On Stability of Distributed-Averaging Proportional-Integral Frequency Control in Power Systems

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***Abstract*—Distributed consensus-based controllers for optimal secondary frequency regulation of microgrids and power systems have received substantial attention in recent years. This paper provides a Lyapunov-based proof that, under a time-scale separation, these control schemes are stabilizing for a wide class of lossless and stable nonlinear ODE power system models, and under weak assumptions on (i) the objective functions used for power allocation, and (ii) the graph topology describing communication between agents in the consensus protocol. The results are illustrated via simulation on a detailed test system.**

***Index Terms*—Distributed control, power systems, smart grid, output regulation, Lyapunov methods**

I. Introduction

THE frequency of operation in an AC power system must remain very close to its nominal set-point value of 50 or 60Hz for most equipment to function properly. Due to a combination of the natural physics of synchronous machines, the aggregate behaviour of motors loads, and the conventional *primary control* loops implemented in the system, there is a roughly linear relationship between the steady-state frequency deviation present in the system and the mismatch between scheduled generation and demand. The problem of *secondary frequency regulation* is to rebalance supply and demand, and thereby eliminate any frequency deviation.

The traditional control architecture [1] for achieving secondary frequency regulation is a straight-forward centralized integral control approach: a frequency deviation measurement is integrated to produce an overall control signal, which is then allocated to the controllable devices in the system according to so-called participation factors. Presently however, the proliferation of distributed energy resources, flexible loads, and high-bandwidth communication throughout modern small and large-scale power systems has prompted the investigation of alternative distributed control architectures which do not require central coordination. Recent surveys of techniques in this direction include [2]–[4].

This note focuses on one such distributed control scheme, known as the *distributed-averaging proportional-integral* (DAPI) controller. Roughly speaking, this controller uses a multi-agent consensus algorithm to distribute integral action across many controllable devices. The controller was proposed independently in [5], [6] as a consensus-based framework for sharing power between generation units and eliminating frequency deviations in microgrids and power systems, respectively. Shortly thereafter, the controller was placed into a distributed optimization framework [7] and experimentally tested for microgrid control [8].

A broad set of literature spanning power electronics, power systems, industrial electronics, and control has subsequently developed around DAPI control; we focus here on the control literature. In terms of stability analysis, [5], [6] showed local exponential stability via linearization for simple first and second-order power system models, respectively. A Lyapunov-based proof of asymptotic stability for a nonlinear swing-type model appeared in [9]. A similar proof method can be extended to include basic voltage dynamics [10], turbine-governor dynamics [11], [12], and to show exponential stability [13]. Other works have analyzed $\mathcal{H}_2$ performance [14]–[16] and performance degradation due to non-cooperative agents [17], designed optimal communication topologies [18], examined stability robustness to communication delays [19], and have placed DAPI within a broad class of optimizing feedback controls [20] for linear time-invariant systems.

All of the stability proofs described above are highly dependent on the particular power system model under consideration, as the Lyapunov functions are constructed to exploit underlying passivity properties of the models. In practice, open-loop AC power system dynamics are stable, but highly uncertain. This fact not only precludes the explicit construction of Lyapunov functions, but additionally forces system operators to always use slow, low-gain secondary control schemes. Our time-scale separation analysis approach in this paper is strongly motivated by this engineering practice, and we do not require explicit construction of any power system Lyapunov function. Additionally, the above quoted stability results are only applicable to the case of the DAPI controller in which the objective functions for resource allocation are quadratic and in which communication between agents is bidirectional (i.e., undirected communication graphs).

Contributions: Our contribution here is to provide a closed-loop stability proof for DAPI control, which holds for quite general open-loop stable power system ODE models and under substantially weakened assumptions on the objective functions and the inter-agent communication topology. Roughly speaking, the criteria are (i) the power system should possess a unique exponentially stable equilibrium when subject to reasonable constant inputs, with the steady-state frequency

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deviation being an affine function of the total injected power, (ii) the objective functions in the DAPI control scheme need only be differentiable and strongly convex (which permits barrier functions), and (iii) the communication graph need only contain a globally reachable node. The key technical insight is that the reduced dynamics obtained after a time-scale separation can be transformed into a nonlinear cascade, which admits a composite-type Lyapunov function. We validate our results via simulation on a detailed 14-machine test system modelling the Australian grid.

Paper Organization: Section II records some necessary material on convex functions, graphs, and Laplacian matrices. Section III describes the power system model, defines the optimal frequency regulation problem, and presents a preliminary lemma. The main stability result is in Section IV. Simulation results on a detailed test system are reported in Section V, with conclusions in Section VI.

