Linear-Convex Optimal Steady-State Control

Liam S. P. Lawrence *Student Member, IEEE*, John W. Simpson-Porco, *Member, IEEE*, and Enrique Mallada *Senior Member, IEEE*

***Abstract*—We consider the problem of designing a feedback controller for a multivariable linear time-invariant system which regulates a system output to the solution of an equality-constrained convex optimization problem despite unknown constant exogenous disturbances; we term this the linear-convex optimal steady-state (OSS) control problem. We introduce the notion of an optimality model, and show that the existence of an optimality model is sufficient to reduce the OSS control problem to a stabilization problem. This yields a constructive design framework for optimal steady-state control that unifies and extends existing design methods in the literature. We illustrate the approach via an application to optimal frequency control of power networks, where our methodology recovers centralized and distributed controllers reported in the recent literature.**

***Index Terms*—Reference tracking and disturbance rejection, output regulation, convex optimization, online optimization**

I. Introduction

Many engineering systems are required to operate at an "optimal" steady-state, specified by the solution of a constrained optimization problem. A traditional approach for achieving this is to compute optimal actuator setpoints offline, using a complete steady-state system model and forecasts of disturbances, and then to feed the resulting setpoints to classical tracking controllers. We refer to this feedforward-type scheduling of setpoints as *offline optimization*. In contrast, one can consider *online optimization* as a feedback counterpart, wherein measurements from the system are processed in real time to inform updates to actuators leading to optimal steady-state operation. Online optimization may therefore be particularly beneficial when a complete offline system model and accurate disturbance forecasts are unavailable.

As an example, consider the problem of optimizing the production setpoints of generators in an electric power system while maintaining supply-demand balance. In the traditional offline approach, optimal generation setpoints are computed in advance using supply and demand forecasts and a complete model of the network. The generation setpoints are then dispatched as reference commands to local controllers at each generation site [1]. This process is repeated periodically, using the best available system model and supply-demand forecasts; a new optimizer is computed, dispatched, and tracked. However, the quality of set-points computed using this offline optimization approach may suffer due to grid model mismatch and demand forecast errors, leading to suboptimal operation and potentially to violation of grid constraints. In an online feedback-based optimization approach, one can expect that the effect of model uncertainty will be mitigated, and that unmeasured disturbances can be rejected. Hence, much recent work in power system control has focused on combining the local controllers at each generation site with a feedback-based online optimization algorithm, so that the optimal operating condition can be tracked in real time [2]–[12].

The same theme of real-time regulation of system variables to optimal values emerges in diverse areas. Fields of application besides the power network control example mentioned already include network congestion management [13], [14], chemical processing [15], wind turbine power capture [16], and temperature regulation in energy-efficient buildings [17]. The breadth of applications motivates the need for a *general* theory and design procedure for controllers that regulate a plant to a maximally efficient operating point defined by an optimization problem, even as the optimizer changes over time due to changing market prices, disturbances to the plant dynamics, and operating constraints that depend on exogenous variables. We refer to the problem of designing such a controller as the *optimal steady-state (OSS) control problem*.

A number of recent publications have formulated problem statements and solutions for variants of the OSS control problem [18]–[27]. Broadly speaking, these design methodologies consist of modifying an off-the-shelf optimization algorithm to accept system measurements; the algorithm then produces a converging estimate of the optimal steady-state control input, yielding a feedback controller. This procedure, while modular, unnecessarily restricts the design space of dynamic controllers. Moreover, none of the reported approaches adequately consider the impact of the system model on the achievable optimal operating points. Our goal in this paper is to address these issues by presenting a framework which widens the design space of optimal steady-state controllers.

Contributions: We consider the *linear-convex OSS control problem*, in which the plant is a finite-dimensional linear time-invariant (LTI) state-space system, the steady-state optimization problem has a convex cost function and affine equality constraints, and the disturbances are constant in time. We introduce the notion of an optimality model, a dynamic filter which reduces the OSS control problem to a nonlinear stabilization problem, which can then be addressed using well-established techniques (e.g., [28]). We provide three explicit designs of optimality models, and highlight how further designs

L. S. P. Lawrence is with the Department of Medical Biophysics, University of Toronto, 101 College Street, Toronto, ON, Canada M5G 1L7. Email: liam.lawrence@mail.utoronto.ca

J. W. Simpson-Porco is with the Department of Electrical and Computer Engineering, University of Toronto, 10 King's College Road, Toronto, ON, M5S 3G4, Canada. Email: jwsimpson@ece.utoronto.ca.

E. Mallada is with the Department of Electrical and Computer Engineering, The Johns Hopkins University, 3400 N. Charles Street, Baltimore, MD 21218, Email: mallada@jhu.edu

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can be obtained. For the specific case of quadratic steady-state optimization problems, we prove that for any of our optimality models, the existence of a stabilizing controller is guaranteed under mild assumptions. Finally, we apply our results to the problem of frequency regulation in power systems, and show that our framework is flexible enough to recover centralized and distributed frequency controllers from the recent literature.

Notation: The symbol $\bullet$ in $\mathbb{R}^\bullet$ indicates that the dimension of the vector space is unspecified. When the arguments of a class C^1 function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are separated by a semicolon, $\nabla f(x; y)$ refers to the gradient of f with respect to its first argument, evaluated at (x, y). The symbol $\mathbb{0}$ denotes a matrix or vector of zeros whose dimensions can be inferred from context. The symbol $\mathbb{1}_n$ denotes the n-vector of all ones. For scalars, vectors, or compatible matrices $\{v_1, v_2, \ldots, v_k\}$, $\mathrm{col}(v_1, v_2, \ldots, v_k)$ is a column matrix obtained by vertical concatenation of $v_1, \ldots, v_k$.

II. Problem Statement

In the linear-convex optimal steady-state control problem, our objective is to design a feedback controller for a linear time-invariant plant so that a specified output is asymptotically driven to a cost-minimizing steady-state, determined by the solution of a convex optimization problem. In contrast to a standard static optimization problem, we must contend with closed-loop stability in addition to optimizing a set of decision variables. The plant is a linear time-invariant system subject to an unknown constant disturbance $w \in \mathbb{R}^{n_w}$

$$\begin{aligned} \dot{x} &= Ax + Bu + B_w w\,, \qquad x(0) \in \mathbb{R}^n\,,\\ y &= Cx + Du + Qw\,,\\ y_{\mathrm{m}} &= h_{\mathrm{m}}(x, u, w). \end{aligned} \tag{1}$$

The measurements y_{m} are permitted to be general nonlinear functions of state, input, and disturbance, to allow for measurement of variables associated with the optimization problem to be presented shortly. The vector $y \in \mathbb{R}^p$ is the *optimization output*, containing states, tracking errors, and control inputs that should be driven to cost-minimizing values in equilibrium.

