Low-Gain Stability of Projected Integral Control for Input-Constrained Discrete-Time Nonlinear Systems (Extended Version)

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***Abstract*— We consider the problem of zeroing an error output of a nonlinear discrete-time system in the presence of constant exogenous disturbances, subject to hard convex constraints on the input signal. The design specification is formulated as a variational inequality, and we adapt a forward-backward splitting algorithm to act as an integral controller which ensures that the input constraints are met at each time step. We establish a low-gain stability result for the closed-loop system when the plant is exponentially stable, generalizing previously known results for integral control of discrete-time systems. Specifically, it is shown that if the composition of the plant equilibrium input-output map and the integral feedback gain is strongly monotone, then the closed-loop system is exponentially stable for all sufficiently small integral gains. The method is illustrated via a simple numerical example.**

I. Introduction

It is a well-known principle of control engineering that regulation of an error signal to zero can be achieved robustly in the presence of model uncertainty and constant references/disturbances only through integral feedback control [1]. The presence of control input constraints however presents challenges to traditional integral controller designs; sufficient actuator authority may not be available to achieve exact regulation for all references/disturbances, and dynamic performance is sometimes degraded through the so-called wind-up phenomenon [2].

There are two broad approaches for accomodating limited actuator authority. The explicit approach is to directly include input constraints into the design, as done in receding-horizon/model-predictive control [3], bounded integral control [4], and in other nonlinear/adaptive approaches [5]. A more traditional implicit approach is to proceed by first designing ignoring the actuator limits, and then to augment or retro-fit the design in order to improve performance in the presence of saturation; this category would include both classic and modern anti-windup design [6]–[8], and reference/command modification [9], [10].

Returning now to the fundamentals of integral control, a commonly encountered case in practice is that the system one wishes to control is complex, and limited dynamic model information is available, but it is however known that the system is stable (possibly achieved via a stabilizing controller design). A general and well-established design philosophy is that asymptotic tracking and disturbance rejection can be guaranteed by adding a supplementary integral control loop, and that the closed-loop stability will be guaranteed if the integral gain is sufficiently low; a famous and widely-deployed example of this design philosophy is the tuning of automatic generation control in power systems [11].

For finite-dimensional multi-input multi-output (MIMO) linear time-invariant (LTI) systems, the fundamental stability result for this approach is due to Davison [12, Lemma 3]; see also [13, Theorem 3] and [14, Lemma 1, A.2, A.3]. While [12] is in continuous-time, the key result is identical in the discrete-time case [15], [16]. Consider the plant model

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + B_w w \\ e_k &= Cx_k + Du_k + D_w w \end{aligned} \tag{1}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, constant disturbance/reference signal $w \in \mathbb{R}^{n_w}$, and error output $e \in \mathbb{R}^p$; we associate a sampling period $T_{\mathrm{s}} > 0$ with (1). Assume that A is Schur stable, and let $G(z) = C(zI_n - A)^{-1}B + D$ denote the transfer matrix of (1) from u to e. One interconnects the system (1) with the integral controller

$$\eta_{k+1} = \eta_k - \tfrac{T_{\mathrm{s}}}{T_{\mathrm{i}}} e_k, \qquad u_k = K\eta_k, \tag{2}$$

where $K \in \mathbb{R}^{m\times p}$ is a gain matrix and $T_{\mathrm{i}} > 0$ is the integral time constant. Davison's low-gain stability result states that if $-G(1)K$ is Hurwitz stable, then there exists $T_{\mathrm{i}}^\star > 0$ such that the closed-loop system is exponentially stable for all $T_{\mathrm{i}} \in (T_{\mathrm{i}}^\star, \infty)$. A substantial literature exists on extensions of this core result to infinite-dimensional systems, including static nonlinearities, sampled-data implementations, and anti-windup compensation; see [17]–[19] and the references therein. Initial extensions to the continuous-time nonlinear case were given in [20]. In [21] the author further generalized these conditions via contraction theory, and provided a LMI-based procedure to design low-gain integral controllers for continuous-time nonlinear systems.

Contributions: In this paper we further contribute to the study of constrained and low-gain integral control. We begin by formulating the error regulation criteria in the presence of input constraints as a variational inequality [22], which leads us to adopt a version of the *projection* or *forward-backward algorithm* [23] as a constrained integral controller. The design is in discrete-time, and is therefore immediately appropriate for digital control implementations. While this design explicitly enforces input constraints at each time instant, it has the following commonality with the more implicit anti-windup approaches: if the input constraints are not encountered during operation, the scheme reduces to the classical integral controller (2). Our main stability

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result (Theorem 3.1) establishes that the “low-gain integral control stability principle” described above also holds for this projected integral controller, which extends the main result of [21] to discrete-time nonlinear systems.

Notation: Given two vectors x and y, $\mathrm{col}(x, y)$ denotes their vertical concatenation. The matrix I_n is the $n \times n$ identity matrix. If P is a symmetric matrix $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote its minimum and maximum eigenvalues. If $P \succ 0$, the inner product induced by P is denoted by $\langle x, y\rangle_P = x^{\mathsf{T}} P y$. A function $f : X \to \mathbb{R}^n$ is Lipschitz continuous on $X \subseteq \mathbb{R}^n$ if there exists $L > 0$ such that $\|f(x) - f(y)\| \leq L\|x - y\|$ for all $x, y \in X$.

II. Problem Formulation

A. Plant Model and Assumptions

We consider a plant described by a finite-dimensional nonlinear time-invariant state-space model

$$x_{k+1} = f(x_k, u_k, w) \tag{3a}$$

$$e_k = h(x_k, u_k, w) \tag{3b}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control input, and $w \in \mathbb{R}^{n_w}$ is a vector of exogenous signals (reference signals and/or disturbances). The signal $e_k \in \mathbb{R}^p$ with $p \leq m$ is an error output to be driven to zero. The model (3) would most commonly arise by discretizing a continuous-time model; we therefore associate a sampling period $T_{\mathrm{s}} > 0$ to (3).

For any fixed w, the possible equilibrium state-input-error triplets $(\bar{x}, \bar{u}, \bar{e})$ are determined by the algebraic equations

$$\bar{x} = f(\bar{x}, \bar{u}, w), \qquad \bar{e} = h(\bar{x}, \bar{u}, w).$$

To capture the steady-state and dynamic behaviour of (3), we assume that there exist convex sets $\mathcal{X}, \mathcal{U}, \mathcal{W}$ such that

(A1) f and h are continuous in all arguments on $\mathcal{X} \times \mathcal{U} \times \mathcal{W}$, f is continuously differentiable with respect to x and u, and f, h, $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial u}$ are all Lipschitz continuous on $\mathcal{X} \times \mathcal{U}$ uniformly in $w \in \mathcal{W}$;

(A2) there is a class C^1 map $\pi_x : \mathcal{U} \times \mathcal{W} \to \mathcal{X}$ which is Lipschitz continuous on $\mathcal{U} \times \mathcal{W}$ such that

$$\pi_x(u, w) = f(\pi_x(u, w), u, w), \qquad (u, w) \in \mathcal{U} \times \mathcal{W}.$$

(A3) the equilibrium $\bar{x} = \pi_x(u, w)$ is exponentially stable, uniformly in $(u, w) \in \mathcal{U} \times \mathcal{W}$.