II. Preliminary Material

A. Strictly convex functions and their conjugates

Let $I \subseteq \mathbb{R}$ be a closed interval with non-empty interior, and let $f : I \to \mathbb{R}$ be continuously differentiable on $\mathrm{interior}(I)$. We say f is *essentially strictly convex* on I if

$$(\nabla f(x) - \nabla f(x'))(x - x') > 0 \tag{1}$$

for all $x, x' \in \mathrm{interior}(I)$ with $x \neq x'$. Note that if f is strictly convex, then $\nabla f : I \to \mathbb{R}$ is injective on $\mathrm{interior}(I)$. Again under continuously differentiability, we say f is *essentially smooth* on I if $|f(x_k)| \to +\infty$ whenever $x_k \to x \in \mathrm{boundary}(I)$. The *conjugate* f^* of f is defined as $f^*(p) = \inf_{x \in I} [f(x) - p^\mathsf{T} x]$. A powerful duality result [21] is that f is essentially strictly convex on I if and only if f^* is essentially smooth on its domain. As a corollary, if f is both essentially strictly convex and essentially smooth on I, then (i) $\mathrm{dom}(f^*) = \mathbb{R}$, (ii) f^* is essentially strictly convex and essentially smooth on $\mathbb{R}$, and (iii) $(\nabla f)^{-1} = \nabla f^*$.

A stronger version of this duality [22] occurs when we consider *strong convexity* and *strong smoothness*. We say $f : I \to \mathbb{R}$ is strongly convex with parameter $\mu > 0$ if

$$(\nabla f(x) - \nabla f(x'))(x - x') \geq \mu |x - x'|^2 \tag{2}$$

for all $x, x' \in \mathrm{interior}(I)$, and when $\mathrm{dom}(f) = \mathbb{R}$, we say that that f is *strongly smooth* with parameter $L > 0$ if $|\nabla f(x) - \nabla f(x')| \leq L|x - x'|$ for all $x, x' \in \mathbb{R}$. A continuously differentiable mapping $f : \mathbb{R} \to \mathbb{R}$ is strongly convex if and only if $f^* : \mathbb{R} \to \mathbb{R}$ is strongly smooth.

B. Directed graphs, connectivity, and the Laplacian matrix

We will require some elements of graph and algebraic graph theory; see [23] for background. A weighted directed graph over m nodes is a triple $\mathcal{G} = (\mathcal{R}, \mathcal{E}, \mathsf{A})$, where $\mathcal{R}$ satisfying $|\mathcal{R}| = m$ is the set of labels for the nodes, $\mathcal{E} \subseteq \mathcal{R} \times \mathcal{R}$ is the set of directed edges specifying the interconnections between nodes, and $\mathsf{A} \in \mathbb{R}^{m \times m}$ is the adjacency matrix, with elements $a_{ij} \geq 0$ satisfying $a_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$. The *Laplacian* matrix $\mathsf{L} \in \mathbb{R}^{m \times m}$ associated with $\mathcal{G}$ is defined element-wise as $\ell_{ij} = -a_{ij}$ if $i \neq j$ and $\ell_{ii} = \sum_{\ell \neq i} a_{i\ell}$. By construction L has zero row-sums ($\mathsf{L}\mathbb{1}_m = 0$), and hence 0 is an eigenvalue of L with right-eigenvector $\mathbb{1}_m = (1, 1, \ldots, 1)^\mathsf{T}$. All non-zero eigenvalues of L have positive real part [23].

The multiplicity of the 0 eigenvalue of L is related to the connections between nodes in $\mathcal{G}$. A *directed path* in $\mathcal{G}$ is an ordered sequence of nodes such that any pair of consecutive nodes in the sequence is a directed edge of $\mathcal{G}$. A node $i \in \mathcal{R}$ is *globally reachable* if for any other node $j \in \mathcal{R} \setminus \{i\}$, there exists a directed path in $\mathcal{G}$ which begins at j and terminates at i. An elegant result is that 0 is a simple eigenvalue of L if and only if $\mathcal{G}$ contains a globally reachable node. In this case, the left-eigenvector $\mathsf{w} \in \mathbb{R}^m$ of L associated with the simple eigenvalue 0 has nonnegative elements, and $\mathsf{w}_i > 0$ if and only if node $i \in \mathcal{R}$ is globally reachable.

III. Power System Model and Optimal Frequency Regulation

A. Power System Model

The precise dynamical model of the network will not be of primary concern to us; we will assume a very generic nonlinear power system model of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), w(t)), \quad x(0) = x_0 \\ \Delta\omega(t) &= h(x(t), u(t), w(t)) \end{aligned} \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the vector of states, $u(t) \in \mathbb{R}^m$ is the vector of control inputs, and $w(t) \in \mathbb{R}^{n_w}$ is the vector of (piecewise) constant reference signals, disturbances, and unknown parameters. The model (3) may describe a microgrid or a transmission system, and may have been obtained from a more general differential-algebraic model under appropriate regularity conditions [24]. The controls u represent total power injection set-points for resources participating in secondary frequency regulation; we let $u^\star$ be the vector of base dispatch points, and let $\mathcal{R}$ be an index set for these resources. The disturbance w models set-point changes to other control loops and unmeasured load and generation changes, e.g., from renewable sources. The measurable output $\Delta\omega(t) \in \mathbb{R}^m$ is the vector of frequency deviations at the secondary control resources.

