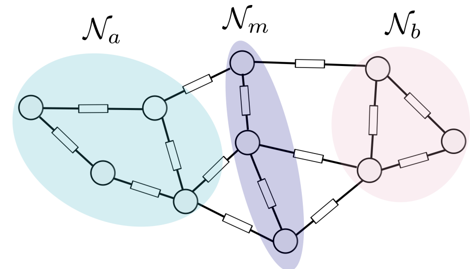

Fig. 2. The three-area case of Section IV, where the middle area $\mathcal{N}_m$ is a nodal cutset between areas $\mathcal{N}_a$ and $\mathcal{N}_b$.

$$\mathcal{N}_{\mathrm{red}} = \mathcal{N}_a \cup \mathcal{N}_b \,,$$
$$\mathcal{E}_{\mathrm{red}} = \mathcal{E}_{\mathrm{red}}^{aa} \cup \mathcal{E}_{\mathrm{red}}^{ab} \cup \mathcal{E}_{\mathrm{red}}^{bb} \,.$$

The current flowing across the cutset from $\mathcal{N}_a$ to $\mathcal{N}_b$ is

$$i_{\mathrm{cut}}^{\mathrm{red}} \triangleq \mathbb{1}_{ab}^{\mathsf{T}} i_{ab}^{\mathrm{red}} \,, \tag{22}$$

where $i^{\mathrm{red}} = (i_{aa}^{\mathrm{red}}, i_{ab}^{\mathrm{red}}, i_{bb}^{\mathrm{red}})$ is a branch current vector satisfying KCL in the reduced network, given by $A_{\mathrm{red}} i^{\mathrm{red}} = I^{\mathrm{red}}$, where A_{red} is the incidence matrix of $\mathcal{G}_{\mathrm{red}}$. With the *conductance of area* $\mathcal{N}_m$ defined as

$$g_{\mathrm{area}} \triangleq -\mathbb{1}_a^{\mathsf{T}} G_{ab}^{\mathrm{red}} \mathbb{1}_b \,, \tag{23}$$

the *voltage across the area* $\mathcal{N}_m$ is defined by

$$v_{\mathrm{area}} \triangleq i_{\mathrm{cut}}^{\mathrm{red}} / g_{\mathrm{area}}. \tag{24}$$

Applying Theorem 3.1 of Section III now yields the following result.

*Theorem 4.1 (***Area Voltage and Mean Voltages):** Consider the three-area circuit described by the nodal equations (19), and let μ_a and μ_b be the mean voltages in areas $\mathcal{N}_a$ and $\mathcal{N}_b$, respectively. Let the reduced quantities be as in (21), with g_{area} and v_{area} as defined in (23),(24), and define the reduced weighted inter-area degree vectors as

$$g_a^{\mathrm{red}} \triangleq -G_{ab}^{\mathrm{red}} \mathbb{1}_b \,, \tag{25a}$$

$$g_b^{\mathrm{red}} \triangleq -G_{ba}^{\mathrm{red}} \mathbb{1}_a \,. \tag{25b}$$

Then

$$v_{\mathrm{area}} = \mu_a - \mu_b - \varepsilon_{\mathrm{area}} \,, \tag{26}$$

where the area voltage error is $\varepsilon_{\mathrm{area}}$ is given by

$$\varepsilon_{\mathrm{area}} \triangleq \frac{1}{g_{\mathrm{area}}} \begin{bmatrix} (g_a^{\mathrm{red}})^{\mathsf{T}} \Pi_a & -(g_b^{\mathrm{red}})^{\mathsf{T}} \Pi_b \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix}. \tag{27}$$

As in Section III, if it is known that the area voltage error $\varepsilon_{\mathrm{area}}$ is small, then the voltage v_{area} across area $\mathcal{N}_m$ can be calculated from pure voltage measurements via (26); surprisingly, these measurements need only come from areas $\mathcal{N}_a$ and $\mathcal{N}_b$, and not from $\mathcal{N}_m$. From Corollary 3.3, we know that $\varepsilon_{\mathrm{area}}$ will in fact vanish if the voltage profiles in areas

‖The exact edges present in $\mathcal{E}_{\mathrm{red}}$ can be determined from the Kron-reduction process; see [14, Theorem III.4] for more information.

$\mathcal{N}_a$ and $\mathcal{N}_b$ are individually uniform, or if $\pi_{\rm red} = \{\mathcal{N}_a, \mathcal{N}_b\}$ is an almost equitable partition of the *reduced* graph $\mathcal{G}_{\rm red}$. We therefore wonder: under what conditions on the original graph $\mathcal{G}$ will $\pi_{\rm red}$ be an almost equitable partition of the reduced graph $\mathcal{G}_{\rm red}$?

*Proposition 4.2 (***AEPs and Kron Reduction***):* Consider the three-area network $\mathcal{G}$ described by the nodal equations (19), and its reduced network $\mathcal{G}_{\rm red}$ described by the reduced nodal equations (20). Then $\pi_{\rm red} = \{\mathcal{N}_a, \mathcal{N}_b\}$ is an almost equitable partition of $\mathcal{G}_{\rm red}$ if $\pi = \{\mathcal{N}_a, \mathcal{N}_m, \mathcal{N}_b\}$ is an almost equitable partition of $\mathcal{G}$.

