Low-Gain Stabilizers for Linear-Convex Optimal Steady-State Control (Extended Version)

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***Abstract*— We consider the problem of designing a feedback controller which robustly regulates an LTI system to an optimal operating point in the presence of unmeasured disturbances. A general design framework based on so-called optimality models was previously put forward for this class of problems, effectively reducing the problem to that of stabilization of an associated nonlinear plant. This paper presents several simple and fully constructive stabilizer designs to accompany the optimality model designs from [1]. The designs are based on a low-gain integral control approach, which enforces time-scale separation between the exponentially stable plant and the controller. We provide explicit formulas for controllers and gains, along with LMI-based methods for the computation of robust/optimal gains. The results are illustrated via an academic example and an application to power system frequency control.**

I. Introduction

Many practical engineering systems are subject to both dynamic response specifications and minimum-cost steady-state operation specifications. Achieving the former typically relies on both good process design and the design of one or more accompanying feedback controllers. In contrast, the latter optimality criterion is most often addressed in a hierarchical fashion, wherein actuator set-points are scheduled using a model and then forwarded to the lower-level feedback controllers as references. The quality of set-points computed via this procedure may however be poor, due to inaccuracies in the steady-state system model and due to discrepancies between forecasted and real-time disturbances. Such inaccuracies will lead to sub-optimal system operation, and may lead to violation of desired operating constraints; these issues motivate the incorporation of additional real-time feedback into the computation of optimal set-points.

Perhaps the most well-known framework which blends the control and optimization layers is model-predictive control (MPC), wherein simultaneous control and optimization is achieved via repeated solution of dynamic optimization problems [2]. An alternative approach — tailored towards applications where low-level dynamic controllers are already in place — is to directly incorporate measurement feedback into the set-point scheduling process, such that the closed-loop system converges to a cost-minimizing operating point. By incorporating feedback, sensitivity to steady-state model uncertainty can be reduced, and constraint violation can be eliminated. While not as universally applicable as MPC, this approach has the benefit of producing simple and explicit controller designs, which are intuitive and interpretable to domain experts in particular applications (e.g., [3]–[5]).

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The general problem described above been studied recently under different names, including dynamic KKT controllers [6], feedback-based optimization [7]–[11], autonomous optimization [12], [13], or — the term we will use here — optimal steady-state (OSS) control [1], [14], [15]; see also [16]–[19]. While the problem setups in the above references vary significantly in terms of the assumptions on the plant model and on the steady-state optimization problem, the general goal is the design of simple feedback laws which compute the set-points required to guide the plant towards a cost-minimizing and constraint-satisfying operating point.

Contributions: This paper continues the development of the OSS control framework put forward in [1], [15], which will be reviewed in Section II. In brief, the framework in [1] was shown to provide substantial flexibility to the designer, enabling customization of the control architecture (centralized vs. distributed) through judicious selection of several gain matrices. However, stabilizability guarantees were provided only for the case of quadratic steady-state optimization problems. Here, focusing on the practically-relevant case of a stable LTI plant, we provide stabilizer designs to accompany the "optimality models" from [1] (Section IV). This renders the approach of [1] fully constructive without the restriction of quadratic costs, and while retaining the design flexibility inherent in [1]. As a secondary contribution, the development of optimality models here (Section III) is more streamlined than in [1]. Taken together, our results provide a library of constructive solutions for solving the OSS control problem in the framework of [1] for exponentially stable plants.

II. Problem Formulation

Consider the finite-dimensional continuous-time LTI plant

$$\begin{aligned} \dot{x} &= Ax + Bu + B_w w \\ z &= Cx + Du + D_w w \end{aligned} \tag{1}$$

with state $x(t) \in \mathbb{R}^n$, measured output $z(t) \in \mathbb{R}^r$, control input $u(t) \in \mathbb{R}^m$, and *constant* exogenous signal $w \in \mathcal{W} \subset \mathbb{R}^{n_w}$, modelling reference signals and (potentially, unmeasured) external disturbances. As we will be subsequently pursuing a low-gain design philosophy [20], [21], we assume that (1) is internally exponentially stable, i.e., A is Hurwitz. This stability may be inherent to the plant model, or may have been achieved through a previous compensator design. In this case, the steady-state input-output mapping of (1) is

$$\bar{z} = G_u \bar{u} + G_w w \tag{2}$$

in which $\bar{z}, \bar{u}$ denote equilibriums value of z and u, and

$$G_u = -CA^{-1}B + D \in \mathbb{R}^{r \times m}$$
$$G_w = -CA^{-1}B_w + D_w \in \mathbb{R}^{r \times n_w}$$

denote the DC gain matrices of (1). Our controller design specification, which we refer to as the *optimal steady-state* (OSS) control problem, roughly follows that in [1].