Motivated by practical systems, our basic assumptions are that (i) the power system converges exponentially to a unique equilibrium when subject to reasonable constant exogenous inputs, and (ii) the steady-state AC frequency is proportional to the net power imbalance.

*Assumption 3.1 (***Power System Model):** For (3) there exist domains $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{I} \subseteq \mathbb{R}^m \times \mathbb{R}^{n_w}$ such that

(A1) f and h are continuously differentiable and Lipschitz continuous on $\mathcal{X} \times \mathcal{I}$;
(A2) there exists a differentiable equilibrium map $\pi_x : \mathcal{I} \to \mathcal{X}$ which is Lipschitz continuous on $\mathcal{I}$ and satisfies $0 = f(\pi_x(u, w), u, w)$ for all $(u, w) \in \mathcal{I}$;
(A3) the equilibrium $\pi_x(u, w) \in \mathcal{X}$ of (3) is exponentially stable, uniformly in $(u, w) \in \mathcal{I}$;
(A4) the equilibrium input-to-frequency map $\Delta\bar{\omega} : \mathcal{I} \to \mathbb{R}^m$ defined by $\Delta\bar{\omega}(u, w) = h(\pi_x(u, w), u, w)$ has the form

$$\Delta\bar{\omega}(u, w) = \tfrac{1}{\beta}\mathbb{1}_m(\mathbb{1}_m^\mathsf{T}(u - u^\star) - d), \tag{4}$$

where $\beta > 0$ and $d \in \mathbb{R}$ is the unscheduled net load.

Assumption (A1)–(A3) above says that associated to each constant control/disturbance pair $(u, w) \in \mathcal{I}$ is a unique (at least, on the set $\mathcal{X}$) exponentially stable equilibrium state $\pi_x(u, w)$. Assumption (A4) specifies that that the network achieves *frequency synchronization* in steady-state, with frequency deviations being equal at all nodes in the system; the expression in (4) is typical for power system models without resistive losses. Inclusion of losses is beyond our scope and deferred to future work. As an example, a simple model which (globally) satisfies Assumption 3.1 is

$$M_i \Delta\dot{\omega}_i = -\sum_{j=1}^{n} T_{ij}(\Delta\theta_i - \Delta\theta_j) - D_i\Delta\omega_i + \Delta P_{\mathrm{m},i} - d_i$$

$$T_i \Delta\dot{P}_{\mathrm{m},i} = -\Delta P_{\mathrm{m},i} - R_{\mathrm{d},i}^{-1}\Delta\omega_i + u_i - u_i^{\star}.$$

for $i \in \{1, \ldots, n\}$ with $\Delta\dot{\theta}_i = \Delta\omega_i$ and $\Delta\theta_1 \equiv 0$. This describes a linearized network-reduced model of synchronous machines with first-order turbine governor models. While we refer the reader to [2] and the references therein for details on these kinds of models, we note that for this particular model, the constant β in (4) is given by $\beta = \sum_{i=1}^{m} D_i + R_{\mathrm{d},i}^{-1}$.

B. Optimal and Distributed Frequency Regulation

For the goal of secondary frequency regulation, a typical power system is highly over-actuated, and the system operator has flexibility in allocating control actions across many actuators. The desired steady-state control set-points $\bar{u}_i$ can be specified via the minimization

$$\underset{\bar{u}\in\mathbb{R}^m}{\text{minimize}} \quad J(\bar{u}) \triangleq \sum\nolimits_{i\in\mathcal{R}} J_i(\bar{u}_i) \tag{5a}$$

$$\text{subject to} \quad 0 = \mathbb{1}_m^{\mathsf{T}}(\bar{u} - u^{\star}) - d \tag{5b}$$

where $J_i : \mathcal{U}_i \to \mathbb{R}$ models the operational disutility of the ith secondary power provider, and includes a (smooth) barrier function for enforcing inequality constraints $\bar{u}_i \in \mathcal{U}_i = (u_i^{\min}, u_i^{\max})$, where $-\infty \leq u_i^{\min} < u_i^{\max} \leq +\infty$. In other words, any limit constraints are directly included in the domain of the function J_i. The constraint (5b) enforces steady-state balance of secondary power injections $\mathbb{1}_m^{\mathsf{T}}(\bar{u} - u^{\star})$ and unscheduled demand d, and by (4), enforces that the steady-state network frequency deviation should be zero. We assume that (5) is strictly feasible, and place the following technical assumptions on J_i.