Assumptions (A1)–(A3) capture the idea that the plant model is sufficiently smooth, and converges exponentially to a locally unique equilibrium when subject to reasonable constant inputs u and w. We call the map

$$\pi : \mathcal{U} \times \mathcal{W} \to \mathbb{R}^m, \quad \pi(\bar{u}, w) := h(\pi_x(\bar{u}, w), \bar{u}, w) \tag{4}$$

the *equilibrium input-to-error map*, which produces the equilibrium error $\bar{e} = \pi(\bar{u}, w)$ associated with the constant control input $\bar{u}$ and disturbance w. When applied to the LTI system (1), (A1)–(A3) simply reduce to A being Schur stable, and the mapping π becomes

$$\pi(\bar{u}, w) = G(1)\bar{u} + G_w(1)w, \tag{5}$$

where $G_w(z) = C(zI_n - A)^{-1}B_w + D_w$ is the transfer matrix from w to e.

B. Constrained Error-Zeroing Specification

Let $\mathcal{C} \subseteq \mathcal{U}$ be a closed non-empty convex set which describes actuator limits. Following (2), the control signal u_k from our new integral controller will be generated as

$$u_k = K\eta_k \tag{6}$$

where $K \in \mathbb{R}^{m \times p}$ is a gain matrix to be designed and $\eta_k \in \mathbb{R}^p$ is the controller state. It follows that the preimage

$$\Gamma := \{\eta \in \mathbb{R}^p \mid K\eta \in \mathcal{C}\}$$

is also a closed and non-empty convex set. For example, in applications $\mathcal{C}$ is often polyhedral, in which case so is Γ.

Our ideal design objective would be to ensure that for any disturbance $w \in \mathcal{W}$, the error signal e_k is asymptotically driven to zero, and that the input constraint $u_k \in \mathcal{C}$ is satisfied at all times. As one might expect however, input constraints may prevent us from exactly zeroing the steady-state error $\bar{e} = \pi(\bar{u}, w)$ for at least some disturbances $w \in \mathcal{W}$. We therefore relax the design objective, and instead seek an equilibrium value $\bar{\eta} \in \Gamma$ for the controller state such that

$$\langle \bar{e}, \eta - \bar{\eta}\rangle_P = \langle \pi(K\bar{\eta}, w), \eta - \bar{\eta}\rangle_P \geq 0, \qquad \forall \eta \in \Gamma, \tag{7}$$

where $\langle x, y\rangle_P = x^{\mathsf{T}} P y$ is the inner product on $\mathbb{R}^p$ induced by some positive definite matrix $P \succ 0$. The inequality (7) is called a *variational inequality* [22], and we notate a solution of the inequality as $\bar{\eta} \in \mathsf{VI}_P(\Gamma, \pi \circ K)$. Note that if $\bar{\eta}$ lies in the interior of the set Γ, then there exists $\tau > 0$ such that $\eta = \bar{\eta} - \tau\bar{e} \in \Gamma$. The inequality (7) then implies that $-\bar{e}^{\mathsf{T}}\bar{e} \geq 0$, implying that $\bar{e} = 0$. In other words, if the input constraints are *strictly feasible*, then (7) is an exact error-zeroing design specification. A geometric interpretation of (7) uses the *normal cone* of the set Γ at $\bar{\eta} \in \Gamma$, defined as

$$\mathcal{N}_\Gamma^P(\bar{\eta}) := \{d \in \mathbb{R}^p \mid \langle d, \eta - \bar{\eta}\rangle_P \leq 0 \text{ for all } \eta \in \Gamma\}.$$

With this, (7) can be equivalently expressed as $-\bar{e} = -\pi(K\bar{\eta}, w) \in \mathcal{N}_\Gamma^P(\bar{\eta})$, as illustrated in Figure 1. The interpretation of Figure 1 is that from the point $\bar{\eta}$, any further attempt to adjust in the direction $-\bar{e} = -\pi(K\bar{\eta}, w)$ will result in constraint violation.

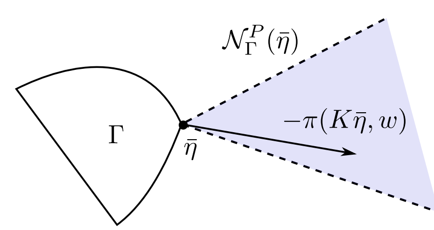

Fig. 1: Illustration of constrained error-zeroing specification.

*Remark 2.1 (***Minimization Interpretation***):* To see how else the design specification (7) could arise, suppose that $\epsilon_k = h_\epsilon(x_k, u_k, w)$ is a measured tracking error of interest for the system (3), with associated equilibrium mapping $\bar{\epsilon} = \pi_\epsilon(\bar{u}, w)$ defined similar to (4). Consider the steady-state minimization problem

$$\underset{\bar{u} \in \mathcal{C}}{\text{minimize}} \;\; J(\bar{\epsilon}) \quad \Leftrightarrow \quad \underset{\bar{\eta} \in \Gamma}{\text{minimize}} \;\; J(\pi_\epsilon(K\bar{\eta}, w)) \tag{8}$$

where $J : \mathbb{R}^p \to \mathbb{R}$ is a class C^1 convex and positive definite function. Critical points of this (generally, non-convex) problem are determined by the inclusion

$$-K^\top \frac{\partial \pi_\epsilon}{\partial \bar{u}}(K\bar{\eta}, w)^\top \nabla J(\pi_\epsilon(K\bar{\eta}, w)) \in \mathcal{N}_\Gamma(\bar{\eta}). \quad (9)$$

If we define the error signal $e_k \coloneqq K^\top \frac{\partial \pi_\epsilon}{\partial \bar{u}}(u_k, w)^\top \nabla J(\epsilon_k)$, then the inclusion (9) is precisely the variational inequality (7). Thus, one could interpret the specification (7) as arising from a steady-state optimization problem where the goal is to minimize a function of the tracking error. This perspective connects our approach directly with recent ideas in autonomous and feedback-based optimization; see [24]–[27] for recent contributions. □

C. The Projection (Forward-Backward) Algorithm

The error-zeroing specification (7) is equivalent to the so-called natural equation [22, Chp. 1.5]

$$\bar{\eta} = \mathrm{Proj}_\Gamma^P(\bar{\eta} - \alpha\pi(K\bar{\eta}, w)) \quad (10)$$

for any $\alpha > 0$, where $\mathrm{Proj}_\Gamma^P : \mathbb{R}^m \to \Gamma$ is the *projection operator*

$$\mathrm{Proj}_\Gamma^P(\eta) = \underset{\nu\in\Gamma}{\mathrm{argmin}} \ \|\eta - \nu\|_P \quad (11)$$

which yields the closest point to η in Γ measured in the norm $\|x\|_P = \sqrt{x^\top P x}$ induced by $P \succ 0$. The equation (10) leads immediately to the classic *projection* or *forward-backward splitting* algorithm [28, Section 25.3].

$$\eta_{k+1} = (1-\lambda)\eta_k + \lambda\,\mathrm{Proj}_\Gamma^P(\eta_k - \alpha\pi(K\eta_k, w)) \quad (12)$$

for solving the variational inequality $\mathsf{VI}_P(\Gamma, \pi \circ K)$, where $\lambda \in (0, 1)$ is a damping parameter. We summarize some well-known conditions which ensure exponential stability of the iteration (12) to a unique equilibrium satisfying (10).