Proof: Let $\pi = \{\mathcal{N}_a, \mathcal{N}_m, \mathcal{N}_b\}$ be an almost equitable partition of $\mathcal{G}$. By Lemma 3.2 then, it holds that

$$G_{am}\mathbb{1}_m \in \operatorname{im}\mathbb{1}_a\,, \quad G_{bm}\mathbb{1}_m \in \operatorname{im}\mathbb{1}_b$$
$$G_{ma}\mathbb{1}_a \in \operatorname{im}\mathbb{1}_m\,, \quad G_{mb}\mathbb{1}_b \in \operatorname{im}\mathbb{1}_m\,.$$

In addition, since the matrix G has zero row sums, we have

$$G_{aa}\mathbb{1}_a \in \operatorname{im}\mathbb{1}_a\,, \quad G_{bb}\mathbb{1}_b \in \operatorname{im}\mathbb{1}_b, \quad G_{mm}\mathbb{1}_m \in \operatorname{im}\mathbb{1}_m\,.$$

The latter implies that $G_{mm}^{-1}\mathbb{1}_m \in \operatorname{im}\mathbb{1}_m$ as well. Therefore, by (21), we find that

$$G_{ab}^{\rm red}\mathbb{1}_b \in \operatorname{im}\mathbb{1}_a, \quad G_{ba}^{\rm red}\mathbb{1}_a \in \operatorname{im}\mathbb{1}_b.$$

Since from (25), $-G_{ab}^{\rm red}\mathbb{1}_b$ and $-G_{ba}^{\rm red}\mathbb{1}_a$ are the weighted inter-area degree vectors of the reduced graph $\mathcal{G}_{\rm red}$, we conclude from Lemma 3.2 that $\pi_{\rm red} = \{\mathcal{N}_a, \mathcal{N}_b\}$ is an almost equitable partition of $\mathcal{G}_{\rm red}$. ■

Applying Proposition 4.2 yields the following corollary.

*Corollary 4.3 (***Zero Area Voltage Error***):* The area voltage error term $\varepsilon_{\rm area}$ vanishes and $v_{\rm area} = \mu_a - \mu_b$ if

(i) $V_a \in \operatorname{im}\mathbb{1}_a$ and $V_b \in \operatorname{im}\mathbb{1}_b$, or
(ii) $\{\mathcal{N}_a, \mathcal{N}_m, \mathcal{N}_b\}$ is an almost equitable partition of $\mathcal{G}$.

Remark 3: The restriction to the two and three area networks considered here is for clarity of exposition: extensions of the results here can be quickly derived for multi-area networks, at the cost of complicated bookkeeping. The results concerning almost equitablity can also be generalized to more general graph structures using the notion of invariant subspaces; see [10, Lemma 5]. □

V. Conclusions

We have shown that the so-called cutset voltage defined in [4] can be written as the difference between the mean voltage levels on either side of the cut, plus an error term which accounts for the distribution of weighted nodal degrees. We then identified two cases where this error term vanishes, implying the model-dependent cutset voltage can be exactly computed using only voltage measurements. Finally, we extended the results to the case of a voltage across an area.

Future work will explore the refinements mentioned in Remarks 1, 2 and 3. It will also be relevant to examine what can be said when voltage measurements are accessible at only a subset of nodes, and examine the size of the error terms $\varepsilon_{\rm cut}$ and $\varepsilon_{\rm area}$ on standard grid data and cutset selections, such as those in [4]. Another open theoretical question is whether the cutset voltage has a useful variational characterization, which would indicate optimality.

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Appendix
Technical Lemmas

Lemma A.1 (Incidence Matrix): For the incidence matrix (3) the following properties hold:

(i) $\mathbb{1}_a^\top A_a^{aa} = \mathbb{0}_{aa}^\top$
(ii) $\mathbb{1}_b^\top A_b^{bb} = \mathbb{0}_{bb}^\top$
(iii) $\mathbb{1}_a^\top A_a^{ab} = -\mathbb{1}_b^\top A_b^{ab} = \mathbb{1}_{ab}^\top\,.$

Proof: The result follows from the partitioning (3) and the fact that $A^\top \mathbb{1}_n = \mathbb{0}_\ell$. ■

Lemma A.2 (Conductance Matrix Submatrices): The submatrices of the conductance matrix in (5) are given by

$$G_{aa} = A_a^{aa}\Lambda_{aa}(A_a^{aa})^\top + A_a^{ab}\Lambda_{ab}(A_a^{ab})^\top \tag{28a}$$
$$G_{ab} = A_a^{ab}\Lambda_{ab}(A_b^{ab})^\top \tag{28b}$$
$$G_{ba} = A_b^{ab}\Lambda_{ab}(A_a^{ab})^\top \tag{28c}$$
$$G_{bb} = A_b^{bb}\Lambda_{bb}(A_b^{bb})^\top + A_b^{ab}\Lambda_{ab}(A_b^{ab})^\top\,. \tag{28d}$$

Proof: The result follows by direct expansion of $G = A\Lambda A^\top$ using (3). ■