*Assumption 3.2 (***Regularity of Objective Functions***):* Each function $J_i : \mathcal{U}_i \to \mathbb{R}_{\geq 0}$ is twice continuously differentiable, strongly convex on $\mathcal{U}_i$ with parameter $\mu_i > 0$, and satisfies the barrier function properties

$$\lim_{\xi \searrow u_i^{\min}} J_i(\xi_i) = +\infty, \quad \lim_{\xi \nearrow u_i^{\max}} J_i(\xi_i) = +\infty.$$

Assumption 3.2 implies that J_i is essentially strictly convex and essentially smooth on $\mathcal{U}_i$ (Section II-A). The control problem of interest is to design a (distributed) feedback controller which drives the system frequency deviation towards zero while simultaneously ensuring the control inputs converge towards the (unique) primal optimizer of (5). The distributed-averaging proportional-integral (DAPI) control scheme combines integral control on local frequency measurements with peer-to-peer communication between secondary control resources to solve this problem. The following preliminary result characterizes the optimal solution of (5).

*Lemma 3.3 (***Distributed Optimality Conditions***):* Consider the optimization problem (5). Let $\bar{u} \in \mathbb{R}^m$, let $\Delta\bar{\omega}$ be as in (4), and let $\mathcal{G} = (\mathcal{R}, \mathcal{E}, \mathsf{A})$ be a weighted directed graph with associated Laplacian matrix L. Assume that $\mathcal{G}$ contains a globally reachable node, and let $\mathsf{w} \in \mathbb{R}_{\geq 0}^m$ be the left-eigenvector of L corresponding to its simple eigenvalue at 0. If $K \succeq 0$ is diagonal matrix such that $\mathsf{w}^{\mathsf{T}} K \mathbb{1}_m > 0$, then the following statements are equivalent:

(i) $\bar{u}$ is the unique primal optimizer of (5);
(ii) there exists a unique vector $\bar{\eta} \in \mathrm{span}(\mathbb{1}_m)$ such that

$$0 = K\Delta\bar{\omega}(\bar{u}, w) + \mathsf{L}\bar{\eta} \tag{6a}$$

$$\bar{u} = \nabla J^*(\bar{\eta}), \tag{6b}$$

where $J^*(\eta) = \sum_{i\in\mathcal{R}} J_i^*(\eta_i)$ is the conjugate of J.

Proof: irst note that since (5) is strictly feasible, $J(\bar{u})$ is strongly convex, and the constraint matrix $\mathbb{1}_m^{\mathsf{T}}$ in (5b) has full row rank, the problem (5) has a unique primal-dual optimal solution $(\bar{u}, \bar{\lambda})$ for some $\bar{\lambda} \in \mathbb{R}$, which satisfies the KKT conditions (5b) and

$$\nabla J(\bar{u}) = \bar{\lambda}\mathbb{1}_m \iff \bar{u} = \nabla J^*(\bar{\lambda}\mathbb{1}_m). \tag{7}$$

Since 0 is a simple eigenvalue of L with right-eigenvector $\mathbb{1}_m$, there exists a unique value $\bar{\lambda}$ satisfying (7) if and only if there exists a unique vector $\bar{\eta} \in \mathrm{span}(\mathbb{1}_m)$ such that

$$0 = \mathsf{L}\bar{\eta} \tag{8a}$$

$$\bar{u} = \nabla J^*(\bar{\eta}). \tag{8b}$$

In addition, trivially, the constraint (5b) holds if and only if

$$0 = \mathbb{1}_m \tfrac{1}{\beta}(\mathbb{1}_m^{\mathsf{T}}(\bar{u} - u^{\star}) - d) = \Delta\bar{\omega}(\bar{u}, w), \tag{9}$$

where we used (4). We now claim that (8a) and (9) hold if and only if (6a) holds. That (8a) and (9) imply (6a) is trivial. For the other direction, left-multiply (6a) by w^{T} to find that

$$0 = \mathsf{w}^{\mathsf{T}}(K\Delta\bar{\omega}(\bar{u}, w) + \mathsf{L}\bar{\eta}) = \mathsf{w}^{\mathsf{T}} K \mathbb{1}_m \tfrac{1}{\beta}(\mathbb{1}_m^{\mathsf{T}}(\bar{u} - u^{\star}) - d).$$

The vector w is non-negative, and is non-zero since the graph $\mathcal{G}$ has a globally reachable node (Section II-B), and by assumption $\mathsf{w}^{\mathsf{T}} K \mathbb{1}_m \neq 0$. We conclude that $\mathbb{1}_m^{\mathsf{T}}(\bar{u} - u^{\star}) - d = 0$, and therefore (9) holds. Substituting this into (6a), it follows that (8a) holds, which completes the proof. □

Lemma 3.3 leads naturally to the DAPI controller

$$\tau\dot{\eta}(t) = -\Delta\omega(t) - \mathsf{L}\eta(t), \quad u(t) = \nabla J^*(\eta(t)), \tag{10}$$

where $\tau > 0$ is a tuning gain. The vector $\eta(t) \in \mathbb{R}^m$ is now the dynamic controller state, and the steady-state frequency vector $\Delta\bar{\omega}$ has been replaced by the real-time frequency measurement vector $\Delta\omega(t)$. In components, (10) is

$$\tau\dot{\eta}_i(t) = -\Delta\omega_i(t) - \sum\nolimits_{j=1}^{m} a_{ij}(\eta_i(t) - \eta_j(t)) \tag{11a}$$