Proposition 2.1: (**Equilibrium and Contraction Properties of Forward-Backward Algorithm**) Let $w \in \mathcal{W}$. If $\eta \mapsto \pi(K\eta, w)$ is μ-strongly monotone on Γ with respect to $\langle\cdot,\cdot\rangle_P$, i.e., if

$$\langle \pi(K\eta, w) - \pi(K\eta', w), \eta - \eta'\rangle_P \geq \mu\|\eta - \eta'\|_P^2$$

for some $\mu > 0$ and all $\eta, \eta' \in \Gamma$, then (12) possesses a unique equilibrium point $\bar{\eta} \in \Gamma$ satisfying (10). If $\eta \mapsto \pi(K\eta, w)$ is additionally L-Lipschitz continuous on Γ with respect to the norm $\|\cdot\|_P$, and α is selected such that $\alpha \in (0, 2\mu/L^2)$, then the following statements hold:

(i) the foward-backward operator

$$\Phi : \Gamma \to \Gamma, \qquad \Phi(\eta) = \mathrm{Proj}_\Gamma^P(\eta - \alpha\pi(K\eta, w)) \quad (13)$$

is a contraction mapping on Γ, satisfying

$$\|\Phi(\eta) - \Phi(\eta')\|_P \leq c_{\mathrm{fb}}\|\eta - \eta'\|_P, \quad \eta, \eta' \in \Gamma,$$

where $c_{\mathrm{fb}} = \sqrt{1 - 2\alpha\mu + \alpha^2 L^2} \in [0, 1)$.

(ii) the damped forward-backward operator

$$\Phi_{\mathrm{d}} : \Gamma \to \Gamma, \qquad \Phi_{\mathrm{d}}(\eta) = (1-\lambda)\eta + \lambda\Phi(\eta)$$

is a contraction mapping on Γ, satisfying

$$\|\Phi_{\mathrm{d}}(\eta) - \Phi_{\mathrm{d}}(\eta')\|_P \leq c_{\mathrm{dfb}}\|\eta - \eta'\|_P.$$

for all $\eta, \eta' \in \Gamma$, where $c_{\mathrm{dfb}} = 1 - \lambda(1 - c_{\mathrm{fb}}) \in (0, 1)$.

Proof: The existence/uniqueness statement is [22, Theorem 2.3.3], and the proof of (i) requires only minor modifications of the proof of [23, Theorem 12.1.2]; (ii) then follows immediately from (i). ■

III. Damped Projected Integral Control and Low-Gain Stability Result

A. Damped Projected Integral Control

We propose adapting the forward-backward splitting algorithm (12) as an integral feedback controller for enforcing the error-zeroing specification (7). Specifically, we propose the *damped projected integral (DP-I) controller*

$$\eta_{k+1} = (1-\lambda)\eta_k + \lambda\,\mathrm{Proj}_\Gamma^P(\eta_k - \tfrac{T_{\mathrm{s}}}{T_{\mathrm{i}}}e_k) \quad (14a)$$

$$u_k = K\eta_k \quad (14b)$$

where $T_{\mathrm{i}} > 0$ is the integral time constant. We make several observations regarding (14):

(i) *Constrained Error-Zeroing:* If (14) is in equilibrium with the plant (3), then it is immediate from (10) that $\bar{\eta} \in \mathsf{VI}_P(\Gamma, \pi \circ K)$.

(ii) *Input Constraint Satisfaction & Windup:* If $\eta_k \in \Gamma$, then $\eta_{k+1} \in \Gamma$, since by (14) η_{k+1} is a convex combination of two points in Γ. Therefore, $u_k = K\eta_k \in \mathcal{C}$ at all points in time. As a result, (14) will never suffer from traditional integrator windup.

(iii) *Reduction to Classical Integral Control:* If $\eta_k \in \Gamma$ and $\eta_k - \frac{T_{\mathrm{s}}}{T_{\mathrm{i}}}e_k \in \Gamma$, then the update (14) reduces to

$$\eta_{k+1} = \eta_k - \tfrac{T_{\mathrm{s}}}{T_{\mathrm{i}}'}e_k, \qquad u_k = K\eta_k \quad (15)$$

where $T_{\mathrm{i}}' = \lambda/T_{\mathrm{i}}$. Thus, when constraints are not encountered, (14) reduces to the integral controller (2).

(iv) *Computation of Projection:* The projection in (14) requires the solution of the convex optimization problem (11), but need only be computed at step k if $\eta_k - \frac{T_{\mathrm{s}}}{T_{\mathrm{i}}}e_k \notin \Gamma$. Projections onto many simple types of constraint sets are computable in closed-form; see, e.g., [29, App. B].

(v) *Alternative Controller:* The controller (14) is based on the natural equation associated with the inequality (7). If one instead uses a *skewed* natural equation (see [22, Chp. 1.5]), one can arrive at the alternative update law

$$\eta_{k+1} = (1-\lambda)\eta_k + \lambda\,\mathrm{Proj}_\Gamma^{I_r}(\eta_k - \tfrac{T_{\mathrm{s}}}{T_{\mathrm{i}}}P^{-1}e_k),$$

where the projection is now with respect to the standard Euclidean norm. In what follows though, we proceed with the formulation (14), mostly due to point (iii) above.

B. Low-Gain Stability with DP-I Control

The closed-loop system consists of the interconnection of the plant (3) and the controller (14); we can now state our main stability result.

*Theorem 3.1 (***Low-Gain Stability with DP-I Control***):* Consider the plant (3) under Assumptions (A1)–(A3) with the DP-I controller (14). Suppose that there exists a matrix $P \succ 0$ and constants $\mu, L > 0$ such that $\eta \mapsto \pi(K\eta, w)$ is μ-strongly monotone and L-Lipschitz continuous on Γ with respect to $\langle \cdot, \cdot \rangle_P$, uniformly in $w \in \mathcal{W}$. Define $T_{\rm i}^\star := T_{\rm s}L^2/2\mu$. Then for any $T_{\rm i} \in (T_{\rm i}^\star, \infty)$, there exists $\lambda^\star \in (0,1)$ such that for any $\lambda \in (0, \lambda^\star)$ and any $w \in \mathcal{W}$, the closed-loop system possesses an exponentially stable equilibrium point $(\bar{x}, \bar{\eta}) \in \mathcal{X} \times \Gamma$ and the pair $(\bar{e}, \bar{\eta}) = (\pi(K\bar{\eta}, w), \bar{\eta})$ satisfies the error-zeroing specification (7).

To interpret the conditions in Theorem 3.1, consider again the LTI case (5). The condition for strong monotonicity requires that there exist $P \succ 0$ satisfying

$$(G(1)K)^{\mathsf{T}} P + PG(1)K \succ 0$$

which is equivalent to the matrix $-G(1)K$ being Hurwitz stable; this is precisely Davison's classical condition, as described in Section I. The main condition required in Theorem 3.1 is that of strong monotonicity of the mapping $\eta \mapsto \pi(K\eta, w)$; as shown in [21], the same condition is sufficient for stability of low-gain integral control applied to continuous-time nonlinear systems. In [21], it was further shown that if π admits a linear fractional representation, methods from robust control and semidefinite programming can be used to certify the monotonicity condition. The same methods can in fact be used to *synthesize* controller gains K which achieve robust performance for the reduced dynamics (12). While we omit the details here due to space limitations, the interested reader will have no trouble adapting the analysis and synthesis results from [21] to the current context.

Proof of Theorem 3.1: The proof is based on a composite Lyapunov construction, and is divided into five steps.