We will explicitly enforce that the optimization of y is consistent with steady-state operation of the plant. Let $\overline{Y}(w)$ be the set of optimization outputs achievable from a forced equilibrium of (1):

$$\begin{aligned} \overline{Y}(w) := \{\bar{y} \in \mathbb{R}^p \mid & \text{ there exists an } (\bar{x}, \bar{u}) \text{ such that}\\ & \mathbb{0} = A\bar{x} + B\bar{u} + B_w w\\ & \bar{y} = C\bar{x} + D\bar{u} + Qw\}\,. \end{aligned} \tag{2}$$

We rewrite $\overline{Y}(w)$ in algebraic form so that we may include membership in $\overline{Y}(w)$ as a constraint of the optimization problem in standard equality form. For each w, the set $\overline{Y}(w)$ is an affine subspace of $\mathbb{R}^p$. It may therefore be written as the sum of a (non-unique) "offset vector" and a unique subspace, which we denote by $\mathbf{sub}(\overline{Y}(w))$. In the following lemma, we construct a matrix G whose columns span this unique subspace.

Lemma 2.1 (**Construction of G**)**:** Fix a $\tilde{y}(w) \in \overline{Y}(w)$. If $\mathcal{N} \in \mathbb{R}^{(n+m)\times\bullet}$ is a matrix such that $\mathbf{range}\,\mathcal{N} = \mathbf{null}\begin{bmatrix} A & B\end{bmatrix}$, then the columns of the matrix

$$G := \begin{bmatrix} C & D\end{bmatrix}\mathcal{N} \in \mathbb{R}^{p\times\bullet} \tag{3}$$

span the subspace $\mathbf{sub}(\overline{Y}(w))$. △

The proof is straightforward and is omitted. Note that when A is invertible, one may select $\mathcal{N} := \mathrm{col}(-A^{-1}B, I_m)$ which yields $G = -CA^{-1}B + D$. This is precisely the DC gain matrix of the $u \to y$ channel for the plant (1). One may think of G in (3) as a generalization of this, which one can compute regardless of whether or not A is invertible.

For fixed $\tilde{y}(w) \in \overline{Y}(w)$, from Lemma 2.1 it follows that

$$\bar{y} \in \overline{Y}(w) \iff \exists\, v \in \mathbb{R}^\bullet \text{ s.t. } \bar{y} = \tilde{y}(w) + Gv\,. \tag{4}$$

With $l := p - \mathbf{rank}\,G$, let $G_\perp \in \mathbb{R}^{l\times p}$ be any full-row-rank matrix satisfying $\mathbf{null}\,G_\perp = \mathbf{range}\,G$. Then from (4), one finds that

$$\overline{Y}(w) = \{\bar{y} \in \mathbb{R}^p \mid G_\perp \bar{y} = b(w)\}\,. \tag{5}$$

where $b(w) := G_\perp \tilde{y}(w)$. We will see shortly that, for our controller design, the matrix $G_\perp$ is important and the vector $b(w)$ is unimportant.

We can now formulate an optimization problem to determine the desired optimal point for $\bar{y}$ as

$$\begin{aligned} \underset{\bar{y}\in\mathbb{R}^p}{\text{minimize}} \quad & f(\bar{y}; w) && \text{(6a)}\\ \text{subject to} \quad & G_\perp \bar{y} = b(w) && \text{(6b)}\\ & H\bar{y} = Lw. && \text{(6c)} \end{aligned}$$

The cost f in (6) is our steady-state performance criterion. The constraint (6b) is the equilibrium constraint just discussed. Note that if the number of control inputs m is greater than or equal to the number of optimization outputs p, then it is possible that $\overline{Y}(w) = \mathbb{R}^p$ — i.e., the set of achievable equilibrium outputs is the entire space — and the constraint (6b) drops from the problem. In terms of the associated matrices in this case, note that $\mathbf{rank}\,G = p$, so $l = p - \mathbf{rank}\,G = 0$, and hence $G_\perp$ is empty. The constraints (6c) represent n_{ec} engineering equality constraints determined by the matrices $H \in \mathbb{R}^{n_{\mathrm{ec}}\times p}$ and $L \in \mathbb{R}^{n_{\mathrm{ec}}\times n_w}$. We make a set of assumptions concerning the optimization problem.

Assumption 2.2 (**Optimization Problem Assumptions**)**:** For the optimization problem (6), we assume f is differentiable and convex in $\bar{y}$ for each w. We further assume that for every w, the problem (6) has a unique optimizer $\bar{y}^\star$, and a feasible region with non-empty relative interior. △

A general nonlinear feedback controller for (1) is given by

$$\begin{aligned} \dot{x}_{\mathrm{c}} &= f_{\mathrm{c}}(x_{\mathrm{c}}, y_{\mathrm{m}})\,, \qquad x_{\mathrm{c}}(0) \in \mathbb{R}^{n_{\mathrm{c}}}\,,\\ u &= h_{\mathrm{c}}(x_{\mathrm{c}}, y_{\mathrm{m}}). \end{aligned} \tag{7}$$

The function f_{c} is assumed to be locally Lipschitz in x_{c} and continuous in y_{m}, while h_{c} is assumed to be continuous. The dynamics of the closed-loop system consist of (1) and (7).

Our objective in linear-convex OSS control (for brevity, we will omit "linear-convex" in the sequel) is to drive the optimization output y of the plant (1) to the solution $\bar{y}^\star(w)$ of the convex optimization problem (6) using a feedback controller while ensuring well-posedness and stability of the closed-loop system. The formal statement is as follows. For a given w, the closed-loop system is said to be *well-posed* if the control input u is uniquely defined for any choice of $(x, x_c) \in \mathbb{R}^n \times \mathbb{R}^{n_c}$, i.e., the equation $u = h_c(x_c, h_m(x, u, w))$ is uniquely solvable in u.

*Problem 2.3 (***OSS Control***):* For the plant (1), design, if possible, a dynamic feedback controller of the form (7) such that for every w:

(i) the closed-loop system is well-posed;
(ii) the closed-loop system possesses a globally asymptotically stable equilibrium point;
(iii) for every initial condition of the closed-loop system, $\lim_{t\to\infty} y(t) = \bar{y}^\star(w)$. △

*Remark 2.4 (***Constant Disturbances***):* We assume throughout that the unmeasured disturbances w are asymptotically constant, which will lead us to incorporate integral action into our controllers; this is by far the most important case in practice. In reality, unmeasured disturbances (and hence, the optimal operating point) will vary over time, and the quality of tracking of the optimizer in (6) will depend on the rate of variation of the disturbance and on the closed-loop bandwidth. For example, if $\dot{w}$ is bounded, the integral-type controllers we develop will track the optimal operating point with bounded error (see, e.g., [25]). This is acceptable in practice, and we defer study of more detailed disturbance models to future work. △