$$u_i(t) = \nabla J_i^*(\eta_i(t)), \tag{11b}$$

which emphasizes that (10) is a distributed controller. To interpret the dispatch rule (11b), consider the objective function

$$J_i(u_i) = \tfrac{1}{2\alpha_i}(u_i - u_i^\star)^2 - \gamma_i[\log(u_i^{\max} - u_i) + \log(-u_i^{\min} + u_i)] \tag{12}$$

where $u_i^\star$ is the base dispatch point of the resource, $\alpha_i > 0$, and $\gamma_i > 0$ is a barrier function parameter. When $\gamma = 0$, (11b) becomes $\nabla J_i^*(\eta_i) = u_i^\star + \alpha_i \eta_i$, which is a linear dispatch rule as used in classic automatic generation control [1]. When $\gamma > 0$, the barrier function enforces limit constraints, and $u_i = \nabla J_i^*(\eta_i)$ yields smooth saturation of u_i to the interval $[u_i^{\min}, u_i^{\max}]$ on top of the simple linear dispatch rule.[1]

*Remark 3.4 (***Generalized DAPI Controllers***):* Lemma 3.3 suggests that one could insert a diagonal matrix $K \succeq 0$ in front of $\Delta\omega(t)$ in (10) to obtain a generalized controller, in which, e.g., only globally reachable nodes require local frequency measurements. For technical reasons however, our Lyapunov analysis to follow is applicable only to (10). □

IV. Main Result: Closed-Loop Exponential Stability with DAPI Control

We now state and prove our main result, that the distributed controller (11) leads to stable and optimal frequency regulation of the power system (3).

*Theorem 4.1 (***Low-Gain Stability with DAPI Control***):* Consider the power system model (3) under Assumption 3.1, interconnected with the DAPI controller (11) under Assumption 3.2. If the communication graph $\mathcal{G}$ contains a globally reachable node, then there exists $\tau^\star > 0$ such that for all $\tau \geq \tau^\star$, the unique equilibrium point $(\bar{x}, \bar{\eta}) \in \mathcal{X} \times \mathbb{R}^m$ of the closed-loop system is exponentially stable and $\bar{u} = \nabla J^*(\bar{\eta})$ is the global optimizer of (5).

Proof of Theorem 4.1: For the closed-loop system (3) and (10), define the new time variable $\ell = t/\tau$, which leads to the singularly perturbed system

$$\begin{aligned} \varepsilon \frac{\mathrm{d}x}{\mathrm{d}\ell} &= f(x, u, w), & \Delta\omega &= h(x, u, w) \\ \frac{\mathrm{d}\eta}{\mathrm{d}\ell} &= -\Delta\omega - \mathsf{L}\eta, & u &= \nabla J^*(\eta), \end{aligned} \tag{13}$$

where $\varepsilon = 1/\tau$. We will apply the quadratic Lyapunov methodology in [25], [26, Chap 7.2], [27, Theorems 11.3, 11.4] to show exponential stability of (13) for sufficiently small ε. Due to Assumption 3.1, the required conditions on the boundary layer system are satisfied, and (3) admits a continuously differentiable Lyapunov function $V_{\mathrm{ps}} : \mathcal{X} \times \mathcal{I} \to \mathbb{R}_{\geq 0}$ certifying exponential stability of $\pi_x(u, w)$ and satisfying the boundedness conditions in [27, Section 11.5] on some neighbourhood of $\pi_x(u, w)$. The reduced dynamics of (13) are

$$\begin{aligned} \dot{\eta} &= -\Delta\bar{\omega}(u, w) - \mathsf{L}\eta \\ u &= \nabla J^*(\eta), \end{aligned} \tag{14}$$

where $\Delta\bar{\omega}$ is as given in (4) and where $\dot{\eta}$ now denotes differentiation with respect to the new temporal variable ℓ. By Lemma 3.3 with $K = I_m$, the system (14) possesses a unique equilibrium point $\bar{\eta} \in \mathrm{span}(\mathbb{1}_m)$. Combining this with Assumption 3.1, it is now clear that (13) possesses a unique equilibrium $(\bar{x}, \bar{\eta}) \in \mathcal{X} \times \mathbb{R}^m$. Eliminating u from (14), the reduced dynamics are given by[2]

$$\dot{\eta} = -\tfrac{1}{\beta}\mathbb{1}_m\mathbb{1}_m^{\mathsf{T}}\nabla J^*(\eta) - \mathsf{L}\eta + \tfrac{1}{\beta}\mathbb{1}_m\tilde{d}, \tag{15}$$

where $\tilde{d} = \mathbb{1}_m^{\mathsf{T}} u^\star + d$. Define the nonsingular transformation matrix $T = \begin{bmatrix} \mathbb{1}_m & V_\perp \end{bmatrix}$, where $V_\perp \in \mathbb{R}^{m\times(m-1)}$ has columns which form an orthonormal basis for the subspace $\{\eta \in \mathbb{R}^m \mid \mathbb{1}_m^{\mathsf{T}}\eta = 0\}$. Consider the change of state variable

$$\eta = T\begin{bmatrix} z \\ \delta \end{bmatrix} = \mathbb{1}_m z + V_\perp \delta, \quad z \in \mathbb{R},\ \delta \in \mathbb{R}^{m-1}.$$