Step #1 — Equilibrium and Error Equations: Let $w \in \mathcal{W}$ and set $\alpha := T_{\rm s}/T_{\rm i}$. Equilibria $(\bar{x}, \bar{\eta})$ are characterized by

$$\begin{aligned} \bar{x} &= f(\bar{x}, \bar{u}, w), \qquad & \bar{\eta} &= \mathrm{Proj}_\Gamma^P(\bar{\eta} - \alpha\bar{e}) \\ \bar{e} &= h(\bar{x}, \bar{\eta}, w), \qquad & \bar{u} &= K\bar{\eta}. \end{aligned} \tag{16}$$

If such an equilibrium exists, then necessarily $\bar{\eta} \in \Gamma$, and hence $\bar{u} = K\bar{\eta} \in \mathcal{C}$. Given any such $\bar{u}$, it follows from (A2) that the first equation in (16) can be solved for $\bar{x} = \pi_x(\bar{u}, w)$; together, (A2)/(A3) imply that $\bar{x}$ is isolated. Eliminating $\bar{x}$ and $\bar{e}$, we obtain the reduced equilibrium equation

$$\bar{\eta} = \mathrm{Proj}_\Gamma^P(\bar{\eta} - \alpha\pi(K\bar{\eta}, w)) = \Phi(\bar{\eta}) = \Phi_{\rm d}(\bar{\eta}) \tag{17}$$

which is equivalent to the error-zeroing specification (7). Since $\eta \mapsto \pi(K\eta, w)$ is μ-strongly monotone on Γ uniformly in w, and Γ is closed, convex, and non-empty, $\mathsf{VI}_P(\Gamma, \pi \circ K)$ admits a unique solution [22, Theorem 2.3.3]. We conclude that the closed-loop system possess a unique equilibrium point $(\bar{x}, \bar{\eta}) \in \mathcal{X} \times \Gamma$ with $\bar{e} = h(\bar{x}, K\bar{\eta}, w) = \pi(K\bar{\eta}, w)$ and control $\bar{u} = K\bar{\eta} \in \mathcal{C}$. Consider the change of state variable

$$\xi_k := x_k - \pi_x(K\eta_k, w).$$

With this, the dynamics (3),(14) become

$$\begin{aligned} \xi_{k+1} &= f(\xi_k + \pi_x(K\eta_k, w), K\eta_k, w) - \pi_x(K\eta_{k+1}, w) \\ e_k &= h(\xi_k + \pi_x(K\eta_k, w), K\eta_k, w) \\ \eta_{k+1} &= (1-\lambda)\eta_k + \lambda\, \mathrm{Proj}_\Gamma^P(\eta_k - \alpha e_k), \end{aligned} \tag{18}$$

and the equilibrium point of interest is $(\xi, \eta) = (0, \bar{\eta})$.

Step #2 — Analyzing the Slow Dynamics: Let $V_{\rm s}(\eta) = \|\eta - \bar{\eta}\|_P^2$. Using Φ and $\Phi_{\rm d}$ from Proposition 2.1, we compute that

$$\begin{aligned} V_{\rm s}(\eta_{k+1})^{\frac{1}{2}} &= \|(1-\lambda)\eta_k + \lambda\,\mathrm{Proj}_\Gamma^P(\eta_k - \alpha e_k) - \bar{\eta}\|_P \\ &= \|(1-\lambda)\eta_k + \lambda\Phi(\eta_k) - \bar{\eta} \\ &\qquad + \lambda(\mathrm{Proj}_\Gamma^P(\eta_k - \alpha e_k) - \Phi(\eta_k))\|_P \\ &= \|\Phi_{\rm d}(\eta_k) - \Phi_{\rm d}(\bar{\eta}) \\ &\qquad + \lambda(\mathrm{Proj}_\Gamma^P(\eta_k - \alpha e_k) - \Phi(\eta_k))\|_P \\ &\leq c_{\rm dfb}\|\eta_k - \bar{\eta}\|_P + \lambda\|\delta\|_P \end{aligned}$$

where $\delta = \mathrm{Proj}_\Gamma^P(\eta_k - \alpha e_k) - \Phi(\eta_k)$. To bound $\|\delta\|_P$ we compute that

$$\begin{aligned} \|\delta\|_P^2 &= \|\,\mathrm{Proj}_\Gamma^P(\eta_k - \alpha e_k) - \mathrm{Proj}_\Gamma^P(\eta_k - \alpha\pi(K\eta_k, w))\|_P^2 \\ &\leq \alpha^2\|e_k - \pi(K\eta_k, w)\|_P^2 \\ &= \alpha^2\|h(\xi_k + \pi_x(K\eta_k, w), K\eta_k, w) \\ &\qquad\qquad - h(\pi_x(K\eta_k, w), K\eta_k, w)\|_P^2 \\ &\leq \alpha^2\lambda_{\max}(P)L_h^2\|\xi_k\|_2^2 \end{aligned}$$

where L_h is the Lipschitz constant of h. Combining the above, with $\Delta V_{\rm s} = V_{\rm s}(\eta_{k+1}) - V_{\rm s}(\eta_k)$, one finds that along trajectories of (18) it holds that

$$\begin{aligned} \Delta V_{\rm s} &\leq (c_{\rm dfb}^2 - 1)\|\eta_k - \bar{\eta}\|_P^2 + \lambda_{\max}(P)L_h^2\alpha^2\lambda^2\|\xi_k\|_2^2 \\ &\qquad + 2\lambda_{\max}(P)^{\frac{1}{2}}\alpha L_h\lambda c_{\rm dfb}\|\eta_k - \bar{\eta}\|_P\|\xi_k\|_2 \\ &= \zeta_k^{\mathsf{T}} Q_{\rm s}\zeta_k \end{aligned}$$

where $\zeta_k = \mathrm{col}(\|\xi_k\|_2, \|\eta_k - \bar{\eta}\|_P)$ and

$$Q_{\rm s} = \begin{bmatrix} q_1\lambda^2 & q_2\lambda \\ q_2\lambda & c_{\rm dfb}^2 - 1 \end{bmatrix}, \qquad \begin{aligned} q_1 &= \lambda_{\max}(P)\alpha^2 L_h^2 \\ q_2 &= \alpha\lambda_{\max}(P)^{1/2}L_h c_{\rm dfb}. \end{aligned}$$

Step #3 — Bounding $\|\eta_{k+1} - \eta_k\|_P$: We compute using the triangle inequality that

$$\begin{aligned} \|\eta_{k+1} - \eta_k\|_P \leq \|\eta_{k+1} &- \Phi_{\rm d}(\eta_k)\|_P \\ &+ \|\Phi_{\rm d}(\eta_k) - \eta_k\|_P. \end{aligned} \tag{19}$$

Using our previous calculations, the first term in (19) can be bounded as

$$\begin{aligned} \|\eta_{k+1} - \Phi_{\rm d}(\eta_k)\|_P &= \lambda\|\,\mathrm{Proj}_\Gamma^P(\eta_k - \alpha e_k) - \Phi(\eta_k)\|_P \\ &= \lambda\|\delta\|_P \\ &\leq \lambda\alpha L_h\lambda_{\max}(P)^{1/2}\|\xi_k\|_2. \end{aligned} \tag{20}$$

To bound the second term in (19), it follows from Proposition 2.1 and the triangle inequality that

$$\begin{aligned}\|\Phi_{\mathrm{d}}(\eta_k) - \eta_k\|_P &= \lambda\|\Phi(\eta_k) - \eta_k\|_P \\ &= \lambda\|(\eta_k - \bar{\eta}) - (\Phi(\eta_k) - \bar{\eta})\|_P \\ &\le \lambda(1 + c_{\mathrm{fb}})\|\eta_k - \bar{\eta}\|_P.\end{aligned} \tag{21}$$

Putting things together we obtain

$$\begin{aligned}\|\eta_{k+1} - \eta_k\|_P \le \lambda L_h \alpha \lambda_{\max}(P)^{1/2}\|\xi_k\|_2 \\ + \lambda(1 + c_{\mathrm{fb}})\|\eta_k - \bar{\eta}\|_P.\end{aligned} \tag{22}$$

Step #4 — Analyzing the Fast Dynamics: Define the deviation vector field $g : \mathbb{R}^n \times \mathcal{U} \times \mathcal{W} \to \mathbb{R}^n$ by

$$\begin{aligned}g(\xi, u, w) &= f(\xi + \pi_x(u, w), u, w) - f(\pi_x(u, w), u, w) \\ &= f(\xi + \pi_x(u, w), u, w) - \pi_x(u, w).\end{aligned}$$