*Remark 2.5 (***Relation to Optimal Control***):* The OSS control problem appears similar to an infinite-horizon optimal tracking control problem; however, the two are distinct in both their assumptions and demands. In the latter, one minimizes a cost functional over system trajectories leading to a HJB equation; determining the optimal feedback policy is computationally expensive and the policy will require state and disturbance measurements. The OSS control problem is much less demanding; we ask only for optimal behaviour asymptotically, not optimal trajectories. As a result, we encounter no computational bottlenecks, and do not need to assume the full plant state and all disturbances are measurable. △

*Remark 2.6 (***Relation to Extremum-Seeking Control***):* The OSS control problem is similar to the online optimization problems considered in the extremum-seeking control literature, e.g., [29], [30]. Extremum-seeking is a model-free control scheme which optimizes a measured objective in steady-state. A sinusoidal probing signal contained in the control input perturbs the plant, allowing estimation of the objective gradient, to which integral control is then applied. In contrast, our OSS controllers will assume partial plant information, and incorporate the steady-state sensitivity matrix (3) between control inputs and measured outputs. This additional information (which can be estimated from step response experiments, e.g. [31]) allows for direct evaluation of the objective function gradient, without the use of a probing signal. Our OSS controllers will therefore be distinct from (and complementary to) those derived via extremum seeking. Indeed, as we will see in Section IV, our framework recovers recent optimizing controllers developed for power system applications which are not based on extremum seeking. △

Under the assumptions on the optimization problem (6), the *Karush-Kuhn-Tucker (KKT) conditions* are necessary and sufficient for optimality [32, Sections 5.2.3 and 5.5.3]. For each w, the optimal solution $\bar{y}^\star \in \mathbb{R}^p$ is characterized as the unique vector such that $\bar{y}^\star$ is feasible for (6) and there exist $\lambda^\star \in \mathbb{R}^r, \mu^\star \in \mathbb{R}^{n_{ec}}$ such that $(\bar{y}^\star, \lambda^\star, \mu^\star)$ satisfies the *gradient condition*

$$\mathbb{0} = \nabla f(\bar{y}^\star; w) + G_\perp^\mathsf{T} \lambda^\star + H^\mathsf{T} \mu^\star \,. \tag{8}$$

III. OSS Controller Design Framework

The main difficulty in solving the OSS control problem is that the optimizer $\bar{y}^\star(w)$ is unknown, and thus the optimality error $y - \bar{y}^\star(w)$ cannot be directly computed. In our design framework, we propose using a dynamic filter called an *optimality model* to convert the OSS control problem to a related *output regulation* problem. One then solves this output regulation problem using an integral controller and a stabilizing controller. An optimality model therefore reduces the OSS control problem to a stabilization problem. For background on output regulation and integral controllers, see [33] and [34, Section 12.3].

A. Optimality Models and Reduction to Stabilization Problem

An optimality model is a filter applied to the measured output y_m of the plant that produces a signal ϵ which acts as a *proxy* for the optimality error $y - \bar{y}^\star(w)$. To make this idea precise, consider a filter (φ, h_ϵ) with state $\xi \in \mathbb{R}^{n_\xi}$, input y_m, output $\epsilon \in \mathbb{R}^{n_\epsilon}$, and dynamics

$$\dot{\xi} = \varphi(\xi, y_m)\,, \quad \epsilon = h_\epsilon(\xi, y_m). \tag{9}$$

*Definition 3.1 (***Optimality Model***):* The filter (9) is said to be an *optimality model* (for the OSS control problem, Problem 2.3) if the following implication holds: if the triple $(\bar{x}, \bar{\xi}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^m$ satisfies

$$\mathbb{0} = A\bar{x} + B\bar{u} + B_w w\,, \qquad \begin{aligned} \mathbb{0} &= \varphi(\bar{\xi}, h_m(\bar{x}, \bar{u}, w)) \\ \mathbb{0} &= h_\epsilon(\bar{\xi}, h_m(\bar{x}, \bar{u}, w)) \end{aligned} \tag{10}$$

then the pair $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfies $\bar{y}^\star(w) = C\bar{x} + D\bar{u} + Qw$. △

In the OSS control framework, the optimality model is cascaded with the plant, and we then attempt to solve the (constant disturbance) output regulation problem with ϵ as the (measurable) error signal. This converts the OSS control problem to stabilization of the *augmented plant*

$$\dot{x} = Ax + Bu + B_w w\,, \tag{11a}$$
$$\dot{\xi} = \varphi(\xi, h_m(x, u, w))\,, \tag{11b}$$
$$\dot{\eta} = \epsilon := h_\epsilon(\xi, h_m(x, u, w)) \tag{11c}$$

using a *stabilizer*

$$\dot{x}_s = f_s(x_s, \eta, \xi, y_m, \epsilon)\,, \tag{12a}$$
$$u = h_s(x_s, \eta, \xi, y_m, \epsilon). \tag{12b}$$

This design framework (Figure 1) is justified by the following theorem, a proof of which may be found in the appendix.

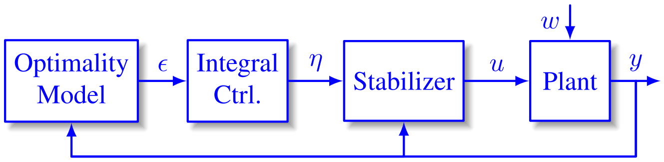

Fig. 1: Block diagram of OSS control architecture.

*Theorem 3.2 (***Reduction of OSS to Stabilization):** Suppose that (φ, h_ϵ) is an optimality model. If the stabilizer $(f_\mathrm{s}, h_\mathrm{s})$ is designed such that the closed-loop system (11)–(12) is well-posed and possesses a globally asymptotically stable equilibrium point for every w, then the controller (11b), (11c), (12a), (12b) solves the OSS control problem. △

Solving the OSS control problem therefore amounts to (i) designing an optimality model and (ii) designing (if possible) a stabilizer for the augmented plant.

B. Optimality Model Design

1) The Gradient Condition: According to Definition 3.1, an optimality model encodes sufficient conditions for optimality when it is in equilibrium with the plant and its output ϵ is held at zero. We can incorporate the KKT conditions — which are sufficient for optimality under our assumptions — into an optimality model for this purpose.

Note that the gradient condition (8) involves the dual variable $\lambda^\star$ associated with the equilibrium constraints. Dual variables associated with equality constraints are typically calculated using an integrator on the constraint violation: see [19, Equation (4a)], or [22, Equation (8f)], for example. Unfortunately integrating the equilibrium constraint violation $G_\perp y - b(w)$ is impossible, since w is unknown. Luckily, doing so is also unnecessary, since the constraint $G_\perp y = b(w)$ is satisfied at any forced equilibrium point of the physical plant, by the definition of $G_\perp$ and $b(w)$; recall (2) and (5). We now describe how to incorporate the gradient condition (8) into an optimality model without calculating $\lambda^\star$.