It is straightforward to see that

$$z = \tfrac{1}{m}\mathbb{1}_m^{\mathsf{T}}\eta, \qquad \delta = V_\perp^{\mathsf{T}}\eta,$$

and by construction, the unique equilibrium $(\bar{z}, \bar{\delta}) = T^{-1}\bar{\eta} = (\frac{1}{m}\mathbb{1}_m^{\mathsf{T}}\bar{\eta}, 0)$ satisfies

$$0 = -\tfrac{1}{\beta}\mathbb{1}_m^{\mathsf{T}}\nabla J^*(\mathbb{1}_m\bar{z}) + \tfrac{1}{\beta}\tilde{d}. \tag{16}$$

In the (z, δ) coordinates, the dynamics (15) become

$$\begin{aligned} \dot{z} &= -\tfrac{1}{\beta}\mathbb{1}_m^{\mathsf{T}}\nabla J^*(\mathbb{1}_m z + V_\perp\delta) - \tfrac{1}{m}\mathbb{1}_m^{\mathsf{T}}\mathsf{L}V_\perp\delta + \tfrac{1}{\beta}\tilde{d} \\ \dot{\delta} &= -V_\perp^{\mathsf{T}}\mathsf{L}V_\perp\delta. \end{aligned} \tag{17}$$

We reformulate the z-dynamics in (17) by adding and subtracting the term $\frac{1}{\beta}\mathbb{1}_m^{\mathsf{T}}\nabla J^*(\mathbb{1}_m z)$ and using the equilibrium equation (16), which yields the equivalent dynamic model

$$\dot{z} = f_1(z, \delta) = \varphi(z) + \psi(z, \delta) \tag{18a}$$

$$\dot{\delta} = f_2(\delta) \tag{18b}$$

where

$$\begin{aligned} \varphi(z) &\triangleq -\tfrac{1}{\beta}\mathbb{1}_m^{\mathsf{T}}\left[\nabla J^*(\mathbb{1}_m z) - \nabla J^*(\mathbb{1}_m\bar{z})\right] \\ \psi(z, \delta) &\triangleq -\tfrac{1}{\beta}\mathbb{1}_m^{\mathsf{T}}\left[\nabla J^*(\mathbb{1}_m z + V_\perp\delta) - \nabla J^*(\mathbb{1}_m z)\right] \\ &\quad - \tfrac{1}{m}\mathbb{1}_m^{\mathsf{T}}\mathsf{L}V_\perp\delta \\ f_2(\delta) &\triangleq -V_\perp^{\mathsf{T}}\mathsf{L}V_\perp\delta. \end{aligned}$$

Note that $\psi(z, 0) = 0$ for all $z \in \mathbb{R}$. By Assumption 3.2, we have from Section II-A that each function J_i^* is essentially strictly convex and strongly smooth with parameter $1/\mu_i$, the latter fact implying the linear bound

$$|\psi(z, \delta)| \leq \underbrace{\left(\tfrac{\sqrt{m}}{\beta}\tfrac{1}{\mu_{\min}}\|V_\perp\|_2 + \tfrac{1}{\sqrt{m}}\|\mathsf{L}V_\perp\|_2\right)}_{\triangleq \kappa}\|\delta\|_2$$

for all $z \in \mathbb{R}$ and $\delta \in \mathbb{R}^{m-1}$, where $\mu_{\min} \triangleq \min_{i \in \mathcal{R}} \mu_i$.

The dynamics (18) are in the form of a *nonlinear cascade*, for which we will construct a composite Lyapunov function which certifies local exponential stability of $(\bar{z}, \bar{\delta})$.[3] First consider the driving system (18b). Since the graph $\mathcal{G}$ contains a globally reachable node, L has a simple eigenvalue at 0 with

[1] The computation of $u_i(t)$ in (11) is done by solving the algebraic constraint $\nabla J_i(u_i(t)) = \eta_i(t)$.

[2] See also [28] for closely related dynamics.