Under Assumptions (A1)–(A3), the conditions of a converse Lyapunov theorem in the appendix are met: there exists a set $\mathcal{Z}$ containing the origin in its interior, positive constants $c_1, c_2, c_3, c_4 > 0$, $\rho_{\mathrm{f}} \in [0, 1)$, and a continuous function

$$V_{\mathrm{f}} : \mathcal{Z} \times \mathcal{U} \times \mathcal{W} \to \mathbb{R}_{\ge 0}, \quad (\xi, u, w) \mapsto V_{\mathrm{f}}(\xi, u, w)$$

satisfying the Lyapunov conditions

$$c_1\|\xi\|_2^2 \le V_{\mathrm{f}}(\xi, u, w) \le c_2\|\xi\|_2^2 \tag{23a}$$

$$V_{\mathrm{f}}(g(\xi, u, w), u, w) - V_{\mathrm{f}}(\xi, u, w) \le -\rho_{\mathrm{f}}\|\xi\|_2^2 \tag{23b}$$

$$|V_{\mathrm{f}}(\xi, u, w) - V_{\mathrm{f}}(\xi', u, w)| \le c_3(\|\xi\|_2 + \|\xi'\|_2)\|\xi - \xi'\|_2 \tag{23c}$$

$$|V_{\mathrm{f}}(\xi, u, w) - V_{\mathrm{f}}(\xi, u', w)| \le c_4\|\xi\|_2^2\|u - u'\|_2 \tag{23d}$$

for all $\xi, \xi' \in \mathcal{Z}$, all $u, u' \in \mathcal{U}$, and all $w \in \mathcal{W}$. Let

$$\Delta V_{\mathrm{f}} = V_{\mathrm{f}}(\xi_{k+1}, \eta_{k+1}, w) - V_{\mathrm{f}}(\xi_k, \eta_k, w)$$

denote the increment of V_{f} along trajectories of (18). Substituting in the dynamics and adding and subtracting terms, we can express the overall change as

$$\Delta V_{\mathrm{f}} = \Delta V_{\mathrm{f}}^1 + \Delta V_{\mathrm{f}}^2 + \Delta V_{\mathrm{f}}^3, \tag{24}$$

where the summands are defined in (25).

We now bound each term in (24) individually. Applying (23c) to $|\Delta V_{\mathrm{f}}^1|$ we obtain

$$\begin{aligned}|\Delta V_{\mathrm{f}}^1| \le c_3\|\pi_x(K\eta_{k+1}, w) - \pi_x(K\eta_k, w)\|_2 \\ \cdot \Big(\|f(\xi_k + \pi_x(K\eta_k, w), K\eta_k, w) - \pi_x(K\eta_{k+1}, w)\|_2 \\ + \|f(\xi_k + \pi_x(K\eta_k, w), K\eta_k, w) - \pi_x(K\eta_k, w)\|_2\Big)\end{aligned} \tag{26}$$

Since π_x is Lipschitz continuous we have

$$\begin{aligned}\|\pi_x(K\eta_{k+1}, w) - \pi_x(K\eta_k, w)\|_2 &\le L_{\pi_x}\|K\|_2\|\eta_{k+1} - \eta_k\|_2 \\ &\le \lambda_{\min}(P)^{-\frac{1}{2}} L_{\pi_x}\|K\|_2\|\eta_{k+1} - \eta_k\|_P.\end{aligned} \tag{27}$$

Since $\pi_x(K\eta, w) = f(\pi_x(K\eta, w), K\eta, w)$ and f is Lipschitz continuous in (x, u) uniformly in w, we have

$$\begin{aligned}&\|f(\xi_k + \pi_x(K\eta_k, w), K\eta_k, w) - \pi_x(K\eta_{k+1}, w)\|_2 \\ &\le L_f \left\| \begin{bmatrix} \xi_k + \pi_x(K\eta_k, w) - \pi_x(K\eta_{k+1}, w) \\ K(\eta_k - \eta_{k+1}) \end{bmatrix} \right\|_2 \\ &\le L_f\|\xi_k\|_2 + L_f(1 + L_{\pi_x})\|K\|_2\lambda_{\min}(P)^{-\frac{1}{2}}\|\eta_{k+1} - \eta_k\|_P\end{aligned} \tag{28}$$

where we used that $\|z\|_2 \le \|z\|_1$ for $z \in \mathbb{R}^n$. Similarly, we compute that

$$\|f(\xi_k + \pi_x(K\eta_k, w), K\eta_k, w) - \pi_x(K\eta_k, w)\|_2 \le L_f\|\xi_k\|_2. \tag{29}$$

Substituting (27)–(29) back into (26), we obtain

$$\begin{aligned}|\Delta V_{\mathrm{f}}^1| \le c_3\lambda_{\min}(P)^{-1}L_f L_{\pi_x}(1 + L_{\pi_x})\|K\|_2^2\|\eta_{k+1} - \eta_k\|_P^2 \\ + 2c_3 L_{\pi_x} L_f\|K\|_2\lambda_{\min}(P)^{-\frac{1}{2}}\|\xi_k\|_2\|\eta_{k+1} - \eta_k\|_P.\end{aligned} \tag{30}$$

Substituting (22) into (30) and collecting terms, we finally obtain the bound

$$|\Delta V_{\mathrm{f}}^1| \le \zeta_k^{\mathsf{T}} Q_1 \zeta_k = \zeta_k^{\mathsf{T}} \begin{bmatrix} k_1\lambda^2 + k_2\lambda & k_3\lambda^2 + k_4\lambda \\ k_3\lambda^2 + k_4\lambda & k_5\lambda^2 \end{bmatrix} \zeta_k \tag{31}$$

where the explicit expressions for the coefficients are

$$\begin{aligned}k_1 &= c_3 L_f L_{\pi_x}(1 + L_{\pi_x})L_h^2\|K\|_2^2\alpha^2\lambda_{\min}(P)^{-\frac{1}{2}} \\ k_2 &= 2c_3 L_{\pi_x} L_f\|K\|_2\alpha\lambda_{\max}(P)^{\frac{1}{2}}\lambda_{\min}(P)^{-\frac{1}{2}} \\ k_3 &= c_3 L_f L_{\pi_x}(1 + L_{\pi_x})L_h\|K\|_2^2\alpha(1 + c_{\mathrm{fb}}) \\ &\qquad \cdot \lambda_{\min}(P)^{-1}\lambda_{\max}(P)^{\frac{1}{2}} \\ k_4 &= c_3 L_{\pi_x} L_f\|K\|_2(1 + c_{\mathrm{fb}})\lambda_{\min}(P)^{-\frac{1}{2}} \\ k_5 &= c_3 L_f L_{\pi_x}(1 + L_{\pi_x})\|K\|_2^2(1 + c_{\mathrm{fb}})^2\lambda_{\min}(P)^{-1}.\end{aligned}$$

To bound $|\Delta V_{\mathrm{f}}^2|$ we may apply (23d) to obtain

$$\begin{aligned}|\Delta V_{\mathrm{f}}^2| &\le c_4\|f(\xi_k + \pi_x(K\eta_k, w), K\eta_k, w) - \pi_x(K\eta_k, w)\|_2^2 \\ &\qquad \cdot \|K(\eta_{k+1} - \eta_k)\|_2 \\ &\le c_4 L_f^2\|K\|_2^2\lambda_{\min}(P)^{-\frac{1}{2}}\|\xi_k\|_2^2\|\eta_{k+1} - \eta_k\|_P,\end{aligned}$$

where in the second line we used (29). Since 0 is an interior point of $\mathcal{Z}$, there exists $r > 0$ such that $\mathcal{B}_r(0) := \{\xi \in \mathbb{R}^n \mid \|\xi\|_2 < r\} \subset \mathcal{Z}$. For $\xi_k \in \mathcal{B}_r(0)$ we therefore have the further bound

$$|\Delta V_{\mathrm{f}}^2| \le c_4 L_f^2\|K\|_2^2\lambda_{\min}(P)^{-\frac{1}{2}} r\|\xi_k\|_2\|\eta_{k+1} - \eta_k\|_P.$$