Let G be the matrix of Lemma 2.1. Recall that $\mathbf{null}\, G_\perp = \mathbf{range}\, G$; by taking the orthogonal complement of both sides, it follows that $\mathbf{range}\, G_\perp^\mathsf{T} = \mathbf{null}\, G^\mathsf{T}$. Rearranging (8) to read $G_\perp^\mathsf{T}(-\lambda^\star) = \nabla f(\bar{y}^\star; w) + H^\mathsf{T}\mu^\star$, we see that $\nabla f(\bar{y}^\star; w) + H^\mathsf{T}\mu^\star \in \mathbf{range}\, G_\perp^\mathsf{T} = \mathbf{null}\, G^\mathsf{T}$. That is, the existence of a triple $(\bar{y}^\star, \lambda^\star, \mu^\star)$ satisfying (8) is equivalent to the existence of a pair $(\bar{y}^\star, \mu^\star)$ satisfying

$$G^\mathsf{T}\left(\nabla f(\bar{y}^\star; w) + H^\mathsf{T}\mu^\star\right) = \mathbb{0}. \tag{13}$$

The left-hand side of this equation is a natural choice for inclusion in the proxy error signal ϵ, since driving ϵ to zero will then enforce the gradient KKT condition.

Note however that there are other equivalent ways we could rewrite the gradient condition. Define a matrix T such that

$$\mathbf{range}\, T = \mathbf{null} \begin{bmatrix} G_\perp \\ H \end{bmatrix}. \tag{14}$$

We have $\mathbf{null}\, T^\mathsf{T} = (\mathbf{range}\, T)^\perp$, which means

$$\mathbf{null}\, T^\mathsf{T} = \left(\mathbf{null} \begin{bmatrix} G_\perp \\ H \end{bmatrix}\right)^\perp = \mathbf{range} \begin{bmatrix} G_\perp^\mathsf{T} & H^\mathsf{T} \end{bmatrix}.$$

Rearrange (8) to read $G_\perp^\mathsf{T}(-\lambda^\star) + H^\mathsf{T}(-\mu^\star) = \nabla f(\bar{y}^\star, w)$, and we see that $\nabla f(\bar{y}^\star, w) \in \mathbf{range} \begin{bmatrix} G_\perp^\mathsf{T} & H^\mathsf{T} \end{bmatrix} = \mathbf{null}\, T^\mathsf{T}$. Therefore, the existence of a triple $(\bar{y}^\star, \lambda^\star, \mu^\star)$ satisfying (8) is equivalent to the existence of a $\bar{y}^\star$ satisfying

$$T^\mathsf{T}\nabla f(\bar{y}^\star; w) = \mathbb{0}. \tag{15}$$

As we did with (13), we can make the expression on the left-hand side of (15) one of the components of an optimality model's error output. The above construction leading to (15) can also be generalized by including only *some* rows of H in the construction of T. For example, writing H and $\mu^\star$ in block form as $H = \mathrm{col}(H_1, H_2)$ and $\mu^\star = \mathrm{col}(\mu_1^\star, \mu_2^\star)$, we can select T_1 such that $\mathbf{range}\, T_1 = \mathbf{null} \left[\begin{smallmatrix} G_\perp \\ H_1 \end{smallmatrix}\right]$ and write the gradient condition as

$$T_1^\mathsf{T}(\nabla f(\bar{y}^\star; w) + H_2^\mathsf{T}\mu_2^\star) = \mathbb{0}\,. \tag{16}$$

2) Optimality Models: We are now ready to construct optimality models for OSS control. The following three propositions present the *output subspace*, *feasible subspace*, and *reduced-error feasible subspace* optimality models. Proving that these filters are indeed optimality models is done by examining the closed-loop equilibria and showing that the resulting equations are equivalent to the KKT conditions; see the extended online version for details [35].

*Proposition 3.3 (***Output Subspace Optimality Model (OS-OM)):** Let G be the matrix of Lemma 2.1. The dynamic filter

$$\begin{aligned} \dot{\mu} &= Hy - Lw \\ \epsilon &= G^\mathsf{T}\left(\nabla f(y; w) + H^\mathsf{T}\mu\right) \end{aligned} \tag{17}$$

is an optimality model for the OSS control problem. △

*Proposition 3.4 (***Feasible Subspace Optimality Model (FS-OM)):** Let T be a matrix satisfying (14). The static filter

$$\epsilon = \begin{bmatrix} Hy - Lw \\ T^\mathsf{T}\nabla f(y; w) \end{bmatrix} \tag{18}$$

is an optimality model for the OSS control problem. △

The alternative KKT condition (16) can also be incorporated into an optimality model, leading to a hybrid between the OS-OM and the FS-OM; the details are omitted. In special circumstances, one can modify the FS-OM above to obtain an optimality model with an error signal of reduced dimension: this reduces the number of integrators required.

*Proposition 3.5 (***Reduced-Error FS-OM (REFS-OM)):** Let G be the matrix of Lemma 2.1 and let T be a matrix satisfying (14). Then the static filter

$$\epsilon = Hy - Lw + T^\mathsf{T}\nabla f(y; w) \tag{19}$$

is an optimality model for the OSS control problem if $\mathbf{range}\, HG \cap \mathbf{range}\, T^\mathsf{T} = \{\mathbb{0}\}$. △

To implement any of these optimality models, the measurement vector y_m must contain the right-hand side of the filter. We assume this is the case for the remainder of the paper. Finally, we note that the only plant model information embedded in the optimality models (17)–(19) comes through the generalized DC gain matrix G. Put differently, the construction of an optimality model requires only knowledge of the *steady-state sensitivity* relationship between control inputs and measured outputs; this is consistent with related studies on feedback-based optimization [10], [24], [25]. In particular, and in contrast to offline optimization, no model is required of the mapping between disturbances and measured outputs.

C. Quadratic Program OSS Control

We now consider the specific case when the optimization problem (6) is an equality-constrained convex quadratic program (QP). We term this variant of the problem QP-OSS control. Under this assumption, the closed-loop system becomes LTI, and we can obtain explicit results on the existence of a stabilizer (Figure 1). Suppose the optimization problem (6) is of the form

$$\begin{aligned} \underset{\bar{y}\in\mathbb{R}^p}{\text{minimize}} \quad & \tfrac{1}{2}\bar{y}^\mathsf{T}\bar{M}\bar{y} - \bar{y}^\mathsf{T}Nw \\ \text{subject to} \quad & G_\perp \bar{y} = b(w) \\ & H\bar{y} = Lw, \end{aligned} \tag{20}$$

where $M \succeq \mathbb{0}$.[1] Since the gradient of the cost function is linear in this case, we can take the available measurements y_m as a linear function of (x, u, w), i.e., $y_\mathrm{m} = C_\mathrm{m}x + D_\mathrm{m}u + Q_\mathrm{m}w$.