[3] The construction additionally certifies global *asymptotic* stability, but this will not be used here.

all other eigenvalues having positive real part (Section II-B). Note that since $\mathsf{L}\mathbb{1}_m = 0$, we have

$$T^{-1}\mathsf{L}T = \begin{bmatrix} \frac{1}{m}\mathbb{1}_m^\mathsf{T} \\ V_\perp^\mathsf{T} \end{bmatrix} \mathsf{L} \begin{bmatrix} \mathbb{1}_m & V_\perp \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m}\mathbb{1}_m^\mathsf{T}\mathsf{L}V_\perp \\ 0 & V_\perp^\mathsf{T}\mathsf{L}V_\perp \end{bmatrix}.$$

It follows that $\mathrm{eig}(\mathsf{L}) = \{0\} \cup \mathrm{eig}(V_\perp^\mathsf{T}\mathsf{L}V_\perp)$ and that all eigenvalues of $-V_\perp^\mathsf{T}\mathsf{L}V_\perp$ have negative real part. By linear Lyapunov theory, there exists $\rho > 0$ and $P \succ 0$ such that with $V_2(\delta) = \delta^\mathsf{T} P \delta$, we satisfy the dissipation inequality

$$\nabla V_2(\delta)^\mathsf{T} f_2(\delta) \leq -\rho\|\delta\|_2^2, \qquad \delta \in \mathbb{R}^{m-1}. \tag{19}$$

For the driven system (18a), consider the continuously differentiable Lyapunov candidate $V_1 : \mathbb{R} \to \mathbb{R}_{\geq 0}$ defined as

$$V_1(z) = \sum\nolimits_{i=1}^m \left[J_i^*(z) - J_i^*(\bar{z}) - \nabla J_i^*(\bar{z})(z - \bar{z})\right].$$

Let $r > 0$ and let $\mathcal{B}_r(\bar{z}) \triangleq [\bar{z} - r, \bar{z} + r]$ be a compact ball around $\bar{z}$; by continuity, we have that $\mathbb{I}_r \triangleq \nabla J_i^*(\mathcal{B}_r(\bar{z})) \subset (u_i^{\min}, u_i^{\max})$ is a compact interval. Since $\nabla^2 J_i$ is continuous, it is bounded on $\mathbb{I}_r$. Therefore, ∇J_i is Lipschitz continuous on $\mathbb{I}_r$, and hence J_i is both strongly convex and strongly smooth on $\mathbb{I}_r$. It follows [22] that each J_i^* is both strongly convex and strongly smooth on $\mathcal{B}_r(\bar{z})$, that $V_1(z)$ inherits these properties [29, Lemma A.2], and that $c_1|z - \bar{z}|^2 \leq V_1(z) \leq c_2|z - \bar{z}|^2$ for some $c_1, c_2 > 0$ and all $z \in \mathcal{B}_r(\bar{z})$. We let $\tilde{\mu}$ and $\tilde{L}$ denote the strong convexity and strong smoothness parameters of V_1 on the set $\mathcal{B}_r(\bar{z})$, and it follows immediately that the gradient $\nabla V_1(z) = \mathbb{1}_m^\mathsf{T}\left[\nabla J^*(\mathbb{1}_m z) - \nabla J^*(\mathbb{1}_m \bar{z})\right]$ satisfies

$$\tilde{\mu}|z - \bar{z}| \leq |\nabla V_1(z)| \leq \tilde{L}|z - \bar{z}|, \quad z \in \mathcal{B}_r(\bar{z}).$$

For $\alpha > 0$, consider now the composite Lyapunov candidate

$$V(z, \delta) = V_1(z) + \alpha V_2(\delta),$$

which is positive definite with respect to $(\bar{z}, 0)$. Easy calculations using (18) now show that

$$\nabla V_1(z)^\mathsf{T}\phi(z) = -\tfrac{1}{\beta}|\nabla V_1(z)|^2 \leq -\tfrac{\tilde{\mu}}{\beta}|z - \bar{z}|^2$$
$$\nabla V_1(z)^\mathsf{T}\psi(z, \delta) \leq \kappa\tilde{L}|z - \bar{z}|\|\delta\|_2$$

for all $z \in \mathcal{B}_r(\bar{z})$ and $\delta \in \mathbb{R}^{m-1}$. Combining these with (19), we find that along trajectories of (18)

$$\dot{V}(z, \delta) \leq -\begin{bmatrix} |z - \bar{z}| \\ \|\delta\|_2 \end{bmatrix}^\mathsf{T} \begin{bmatrix} \tilde{\mu}/\beta & -\tilde{L}\kappa/2 \\ -\tilde{L}\kappa/2 & \alpha\rho \end{bmatrix} \begin{bmatrix} |z - \bar{z}| \\ \|\delta\|_2 \end{bmatrix}.$$

Selecting $\alpha > \beta\tilde{L}^2\kappa^2/(4\tilde{\mu}\rho)$, the right-hand side becomes a negative definite with respect to the equilibrium $(\bar{z}, 0)$ for all $(z, \delta) \in \mathcal{B}_r(\bar{z}) \times \mathbb{R}^{m-1}$. We conclude via [27, Theorem 4.10] that the equilibrium $(\bar{z}, 0)$ of (18) — or equivalently, the equilibrium $\bar{\eta}$ of the reduced dynamics (15) — is exponentially stable. One may now proceed using a singular perturbation composite Lyapunov construction as in, e.g., [27, Theorem 11.4] to complete the proof; the details are standard and are omitted due to space limitations. □

An interesting aspect of Theorem 4.1 is that it imposes the weakest possible time-invariant connectivity assumption one can place on the communication graph $\mathcal{G}$ to ensure consensus, namely the existence of a globally reachable node [23]. Indeed, if $\mathcal{G}$ does not contain a globally reachable node, then L has at least two eigenvalues at 0, and the reduced dynamics (15) contain a marginally stable mode. Strongly connected, weight-balanced, and undirected communication graphs [23] are all covered as special cases. The proof is based on singular perturbation time-scale separation, along with a Lyapunov construction for the reduced dynamics. By leveraging loop transformations, an alternative small-gain proof of Theorem 4.1 also appears feasible, with [30, Lem. 1] as a possible starting point.