Substituting (22) into this, we finally obtain

$$|\Delta V_{\mathrm{f}}^2| \le \zeta_k^{\mathsf{T}} Q_2 \zeta_k = \zeta_k^{\mathsf{T}} \begin{bmatrix} k_6\lambda & k_7\lambda \\ k_7\lambda & 0 \end{bmatrix} \zeta_k \tag{32}$$

where

$$\begin{aligned}k_6 &= c_4 L_f^2\|K\|_2^2\lambda_{\min}(P)^{-\frac{1}{2}} r L_h\alpha\lambda_{\max}(P)^{\frac{1}{2}} \\ k_7 &= \tfrac{1}{2}c_4 L_f^2\|K\|_2^2\lambda_{\min}(P)^{-\frac{1}{2}} r(1 + c_{\mathrm{fb}}).\end{aligned}$$

To bound ΔV_{f}^3 we apply (23b) to obtain

$$\Delta V_{\mathrm{f}}^3 \le -\rho_{\mathrm{f}}\|\xi_k\|_2^2. \tag{33}$$

$$
\begin{aligned}
\Delta V_{\mathrm{f}}^{1} &\coloneqq V_{\mathrm{f}}(f(\xi_k + \pi_x(K\eta_k, w), K\eta_k, w) - \pi_x(K\eta_{k+1}, w), K\eta_{k+1}, w) \\
&\quad - V_{\mathrm{f}}(f(\xi_k + \pi_x(K\eta_k, w), K\eta_k, w) - \pi_x(K\eta_k, w), K\eta_{k+1}, w) \\
\Delta V_{\mathrm{f}}^{2} &\coloneqq V_{\mathrm{f}}(f(\xi_k + \pi_x(K\eta_k, w), K\eta_k, w) - \pi_x(K\eta_k, w), K\eta_{k+1}, w) \\
&\quad - V_{\mathrm{f}}(f(\xi_k + \pi_x(K\eta_k, w), K\eta_k, w) - \pi_x(K\eta_k, w), K\eta_k, w) \\
\Delta V_{\mathrm{f}}^{3} &\coloneqq V_{\mathrm{f}}(f(\xi_k + \pi_x(K\eta_k, w), K\eta_k, w) - \pi_x(K\eta_k, w), K\eta_k, w) - V_{\mathrm{f}}(\xi_k, K\eta_k, w).
\end{aligned}
\tag{25}
$$

Substituting the individual bounds for $|\Delta V_{\mathrm{f}}^1|$, $|\Delta V_{\mathrm{f}}^2|$, and ΔV_{f}^3 from (31), (32), and (33) back into (24), we obtain the overall bound

$$\Delta V_{\mathrm{f}} \leq \zeta_k Q_{\mathrm{f}} \zeta_k$$

where $Q_{\mathrm{f}} \coloneqq Q_1 + Q_2 + Q_3$ evaluates to

$$Q_{\mathrm{f}} = \begin{bmatrix} -\rho_{\mathrm{f}} + k_1\lambda^2 + (k_2 + k_6)\lambda & k_3\lambda^2 + (k_4 + k_7)\lambda \\ k_3\lambda^2 + (k_4 + k_7)\lambda & k_5\lambda^2 \end{bmatrix}.$$

Step #5 – Putting the Pieces Together: Define the composite Lyapunov candidate $V(\xi, \eta, w) = V_{\mathrm{s}}(\eta) + V_{\mathrm{f}}(\xi, \eta, w)$. Along trajectories of (18), we combine the previous inequalities to compute that

$$\Delta V = V(\xi_{k+1}, \eta_{k+1}, w) - V(\xi_k, \eta_k, w) \leq \zeta_k^{\mathsf{T}} Q \zeta_k$$

holds for all $(\xi_k, \eta_k, w) \in \mathcal{B}_r(0) \times \Gamma \times \mathcal{W}$, where

$$Q = \begin{bmatrix} -\rho_{\mathrm{f}} + (k_1 + q_1)\lambda^2 + \tilde{k}_2\lambda & k_3\lambda^2 + \tilde{k}_4\lambda \\ k_3\lambda^2 + \tilde{k}_4\lambda & -(1 - c_{\mathrm{dfb}}^2) + k_5\lambda^2 \end{bmatrix}$$

and where for compactness we set $\tilde{k}_2 = k_2 + k_6$ and $\tilde{k}_4 = k_4 + k_7 + q_2$. Note that the $(1,1)$ element of Q is negative and $\mathcal{O}(1)$ as $\lambda \to 0^+$[1]. From Proposition 2.1, we have that

$$1 - c_{\mathrm{dfb}}^2 = 2\lambda(1 - c_{\mathrm{fb}}) - \lambda^2(1 - c_{\mathrm{fb}})^2,$$

with $c_{\mathrm{fb}} \in (0,1)$, and therefore the $(2,2)$ element of Q is negative and $\mathcal{O}(\lambda)$ as $\lambda \to 0$. Since the off diagonal elements are $\mathcal{O}(\lambda)$ as $\lambda \to 0$, it is straightforward to argue that there exists some $\lambda^\star > 0$ such that $Q \prec 0$ for all $\lambda \in (0, \lambda^\star)$. Using (23a), there therefore exists $\varepsilon > 0$ such that $\Delta V(\xi_k, \eta_k, w) \leq -\varepsilon V(\xi_k, \eta_k, w)$ for all $(\xi_k, \eta_k, w) \in \mathcal{B}_r(0) \times \Gamma \times \mathcal{W}$. Standard arguments (e.g., [30, Thm. 13.2] now complete the proof. □

IV. Simulation Example

We now illustrate the action of the controller via an academic example. For simplicity, we generate a random internally stable discrete-time LTI system with sampling period $T_{\mathrm{s}} = 1$, $n = 5$ states, $r = 2$ measurements, $m = 3$ control inputs, and $n_w = 2$ exogenous reference signals, and consider a standard reference-tracking control arrangement where $e_k = y_k - w_k$. For this system, the DC gain matrix $G(1) \in \mathbb{R}^{r \times m}$ has full row rank. For the DP-I controller (14), the gain K is selected as $K = G(1)^\dagger$, and hence the monotonicity and Lipschitz conditions in Theorem 3.1 are satisfied with $P = I_2$ and $\mu = L = 1$; the remaining parameters are selected as $T_{\mathrm{i}} = 30$, and $\lambda = 0.9$. The inputs $u_k \in \mathbb{R}^3$ are constrained to the convex set

$$\mathcal{C} = \{u \in \mathbb{R}^3 \mid |u_1| + |u_2| \leq 1,\ |u_3| \leq 2\}.$$

For comparison purposes, we simulate (i) the response of the closed-loop system *without constraints* with the basic integral controller (2), and (ii) the response of the closed-loop system *with constraints* with the DP-I controller (14). Figure 2 shows the response under several step changes to the two reference signals. In the unconstrained case, as expected, the reference signals are tracked after the initial transient response. In the constrained case, sufficient actuator capacity is not available, and exact tracking is not achieved. Figure 3 shows the control signal, projected onto the u_1–u_2 plane; the two responses are identical when the projection operation is inactive (see Section III-A), but as expected $u_k \in \mathcal{C}$ at all times in the constrained case.