Under mild assumptions below, we can ensure that the augmented plant (11) arising from the FS-OM, OS-OM, or REFS-OM is both stabilizable and detectable, which in turn guarantees that a solution of the OSS control problem exists and can be found using standard LTI design methods.

*Theorem 3.6 (***Solvability of QP-OSS Control***):* The QP-OSS control problem is solvable when

(i) (C_m, A, B) is stabilizable and detectable,
(ii) a unique primal solution to (20) exists, and
(iii) at least one of the following holds:
  a) a unique dual solution to (20) exists;
  b) $\mathbf{range}\, HG \cap \mathbf{range}\, T^\mathsf{T} = \{\mathbb{0}\}$ and $(\mathbf{range}\, HG)^\perp \cap (\mathbf{range}\, T^\mathsf{T})^\perp = \{\mathbb{0}\}$. △

Proof: When (i), (ii), and (iii)(a) hold, one can show that the augmented plant arising from the FS-OM or OS-OM may be made stabilizable and detectable. When (i), (ii), and (iii)(b) hold, one can show that the augmented plant arising from the REFS-OM is stabilizable and detectable. A proof of the first case for the FS-OM may be found in the appendix; the remaining proofs may be found in the online version [35]. □

We close this section by making explicit a simple design procedure for OSS controllers, motivated by Figure 1.

(i) Generate optimality models, such as those given in (17)–(19), or other hybrid optimality models associated with the generalized KKT conditions (16).
(ii) Select one of the candidate optimality models, and design a stabilizer which renders the closed-loop system internally stable, possibly while minimizing a desired dynamic performance criteria (e.g., [28]).
(iii) If closed-loop performance is unsatisfactory, repeat step (ii) with a different optimality model.

[1] Any constant term of the form $\bar{y}^\mathsf{T}c$ with $c \in \mathbb{R}^p$ may be included in the term $\bar{y}^\mathsf{T}Nw$ by appropriate redefinition of N and w.

Regarding step (ii) in the above procedure, Theorem 3.6 covers the case of quadratic OSS cost functions f, for each of the optimality models (17)–(19). Due to space limitations, here we do not study the case of generic convex functions f, which result in nonlinear stabilization problems. One approach to such problems is to apply robust synthesis techniques [28], as ∇f will typically satisfy sector-boundedness conditions. Another approach is to leverage recently developed results on low-gain integral control [36]; this will be a topic of future work.

IV. Case Study: Optimal Frequency Regulation in Power Systems

This final section illustrates the application of our theory to a power system control problem. Our main objective is to work through the constructions presented in Section III, and to simultaneously illustrate the many sources of design flexibility within our proposed framework. In particular, we will show that centralized and distributed frequency controllers proposed in the literature are recoverable as special cases of our framework.

The dynamics of synchronous generators in a connected AC power network with n buses and n_t transmission lines is modelled in a reduced-network framework by the *swing equations*. The vectors of angular frequency (deviations from nominal) $\omega \in \mathbb{R}^n$ and real power flows $p \in \mathbb{R}^{n_\mathrm{t}}$ along the transmission lines obey the dynamic equations

$$M\dot{\omega} = P^\star - D\omega - \mathcal{A}p + u, \qquad \dot{p} = \mathcal{B}\mathcal{A}^\mathsf{T}\omega, \tag{21}$$

in which $M \succ \mathbb{0}$ is the (diagonal) inertia matrix, $D \succ \mathbb{0}$ is the (diagonal) damping matrix, $\mathcal{A} \in \{0, 1, -1\}^{n\times n_\mathrm{t}}$ is the signed node-edge incidence matrix of the network, $\mathcal{B} \succ \mathbb{0}$ is the diagonal matrix of transmission line susceptances, $P^\star \in \mathbb{R}^n$ is the vector of uncontrolled power injections (generation minus demand) at the buses, and $u \in \mathbb{R}^n$ is the controllable reserve power produced by the generators. The incidence matrix satisfies $\mathbf{null}\, \mathcal{A}^\mathsf{T} = \mathrm{span}(\mathbb{1}_n)$, and strictly for simplicity we assume that the network is acyclic, in which case $n_\mathrm{t} = n-1$ and $\mathbf{null}\, \mathcal{A} = \{\mathbb{0}\}$. We refer to [37, Section VII] for a first-principles derivation of this model, and remark that our calculations to follow extend without issue to more complex models which include turbine-governor dynamics.

We consider the *optimal frequency regulation problem* (OFRP), wherein we minimize the total cost $\sum_i J_i(\bar{u}_i)$ of steady-state reserve power production in the system subject to system equilibrium and zero steady-state frequency deviations:

$$\begin{aligned} \underset{\bar{u}\in\mathbb{R}^n, \bar{\omega}\in\mathbb{R}^n}{\text{minimize}} \quad & J(\bar{u}) := \sum\nolimits_{i=1}^n J_i(\bar{u}_i) \\ \text{subject to} \quad & G_\perp \,\mathrm{col}(\bar{u}, \bar{\omega}) = b(w) \\ & F\bar{\omega} = \mathbb{0}. \end{aligned} \tag{22}$$

We shall compute the matrix $G_\perp$ of the equilibrium constraints shortly; the vector $b(w)$ is unimportant for controller design. The matrix F encodes the steady-state frequency constraint. We will specify the requirements on F later in this section.

With state vector $x := \mathrm{col}(\omega, p)$, the dynamics (21) can be put into the standard LTI form (1) with matrices

$$A := \begin{bmatrix} -M^{-1}D & -M^{-1}\mathcal{A} \\ \mathcal{B}\mathcal{A}^\mathsf{T} & \mathbb{0} \end{bmatrix}, \; B = B_w := \begin{bmatrix} M^{-1} \\ \mathbb{0} \end{bmatrix}.$$

We select the optimization output as $y := \mathrm{col}(u, \omega)$, so that

$$C := \begin{bmatrix} \mathbb{0} & \mathbb{0} \\ I_n & \mathbb{0} \end{bmatrix} \quad D := \begin{bmatrix} I_n \\ \mathbb{0} \end{bmatrix}, \tag{23}$$

and we take the measured output as $y_{\mathrm{m}} = \mathrm{col}(u, F\omega)$.