V. Simulation on Australian Test System

We illustrate our result by simulating the controller (11) on a highly detailed dynamic power system model based on the south eastern Austalian system [31]. The model contains 59 buses and 14 synchronous generators, with full-order turbine-governor, excitation, and PSS models. We will use 5 of these generators (buses 201, 301, 401, 403, and 503) as controllable for secondary frequency regulation, with the inputs u_i being the power set-points to their turbine-governor systems.

To exploit the full flexibility of the theoretical result, we consider heterogeneous objective functions in (12); the parameters are listed in Table I, with an aggressive time constant tuning of $\tau = 15$s obtained following classical regulator tuning procedures (e.g., [32]). The upper and lower power limits for the resources were set as ± 0.1 p.u. from their respective dispatch points. The communication graph $\mathcal{G}$ is a directed line graph (with weights $a_{ij} = 1$) connecting the five controllable machines; bus 503 is the unique globally reachable node.

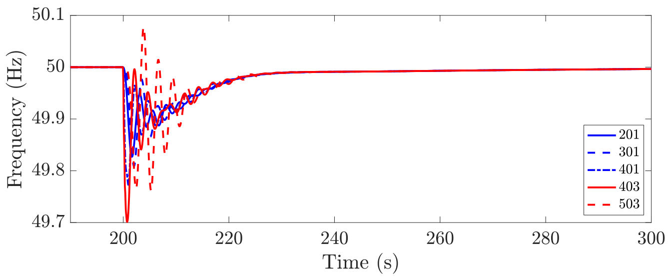
	G201	G301	G401	G403	G503
α_i	0.15	0.15	0.15	0.15	0.4
$u_i^\star$ (p.u.)	0.9	0.9	0.787	0.787	0.6539

TABLE I: Parameters for simulation study; $\gamma_i = 50$.

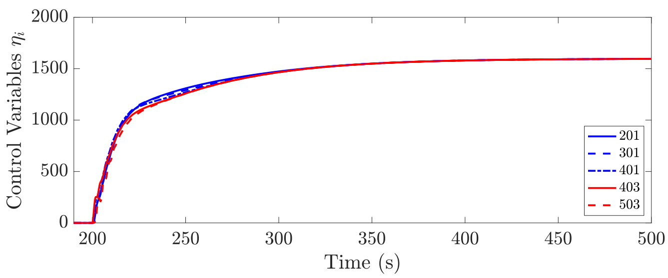
Figure 1 shows the closed-loop response when a 750 MW load is applied at bus 406 at time $t = 200$. This is a large disturbance, and inter-area modes are visible in the frequency plot of Figure 1a; as expected though, the frequency deviation is asymptotically eliminated. The consensus action in (11) drives the marginal cost variables η_i to agreement (Figure 1b). Figure 1c shows the set-points u_i sent to the resources. Note that G403, G404, and G503 all reach their upper production limits and are held there, which indicates that the log barrier functions are ensuring $u_i \in (u_i^{\min}, u_i^{\max})$ as expected.

VI. Conclusions

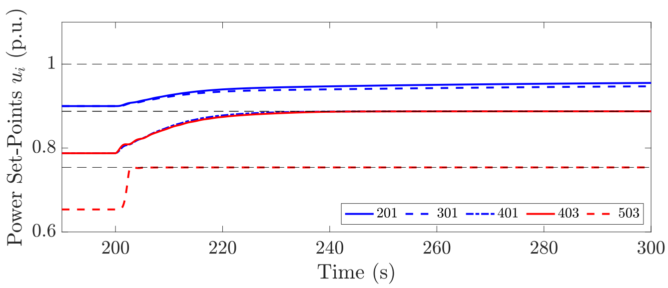
We have presented a proof that distributed-averaging proportional-integral (DAPI) optimal frequency control is stabilizing for open-loop stable nonlinear power system models and under weak assumptions on the objective functions and the inter-agent communication topology uses for consensus. Unresolved questions include extensions to lossy DAE networks, and how far the convexity assumptions on the objective functions J_i can be relaxed.



(a) Frequency measurements from controllable resources.



(b) Marginal cost variables from (11).



(c) Set-points u_i for generators; black dashed lines are upper limits $u_{201}^{\max} = u_{301}^{\max} = 1$, $u_{401}^{\max} = u_{403}^{\max} = 0.887$, and $u_{503}^{\max} = 0.7539$.

Fig. 1: Australian 59-bus 14-machine system with DAPI control.

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