*Problem 2.1 (***OSS Control***):* For the plant (1), design an output-feedback controller such that for each $w \in \mathcal{W}$, the closed-loop system possesses a unique locally exponentially stable equilibrium point, and such that the equilibrium pair $(\bar{u}, \bar{z})$ is an optimal point of the problem

$$\underset{\bar{z}, \bar{u}}{\text{minimize}} \quad f_0(\bar{u}) + g_0(\bar{z}) \tag{3a}$$
$$\text{subject to} \quad \bar{z} = G_u \bar{u} + G_w w \tag{3b}$$
$$0 = H_z \bar{z} + H_u \bar{u} + H_w w. \tag{3c}$$

In the optimization problem (3), $f_0 : \mathcal{U} \to \mathbb{R}$ and $g_0 : \mathcal{Z} \to \mathbb{R}$ are convex objective functions to be minimized (e.g., operational costs, measures of tracking error), where $\mathcal{U} \subseteq \mathbb{R}^m$ and $\mathcal{Z} \subseteq \mathbb{R}^r$ are closed and non-empty convex sets. The functions f_0, g_0 may incorporate barrier or penalty terms for enforcement of inequality constraints. The constraint (3b) is the steady-state constraint imposed by the dynamic system (1). Finally, (3c) represents n_c additional affine *engineering constraints* imposed by the designer, with $H_z \in \mathbb{R}^{n_c \times r}$, $H_u \in \mathbb{R}^{n_c \times m}$, and $H_w \in \mathbb{R}^{n_c \times n_w}$. The goal is to minimize the objective function, subject to the physical limitations imposed by the plant in steady-state, and subject to the engineering constraints imposed by the designer. We make the following standing assumptions on the problem data.

*Assumption 2.1 (***Problem Data***):* On the interiors of their domains, the maps f_0 and g_0 are convex, continuously differentiable, and have at least locally Lipschitz continuous gradients. The set of disturbances $\mathcal{W}$ is convex and compact. The problem (3) is strictly feasible and has an optimal solution for all $w \in \mathcal{W}$. Both $z(t)$ and the engineering constraint violation $H_z z(t) + H_u u(t) + H_w w(t)$ are measurable.

Uniqueness of a primal solution to (3) is guaranteed if $u \mapsto f_0(u) + g_0(G_u u + G_w w)$ is strongly convex on $\mathcal{U} \cap \{u \in \mathcal{U} \mid G_u u + G_w w \in \mathcal{Z}\}$ for each $w \in \mathcal{W}$; this holds, for instance, if f_0 is strongly convex on $\mathcal{U}$. The dual variable μ associated with the linear constraint

$$0 = H_z(G_u \bar{u} + G_w w) + H_u \bar{u} + H_w w$$

will be unique if $H_z G_u + H_u$ has full row rank; ensuring this is often simply a matter of the designer properly specifying the desired additional constraints (3c).

A. Review of OSS Control

In [1] a framework was developed for the construction of controllers solving Problem 2.1. The key idea introduced was that of an *optimality model*, which is a (potentially, dynamic) nonlinear filter of the form

$$\tau \dot{\mu} = \varphi(\mu, z, u), \qquad e = h(\mu, z, u) \tag{4}$$

with state μ, output e, and time constant $\tau > 0$. The filter (4) is said to be an optimality model if the following statement holds: if $(\bar{x}, \bar{\mu}, \bar{u}, \bar{z})$ is an equilibrium of (1), (4) satisfying $0 = h(\bar{\mu}, \bar{z}, \bar{u})$, then $(\bar{z}, \bar{u})$ is an optimal solution of (3) The idea is that $\dot{\mu}$ and the *error* variable e in (4) should together provide a measure of the optimality gap. Specifically, $\dot{\mu} = 0$ along with regulation of e to zero will result in regulation of $(\bar{z}, \bar{u})$ to an optimal solution of (3), since the plant itself will enforce the constraint (3b) in equilibrium. To this end, the error output e from (4) is fed to an integral controller

$$\tau \dot{\eta} = e. \tag{5}$$

The optimality model is appended to the output of the plant (1), creating a cascade of the plant, the optimality model, and a bank of integrators. If a stabilizer processing (z,μ,η) can be designed to ensure that the cascade (1), (4), (5) possess a unique exponentially stable equilibrium point, then the optimality model, integral controller, and stabilizer together constitute a solution to the OSS control problem (Figure 1).