We will demonstrate the use of the feasible subspace and reduced-error feasible subspace optimality models of Propositions 3.4 and 3.5. We begin by constructing the matrix G of Lemma 2.1 and a matrix T satisfying (14). We first construct a matrix $\mathcal{N}$ satisfying $\mathbf{range}\,\mathcal{N} = \mathbf{null}\begin{bmatrix} A & B \end{bmatrix}$. One may verify that choosing

$$\mathcal{N} := \begin{bmatrix} \mathbb{1}_n & \mathbb{0} \\ \mathbb{0} & I_n \\ D\mathbb{1}_n & \mathcal{A} \end{bmatrix} \tag{24}$$

yields the required property. Using (24) and (23), we calculate $G = \begin{bmatrix} C & D \end{bmatrix} \mathcal{N}$ to be

$$G = \begin{bmatrix} D\mathbb{1}_n & \mathcal{A} \\ \mathbb{1}_n & \mathbb{0} \end{bmatrix}.$$

Next, we construct a full-row-rank matrix $G_\perp \in \mathbb{R}^{n \times 2n}$ satisfying $\mathbf{null}\, G_\perp = \mathbf{range}\, G$. We find that selecting

$$G_\perp := \begin{bmatrix} \mathbb{1}_n \mathbb{1}_n^{\mathsf{T}} & -(\mathbb{1}_n^{\mathsf{T}} D \mathbb{1}_n) I_n \end{bmatrix}$$

yields the required property. We identify the matrix H of the engineering equality constraints in (6) for the problem (22) as $H := \begin{bmatrix} \mathbb{0} & F \end{bmatrix}$. Following (14), we select a matrix T satisfying

$$\mathbf{range}\, T = \mathbf{null} \begin{bmatrix} \mathbb{1}_n \mathbb{1}_n^{\mathsf{T}} & -(\mathbb{1}_n^{\mathsf{T}} D \mathbb{1}_n) I_n \\ \mathbb{0} & F \end{bmatrix}. \tag{25}$$

The null space on the right-hand side of (25) is spanned by vectors of the form $\mathrm{col}(v, \mathbb{0})$ where $\mathbb{1}_n^{\mathsf{T}} v = 0$. Inspired by approaches in multi-agent control, we introduce a connected, weighted and directed communication graph $\mathcal{G}_{\mathrm{c}} = (\{1, \ldots, n\}, \mathcal{E}_{\mathrm{c}})$ between the buses, with associated *Laplacian matrix* $L_{\mathrm{c}} \in \mathbb{R}^{n \times n}$. We assume the directed graph $\mathcal{G}_{\mathrm{c}}$ contains a globally reachable node.[2] Under this assumption, we have that $\mathbf{rank}(L_{\mathrm{c}}) = n - 1$ with $\mathbf{null}(L_{\mathrm{c}})$ spanned by $\mathbb{1}_n$. It follows that (25) holds with $T = \begin{bmatrix} L_{\mathrm{c}}^{\mathsf{T}} \\ \mathbb{0} \end{bmatrix}$.

It further holds that L_{c} has a left null space of dimension one spanned by a nonnegative and non-zero vector $\mathsf{w} \in \mathbb{R}^n$. Assuming that F is selected such that $\mathsf{w}^{\mathsf{T}} F \mathbb{1}_n \neq \mathbb{0}$, the range condition of Proposition 3.5 is satisfied, and we may apply the REFS-OM (19) to obtain the optimality model

$$\epsilon = F\omega + L_{\mathrm{c}} \nabla J(u). \tag{26}$$

Therefore, one option for an OSS controller is

$$\dot{\eta} = F\omega + L_{\mathrm{c}} \nabla J(u) \tag{27a}$$

$$u = -K_{\mathrm{p}}\omega - K_{\mathrm{i}}\eta, \tag{27b}$$

where $K_{\mathrm{p}}, K_{\mathrm{i}}$ are gain matrices that should be selected for closed-loop stability/performance. With $F := I_n$, $K_{\mathrm{p}} = \mathbb{0}$ and $K_{\mathrm{i}} = \frac{1}{k} I_n$ for $k > 0$, this design reduces to the *distributed-averaging proportional-integral* (DAPI) frequency control scheme; see [4], [39], [40].

[2]See [38, Chapter 6] for details.

We can obtain several other control schemes by instead applying the FS-OM as our optimality model. Let $F := \mathsf{c}^{\mathsf{T}}$, where c is a vector of convex combination coefficients satisfying $\mathsf{c}_i \geq 0$ and $\sum_{i=1}^n \mathsf{c}_i = 1$. Define $\widetilde{L}_{\mathrm{c}} \in \mathbb{R}^{(n-1) \times n}$ as the matrix obtained by eliminating the first row from L_{c} and set $T := \begin{bmatrix} \widetilde{L}_{\mathrm{c}}^{\mathsf{T}} \\ \mathbb{0} \end{bmatrix}$. This choice of T also satisfies (25). The FS-OM (18) yields the optimality model

$$\epsilon = \begin{bmatrix} \mathsf{c}^{\mathsf{T}} \omega \\ \widetilde{L}_{\mathrm{c}} \nabla J(u) \end{bmatrix}. \tag{28}$$

It follows that one option for an OSS controller is

$$\dot{\eta}_1 = \mathsf{c}^{\mathsf{T}} \omega \tag{29a}$$

$$\dot{\eta}_2 = \widetilde{L}_{\mathrm{c}} \nabla J(u) \tag{29b}$$

$$u = -K_{\mathrm{p}}\omega - K_1\eta_1 - K_2\eta_2. \tag{29c}$$

where again K_{p}, K_1, K_2 are gain matrices. The interpretation of this (novel) controller is that one agent collects frequency measurements and implements the integral control (29a), while the other agents average their marginal costs via (29b).

If the objective function J is a positive definite quadratic, one can use Theorem 3.6 to show that a solution to the present OSS control problem is guaranteed to exist. Specifically, for $F := I_n$, one uses Theorem 3.6 with condition (iii)b, and for $F := \mathsf{c}^{\mathsf{T}}$, one uses Theorem 3.6 with condition (iii)a. Moreover, the augmented plant defined by the use of either (27a) or (29a)-(29b) can be shown to be stabilizable and detectable using the proof of Theorem 3.6.

As a final example, we can recover the "gather-and-broadcast" scheme of [6] from the optimality model (28) as follows. Assume that each J_i is strictly convex, and retain the integral controller (29a). Next, using the fact that $\mathbf{null}\, \widetilde{L}_{\mathrm{c}} = \mathrm{span}(\mathbb{1}_n)$, select the input u to zero the second component of ϵ:

$$\begin{aligned} \widetilde{L}_{\mathrm{c}} \nabla J(u) = \mathbb{0} \quad &\Longleftrightarrow \quad \nabla J(u) = \alpha \mathbb{1}_n \text{ for all } \alpha \in \mathbb{R} \\ &\Longleftrightarrow \quad u = (\nabla J)^{-1}(\alpha \mathbb{1}_n), \text{ for all } \alpha \in \mathbb{R}. \end{aligned}$$

Selecting $\alpha = \eta$ leads to the gather-and-broadcast controller

$$\dot{\eta} = \sum\nolimits_{i=1}^n \mathsf{c}_i \omega_i, \quad u_i = (\nabla J_i)^{-1}(\eta). \tag{30}$$

In summary, several recent frequency control schemes, and the novel scheme (29), can be recovered as special cases of our general control framework. The full potential of our methodology for the design of improved power system control will be an area for future study.