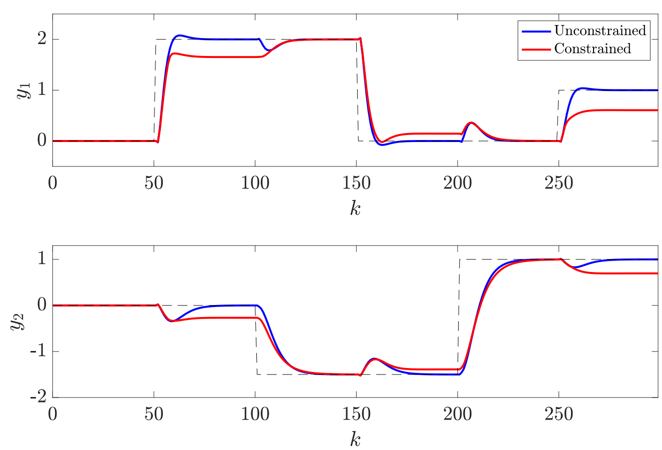

Fig. 2: Tracking response of closed-loop system; dashed black lines denote reference set-points.

V. Conclusions

We have formulated an approximate tracking specification as a variational inequality, and designed projected integral controller to meet this specification while maintaining arbitrary convex constraints on the input signal at all times. In the absence of constraints, the approximate tracking specification reduces to an exact tracking specification, and the projected integral controller reduces to a classical integral controller. The controller inherits what is perhaps the most important stability property of traditional integral control; under mild monotonicity conditions on the plant equilibrium mapping, closed-loop stability can be guaranteed when the plant is exponentially stable and the integral gain is sufficiently low.

[1]For a function $g : \mathbb{R} \to \mathbb{R}$ which is positive definite with respect to 0, a function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{O}(g(\lambda))$ as $\lambda \to 0$ if $\lim_{\lambda \to 0} |f(\lambda)|/g(\lambda) < \infty$.

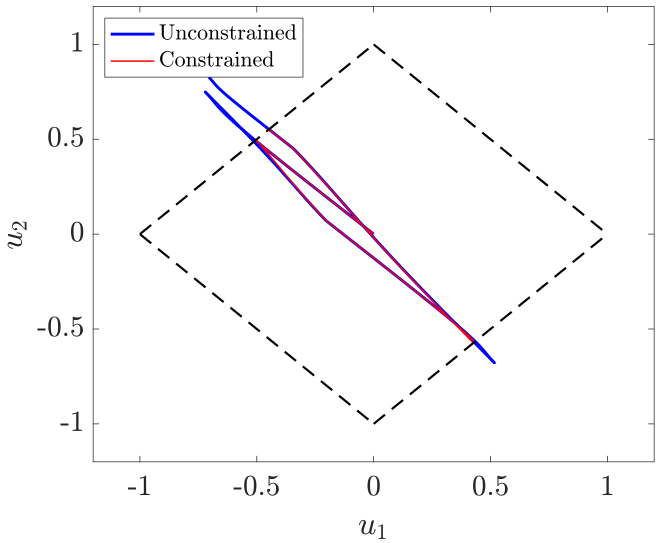

Fig. 3: Control inputs; dashed black lines denote constraints.

Future work will consider applications of this control scheme, the extension of this scheme to a projected PID controller, and the extension to more general discrete-time output-regulating controllers which admit a representation in incremental form.

Appendix

Consider the discrete-time nonlinear system

$$x_{k+1} = f(x_k, u) \tag{34}$$

where $f : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$, with $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ being convex sets. Assume that

(i) f is continuously differentiable on $\mathcal{X} \times \mathcal{U}$,
(ii) $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}$ are Lipschitz continuous in x on the set $\mathcal{X}$, uniformly in the inputs $u \in \mathcal{U}$,
(iii) there exists a continuously differentiable map $\pi_x : \mathcal{U} \to \mathcal{X}$ which is Lipschitz continuous on $\mathcal{U}$ and satisfies $\pi_x(u) = f(\pi_x(u), u)$ for all $u \in \mathcal{U}$.

Defining the new state variable $z = x - \pi_x(u)$, we obtain the deviation model

$$z_{k+1} = f(z_k + \pi_x(u), u) - \pi_x(u) := g(z_k, u), \tag{35}$$

where now $z = 0$ is an equilibrium point of (35) for all $u \in \mathcal{U}$.

Theorem A1: Consider the dynamics (35) under the previous assumptions. If the origin $z = 0$ is locally exponentially stable uniformly in $u \in \mathcal{U}$, then there exists a set $\mathcal{Z}$ containing 0 in its interior, a continuous function $V : \mathcal{Z} \times \mathcal{U} \to \mathbb{R}$, and constants $c_1, c_2, c_3, c_4 > 0$ and $\rho_{\mathrm{f}} \in (0, 1)$ such that

$$c_1\|z\|_2^2 \leq V(z, u) \leq c_2\|z\|_2^2 \tag{36a}$$
$$V(g(z, u), u) - V(z, u) \leq -\rho_{\mathrm{f}}\|z\|_2^2 \tag{36b}$$
$$|V(z, u) - V(z', u)| \leq c_3(\|z\|_2 + \|z'\|_2)\|z - z'\|_2 \tag{36c}$$
$$|V(z, u) - V(z, u')| \leq c_4\|z\|_2^2\|u - u'\|_2 \tag{36d}$$

for all $z, z' \in \mathcal{Z}$ and all $u, u' \in \mathcal{U}$.

Proof of Theorem A1: Let $\phi_k(z; u)$ denote the solution of (35) from initial condition $z_0 = z$ and with input $u \in \mathcal{U}$. By uniform exponential stability, there exist constants $M \geq 1$, $\rho \in (0, 1)$ and $\delta > 0$ such that

$$\|z\|_2 < \delta \implies \|\phi_k(z; u)\|_2 \leq M\rho^k\|z\|_2 \tag{37}$$

for all $k \geq 0$ and for any $u \in \mathcal{U}$. In particular then, $\|\phi_k(z; u)\|_2 \leq M\delta := \epsilon$ for all $k \geq 0$. Let $\mathcal{B}_\delta(0) = \{\xi \in \mathbb{R}^n \mid \|\xi\|_2 < \delta\}$ and define

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^n \mid \exists k \geq 0, z \in B_\delta(0), u \in \mathcal{U} \text{ s.t. } \phi_k(z; u) = \zeta\}.$$

In other words, $\mathcal{Z}$ is the image of all solutions with initial conditions in $B_\delta(0)$ and inputs in $\mathcal{U}$. Due to uniform exponential stability, we have that $\mathcal{Z} \subseteq B_\epsilon(0)$, so $\mathcal{Z}$ is a bounded set. Moreover, by definition, $\mathcal{Z}$ is positively invariant. Finally, note that $B_\delta(0) \subseteq \mathcal{Z}$, so 0 is in the interior of $\mathcal{Z}$. With this refined notation, we have that if $z \in B_\delta(0)$, then $\phi_k(z; u) \in \mathcal{Z}$ for any $k \geq 0$ and any $u \in \mathcal{U}$.

Next, define the function $V : \mathcal{Z} \times \mathcal{U} \to \mathbb{R}$ by

$$V(z, u) = \sum\nolimits_{k=0}^{N-1} \phi_k(z; u)^\mathsf{T}\phi_k(z; u)$$

for some $N \geq 1$ to be determined. Since g is Lipschitz continuous in both arguments, so is ϕ_k, so V is continuous in both arguments. We first observe that

$$\begin{aligned} V(z, u) &= \phi_0(z; u)^\mathsf{T}\phi_0(z; u) + \sum\nolimits_{k=1}^{N-1} \phi_k(z; u)^\mathsf{T}\phi_k(z; u) \\ &\geq \phi_0(z; u)^\mathsf{T}\phi_0(z; u) \\ &= \|z\|_2^2. \end{aligned}$$

By uniform exponential stability, it also holds that

$$V(z, u) \leq \sum\nolimits_{k=0}^{N-1} M^2\rho^{2k}\|z\|_2^2,$$

and hence (36a) holds for all $(z, u) \in \mathcal{Z} \times \mathcal{U}$ with $c_1 = 1$ and $c_2 = \sum_{k=0}^{N-1} M^2\rho^{2k}$. Next, note that since $\mathcal{Z}$ is forward invariant, we have that $g(z, u) \in \mathcal{Z}$ for any $z \in \mathcal{Z}$ and any $u \in \mathcal{U}$. We may therefore compute for $z \in \mathcal{Z}$ that

$$\begin{aligned} &V(g(z, u), u) - V(z, u) \\ &\quad = \sum_{k=0}^{N-1} \|\phi_k(g(z, u); u)\|_2^2 - \|\phi_k(z; u)\|_2^2 \end{aligned}$$

By the semi-group property of solutions, we have

$$\phi_k(g(z, u); u) = \phi_{k+1}(z; u).$$

The sum therefore telescopes, and we find that

$$\begin{aligned} V(g(z, u)) - V(z, u) &= \|\phi_N(z; u)\|_2^2 - \|z\|_2^2 \\ &\leq -\left(1 - M^2\rho^{2N}\right)\|z\|_2^2, \end{aligned}$$

where we have used (37). Selecting N large enough such that $M^2\rho^{2N} < 1$, we define $\rho_{\mathrm{f}} = 1 - M^2\rho^{2N} \in (0, 1)$ and we obtain (36b) for all $(z, u) \in \mathcal{Z} \times \mathcal{U}$. To show (36c) we first compute for $z, z' \in \mathcal{Z}$ that

$$\begin{aligned} &\|g(z, u) - g(z', u)\|_2 \\ &\quad \leq \|f(z + \pi_x(u), u) - f(z' + \pi_x(u), u)\| \\ &\quad \leq L_1\|z - z'\|_2 \end{aligned}$$

where $L_1 > 0$ is the Lipschitz constant of f. It quickly follows then that $\|\phi_k(z;u) - \phi_k(z';u)\|_2 \leq L_1^k \|z - z'\|_2$. We may now compute that

$$\begin{aligned}|V(z,u) - V(z',u)| &= \Big|\sum\nolimits_{k=0}^{N-1} \phi_k(z;u)^\mathsf{T}\phi_k(z;u) \\ &\quad - \sum\nolimits_{k=0}^{N-1} \phi_k(z';u)^\mathsf{T}\phi_k(z';u)\Big| \\ &= \Big|\sum\nolimits_{k=0}^{N-1} \phi_k(z;u)^\mathsf{T}(\phi_k(z;u) - \phi_k(z';u)) \\ &\quad - \sum\nolimits_{k=0}^{N-1} \phi_k(z';u)^\mathsf{T}(\phi_k(z';u) - \phi_k(z;u))\Big| \\ &\leq \sum_{k=0}^{N-1} L_1^k(\|\phi_k(z;u)\|_2 + \|\phi_k(z';u)\|_2)\|z - z'\|_2 \\ &\leq \sum_{k=0}^{N-1} L_1^k M\rho^k(\|z\|_2 + \|z'\|_2)\|z - z'\|_2\end{aligned}$$

so (36c) holds with $c_4 = \sum_{k=0}^{N-1} L_1^k M\rho^k$. To show (36d), begin by computing (35) that

$$\begin{aligned}\tfrac{\partial g}{\partial u}(z,u) &= \tfrac{\partial f}{\partial x}(z + \pi_x(u), u)\tfrac{\partial \pi_x}{\partial u}(u) \\ &\quad + \tfrac{\partial f}{\partial u}(z + \pi_x(u), u) - \tfrac{\partial \pi_x}{\partial u}(u).\end{aligned} \tag{38}$$

Next, since $\pi_x(u) = f(\pi_x(u), u)$ for any $u \in \mathcal{U}$, we have by continuous differentiability that

$$\tfrac{\partial \pi_x}{\partial u}(u) = \tfrac{\partial f}{\partial x}(\pi_x(u), u)\tfrac{\partial \pi_x}{\partial u}(u) + \tfrac{\partial f}{\partial u}(\pi_x(u), u). \tag{39}$$

Inserting (39) into (38) we find that

$$\begin{aligned}\tfrac{\partial g}{\partial u}(z,u) &= \left[\tfrac{\partial f}{\partial x}(z + \pi_x(u), u) - \tfrac{\partial f}{\partial x}(\pi_x(u), u)\right]\tfrac{\partial \pi_x}{\partial u}(u) \\ &\quad + \left[\tfrac{\partial f}{\partial u}(z + \pi_x(u), u) - \tfrac{\partial f}{\partial u}(\pi_x(u), u)\right].\end{aligned}$$

Using Lipschitz continuity of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial u}$, and π_x, we therefore obtain the bound

$$\|\tfrac{\partial g}{\partial u}(z,u)\|_2 \leq \ell\|z\|_2.$$

for some $\ell > 0$. By convexity of $\mathcal{U}$ and the multivariable mean-value theorem, this immediately implies that

$$\|g(z,u) - g(z,u')\|_2 \leq \ell\|z\|_2\|u - u'\|_2. \tag{40}$$

Let $\Delta_k = \phi_k(z;u) - \phi_k(z;u')$ denote the difference between the flows given different inputs. We bound this as

$$\begin{aligned}\|\Delta_k\|_2 &= \|g(\phi_{k-1}(z;u), u) - g(\phi_{k-1}(z;u'), u')\|_2 \\ &= \|g(\phi_{k-1}(z;u), u) - g(\phi_{k-1}(z;u'), u) \\ &\quad - [g(\phi_{k-1}(z;u'), u') - g(\phi_{k-1}(z;u'), u)]\|_2 \\ &\leq L_1\|\Delta_{k-1}\|_2 + L_1\ell\|\phi_{k-1}(z,u')\|_2\|u - u'\|_2 \\ &\leq L_1\|\Delta_{k-1}\|_2 + L_1\ell M\rho^{k-1}\|z\|_2\|u - u'\|_2\end{aligned}$$

where we added and subtracted terms and then bounded using (40) and (37). Using this bound recursively and noting that $\Delta_0 = 0$, we obtain

$$\|\Delta_k\|_2 \leq \|z\|_2\|u - u'\|_2 \underbrace{\sum\nolimits_{m=0}^{k-1} L_1^{k-m}\ell M\rho^m}_{:=\kappa}.$$

We are now ready to compute that

$$\begin{aligned}|V(z,u) - V(z,u')| &= \Big|\sum\nolimits_{k=0}^{N-1} \phi_k(z;u)^\mathsf{T}\phi_k(z;u) \\ &\quad - \sum\nolimits_{k=0}^{N-1} \phi_k(z;u')^\mathsf{T}\phi_k(z;u')\Big| \\ &= \sum\nolimits_{k=0}^{N-1} \phi_k(z;u)^\mathsf{T}[\phi_k(z;u) - \phi_k(z;u')] \\ &\quad - \sum\nolimits_{k=0}^{N-1} \phi_k(z;u')^\mathsf{T}[\phi_k(z;u') - \phi_k(z;u)]\Big| \\ &\leq \sum\nolimits_{k=0}^{N-1} (\|\phi_k(z;u)\|_2 + \|\phi_k(z;u')\|_2)\|\Delta_k\|_2 \\ &\leq \kappa\|z\|_2^2\|u - u'\|_2 \sum\nolimits_{k=0}^{N-1} 2M\rho^k\end{aligned}$$

which completes the proof with $c_5 = \sum_{k=0}^{N-1} 2\kappa M\rho^k$. □

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