V. Conclusions

We have studied in detail the linear-convex OSS control problem, wherein we design a controller to guide an LTI system to the solution of an optimization problem despite unknown, constant exogenous disturbances. We introduced the idea of an optimality model, the existence of which allows us to reduce the OSS control problem to a stabilization problem. Several candidate filters were presented, which under weak conditions are indeed optimality models, and in Theorem 3.6 we provided natural conditions under which an associated stabilizer exists for any of these optimality models. The possibility of constructing multiple optimality models for a single OSS control problem is a source of flexibility in our framework. This was illustrated through an example arising in power systems control, where we are able to recover several existing controllers from the literature.

Future work will present the analogous discrete-time and sampled-data OSS control problems, along with a more detailed study of applications in power system control. A large number of open problems and directions exist, including but not limited to: OSS control for nonlinear systems subject to time-varying disturbances, flexibility of the framework for distributed/decentralized control, formulations and solutions of hierarchical, competitive, and approximate OSS control problems, and the application of the OSS control framework to the design of new optimization algorithms.

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Appendix

Proof of Theorem 3.2: By assumption, the closed-loop system (11)–(12) is well-posed and possesses a globally asymptotically stable equilibrium point for each w; hence, the first two requirements of Problem 2.3 are satisfied. It remains to show that $\lim_{t\to\infty} y(t) = y^\star(w)$ for each w and every initial condition. Since the closed-loop system possesses a globally asymptotically stable equilibrium point for each w, there exists a unique solution $(\bar{x}, \bar{\xi}, \bar{\eta}, \bar{x}_{\mathrm{s}})$ to the steady-state equations

$$\begin{aligned} \mathbb{0} &= A\bar{x} + B\bar{u} + B_w w & \qquad \mathbb{0} &= \varphi(\bar{\xi}, \bar{y}_{\mathrm{m}}) \\ \bar{y}_{\mathrm{m}} &= h_{\mathrm{m}}(\bar{x}, \bar{u}, w) & \qquad \mathbb{0} &= h_{\epsilon}(\bar{\xi}, \bar{y}_{\mathrm{m}}) \end{aligned}$$

$$
\begin{aligned}
\mathbb{0} &= f_{\mathrm{s}}(\bar{x}_{\mathrm{s}}, \bar{\eta}, \bar{\xi}, \bar{y}_{\mathrm{m}}, \mathbb{0}) \\
\bar{u} &= h_{\mathrm{s}}(\bar{x}_{\mathrm{s}}, \bar{\eta}, \bar{\xi}, \bar{y}_{\mathrm{m}}, \mathbb{0})
\end{aligned}
$$

for each w. Since (φ, h_ϵ) is an optimality model, the pair $(\bar{x}, \bar{u})$ satisfies $\bar{y}^\star(w) = C\bar{x} + D\bar{u} + Qw$. Because this equilibrium point attracts all trajectories of the closed-loop system and $y(t)$ is continuous, it must be the case that $\lim_{t\to\infty} y(t) = y^\star(w)$ for every w and every initial condition. Therefore, the controller (11b), (11c), (12a), (12b) solves the OSS control problem. □

Proof of Theorem 3.6: We will apply the classic result [33, Theorem 1] which provides necessary and sufficient conditions for stabilizability and detectability of the augmented plant (11). We first require the following lemmas.

*Lemma A.1 (***Unique Primal Solution***):* Suppose the optimization problem (20) is feasible, and let $T \in \mathbb{R}^{p\times\bullet}$ be any matrix satisfying $\mathbf{range}\, T = \mathbf{null} \left[\begin{smallmatrix} G_\perp \\ H \end{smallmatrix}\right]$. Then (20) has a unique optimizer if and only if $v^\mathsf{T} M v > 0$ on $\mathbf{range}\, T$.

Proof: Fix a member $\tilde{y}(w)$ of the feasible set of (20). Since $\mathbf{range}\, T = \mathbf{null} \left[\begin{smallmatrix} G_\perp \\ H \end{smallmatrix}\right]$, we can rewrite the optimization problem (20) as $\mathrm{minimize}_{\bar{y}\in\mathbb{R}^p,\, v\in\mathbf{range}\, T}\ \frac{1}{2}\bar{y}^\mathsf{T} M \bar{y} - \bar{y}^\mathsf{T} N w$ subject to the constraint $\bar{y} = \tilde{y}(w) + v$. Eliminating $\bar{y}$ and writing $v = T' r$, where T' is a full-column-rank matrix satisfying $\mathbf{range}\, T' = \mathbf{range}\, T$ and $r \in \mathbb{R}^\bullet$ is a new decision variable, we obtain the equivalent problem $\mathrm{minimize}_{r\in\mathbb{R}^\bullet}\ \frac{1}{2} r^\mathsf{T} T'^\mathsf{T} M T' r + r^\mathsf{T} T'^\mathsf{T} (M\tilde{y}(w) - Nw) + \tilde{y}(w)^\mathsf{T} (M\tilde{y}(w) - Nw)$. This unconstrained QP has a unique optimizer $r^\star$ if and only if $T' M T' \succ \mathbb{0}$, which is equivalent to M being positive definite on $\mathbf{range}\, T$. □

*Lemma A.2 (***Unique Dual Solution***):* Suppose the optimization problem (20) has a unique primal solution $\bar{y}^\star$. The corresponding dual solution is unique if and only if the matrix $\left[\begin{smallmatrix} G_\perp \\ H \end{smallmatrix}\right]$ is full row rank.

Proof: Let $\bar{y}^\star$ denote the unique primal solution of (20). Under our assumptions, the pair $(\lambda^\star, \mu^\star)$ is a dual solution if and only if $(\lambda^\star, \mu^\star)$ satisfies the gradient KKT condition (8), which in the present context is given by $\mathbb{0} = M\bar{y}^\star - Nw + G_\perp^\mathsf{T} \lambda^\star + H^\mathsf{T} \mu^\star$. The assumption of a primal solution implies that at least one dual solution $(\lambda^\star, \mu^\star)$ to the preceding exists; this solution is unique if and only if $\begin{bmatrix} G_\perp^\mathsf{T} & H^\mathsf{T} \end{bmatrix}$ is full column rank. □

We move on to the main proof; we will show that when using the FS-OM (18), the OSS control problem is solvable if and only if the stated conditions (i),(ii),(iii)a hold; a similar argument can be made for the OS-OM. A modified version of the same argument can be made for the REFS-OM when (i),(ii),(iii)b hold. Condition (i) is exactly conditions (a) and (b) of [33, Theorem 1]; condition (e) of [33, Theorem 1] is automatically satisfied here. We show conditions (ii) and (iii)a are equivalent to conditions (c) and (d) of [33, Theorem 1]. Define the matrices $\mathcal{N}$, G, and $G_\perp$ as done in Section II, and without loss of generality, assume T in (14) is selected to have full column rank. The augmented plant using the FS-OM is

$$
\begin{aligned}
\dot{x} &= Ax + Bu + B_w w \\
\dot{\eta} &= \begin{bmatrix} HC \\ T^\mathsf{T} MC \end{bmatrix} x + \begin{bmatrix} HD \\ T^\mathsf{T} MD \end{bmatrix} u - \begin{bmatrix} L \\ T^\mathsf{T} N \end{bmatrix} w.
\end{aligned}
$$

Following [33, Equation (13)], we check whether

$$
\mathcal{R}_{\mathrm{FS}} := \left[\begin{array}{c|c} I_n & \mathbb{0} \\ \hline \mathbb{0} & \left[\begin{smallmatrix} H \\ T^\mathsf{T} M \end{smallmatrix}\right] \end{array}\right] \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{31}
$$

has full row rank. Let $\mathrm{col}(\alpha, \beta, \gamma) \in \mathbf{null}\, \mathcal{R}_{\mathrm{FS}}^\mathsf{T}$, so that

$$
\begin{bmatrix} \alpha \\ H^\mathsf{T}\beta + MT\gamma \end{bmatrix}^\mathsf{T} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \mathbb{0}. \tag{32}
$$

Multiplying on the right by $\mathcal{N}$ and recalling that $\mathbf{range}\, \mathcal{N} = \mathbf{null} \begin{bmatrix} A & B \end{bmatrix}$ and also that $G = \begin{bmatrix} C & D \end{bmatrix} \mathcal{N}$, we find

$$
(H^\mathsf{T}\beta + MT\gamma)^\mathsf{T} G = \mathbb{0}. \tag{33}
$$

Hence, $H^\mathsf{T}\beta + MT\gamma \in (\mathbf{range}\, G)^\perp$. Because $(\mathbf{range}\, G)^\perp = \mathbf{range}\, G_\perp^\mathsf{T}$ by the definition of $G_\perp$, the above is equivalent to the existence of a vector v such that

$$
H^\mathsf{T}\beta + MT\gamma = G_\perp^\mathsf{T} v. \tag{34}
$$

Recall that $\mathbf{range}\, T = (\mathbf{null}\, G_\perp) \cap (\mathbf{null}\, H)$, so $G_\perp T = \mathbb{0}$ and $HT = \mathbb{0}$. Multiplying (34) on the left by $\gamma^\mathsf{T} T^\mathsf{T}$ we find

$$
\gamma^\mathsf{T} T^\mathsf{T} MT\gamma = 0. \tag{35}
$$

For the sufficient direction, we show that if conditions (ii) and (iii)a hold, then $\mathrm{col}(\alpha, \beta, \gamma) = \mathbb{0}$. From condition (ii), it follows by Lemma A.1 that the matrix $T^\mathsf{T} MT$ is positive definite and hence from (35) that $\gamma = \mathbb{0}$. Equation (34) then implies that

$$
\begin{bmatrix} v & -\beta \end{bmatrix}^\mathsf{T} \begin{bmatrix} G_\perp \\ H \end{bmatrix} = \mathbb{0}. \tag{36}
$$

By condition (iii)a and Lemma A.2, the coefficient matrix (36) in has full row rank, and hence (36) implies that $v = \mathbb{0}$ and $\beta = \mathbb{0}$. Equation (32) then implies that $\alpha^\mathsf{T} \begin{bmatrix} A & B \end{bmatrix} = \mathbb{0}$. Since (A, B) is stabilizable, the left null space of $\begin{bmatrix} A & B \end{bmatrix}$ is empty. Therefore $\alpha = \mathbb{0}$, we conclude that $\mathcal{R}_{\mathrm{FS}}$ has full row rank, and thus by [33, Theorem 1] the augmented plant is stabilizable/detectable; it follows by Theorem 3.2 that the OSS control problem is solvable.

For the necessary direction, we show that if either of conditions (ii) or (iii)a fails, then we can construct $\mathrm{col}(\alpha, \beta, \gamma) \neq \mathbb{0}$ satisfying (32), which in turn will violate the transmission zero condition in [33, Theorem 1] and show the augmented plant is not stabilizable. Suppose (iii)a fails. Then by Lemma A.2 there exists a nonzero solution to (36). It cannot be the case that $\beta = \mathbb{0}$, for then v would be zero since $G_\perp$ is full row rank by construction. As a result, if we set $\gamma := \mathbb{0}$, (33) implies that there exists a $\bar{\beta} \neq \mathbb{0}$ such that $\bar{\beta}^\mathsf{T} HG = \mathbb{0}$. We observe $\bar{\beta}^\mathsf{T} HG = \begin{bmatrix} \bar{\beta}^\mathsf{T} HC & \bar{\beta}^\mathsf{T} HD \end{bmatrix} \mathcal{N} = \mathbb{0}$. Since $\mathbf{range}\, \mathcal{N} = \mathbf{null} \begin{bmatrix} A & B \end{bmatrix}$, the preceding implies that

$$
\begin{bmatrix} C^\mathsf{T} H^\mathsf{T} \bar{\beta} \\ D^\mathsf{T} H^\mathsf{T} \bar{\beta} \end{bmatrix} \in \left(\mathbf{null} \begin{bmatrix} A & B \end{bmatrix}\right)^\perp = \mathbf{range} \begin{bmatrix} A^\mathsf{T} \\ B^\mathsf{T} \end{bmatrix}.
$$

As a result, a solution $\bar{\alpha}$ exists to $\left[\begin{smallmatrix} C^\mathsf{T} H^\mathsf{T} \bar{\beta} \\ D^\mathsf{T} H^\mathsf{T} \bar{\beta} \end{smallmatrix}\right] = \left[\begin{smallmatrix} A^\mathsf{T} \\ B^\mathsf{T} \end{smallmatrix}\right] \bar{\alpha}$. Let $\bar{\alpha}$ satisfy the preceding. Then $\mathrm{col}(\alpha, \beta, \gamma) := \mathrm{col}(-\bar{\alpha}, \bar{\beta}, \mathbb{0})$ satisfies (32). Now suppose instead condition (ii) fails. Then by Lemma A.1 there exists a $\bar{\gamma} \neq \mathbb{0}$ such that $\bar{\gamma}^\mathsf{T} T^\mathsf{T} MT\bar{\gamma} = 0$, which by positive semidefiniteness of $T^\mathsf{T} MT$ implies that $T^\mathsf{T} MT\bar{\gamma} = \mathbb{0}$, and hence that $MT\bar{\gamma} = \mathbb{0}$. It follows that the vector $\mathrm{col}(\alpha, \beta, \gamma) := \mathrm{col}(\mathbb{0}, \mathbb{0}, \bar{\gamma})$ satisfies (32). □