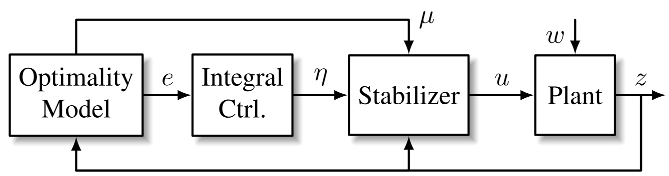

Fig. 1: Block diagram of general OSS control architecture [1].

In [1] several optimality models were designed based on variations of the KKT conditions for (3), and results on stabilizability and detectability of the cascaded system (1), (4), (5) were presented for the case of *quadratic* objective functions f_0, g_0. When the objective functions are non-quadratic, the design of a stabilizer becomes more challenging, and no guarantees or constructive results were provided in [1]. The goal of this paper is to address this gap for the practically-relevant case of an exponentially stable plant.

III. Development of Optimality Models

Under Assumption 2.1 the KKT conditions provide a necessary and sufficient characterization of optimal points of (3) and can be used to construct optimality models. Different optimality models can be derived by manipulating the optimization problem (3) and the resulting KKT conditions, and we now pursue this direction. Section III-A is conceptually identical to [1, Prop. 3.3], while Section III-B and Section III-C contain new results extending the ideas in [1, Prop. 3.4] and [1, Prop. 3.5], respectively.

A. Optimality Model #1

The most obvious initial step is to eliminate the variable $\bar{z}$ from the problem (3). Doing so, the resulting Lagrangian is

$$L(\bar{u}, \bar{\mu}) = f_0(\bar{u}) + g_0(G_u \bar{u} + G_w w) + \bar{\mu}^{\mathsf{T}}(H_z(G_u \bar{u} + G_w w) + H_u \bar{u} + H_w w)$$

with multiplier $\bar{\mu} \in \mathbb{R}^{n_c}$, leading to the KKT conditions

$$\begin{aligned} 0 &= \nabla f_0(\bar{u}) + G_u^\mathsf{T} \nabla g_0(G_u\bar{u} + G_w w) \\ &\quad + (H_z G_u + H_u)^\mathsf{T} \mu \\ 0 &= H_z(G_u\bar{u} + G_w w) + H_u \bar{u} + H_w w. \end{aligned} \tag{6}$$

The approach to constructing an optimality model is now based on replacement of the steady-state output value $\bar{z} = G_u\bar{u} + G_w w$ by the real-time measurement $z(t)$. Following this idea leads to the optimality model

$$\tau\dot{\mu} = H_z z + H_u u + H_w w \tag{7a}$$

$$e = \nabla f_0(u) + G_u^\mathsf{T} \nabla g_0(z) + (H_z G_u + H_u)^\mathsf{T} \mu \tag{7b}$$

where $\tau > 0$ is a tuning parameter. In [1], (7) was referred to as the *output subspace* optimality model; integration of the constraint violation in (7a) will ensure primal feasibility, while $e = 0$ in (7b) will ensure stationarity.

B. Optimality Model #2

A quite different optimality model can be obtained via a more involved reduction of the linear constraints (3b)–(3c). First note that (3b)–(3c) can be expressed as

$$\begin{bmatrix} I_r & -G_u \\ H_z & H_u \end{bmatrix} \begin{bmatrix} \bar{z} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} G_w \\ H_w \end{bmatrix} w. \tag{8}$$

Let $T = \left[\begin{smallmatrix} T_z \\ T_u \end{smallmatrix}\right] \in \mathbb{R}^{(r+m)\times q}$ be a matrix such that

$$\mathrm{range}(T) = \mathrm{null} \begin{bmatrix} I_r & -G_u \\ H_z & H_u \end{bmatrix}. \tag{9}$$

This allows the constraints (3b)–(3c) to be expressed as

$$\begin{bmatrix} \bar{z} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} T_z \\ T_u \end{bmatrix} \bar{\xi} + \begin{bmatrix} z_0(w) \\ u_0(w) \end{bmatrix}, \tag{10}$$

where $(z_0(w), u_0(w))$ is any solution of (8) and $\bar{\xi} \in \mathbb{R}^q$ is free. The basic idea is that $\mathrm{range}(T)$ captures the subspace on which the physical steady-state plant constraints intersect with the specified steady-state engineering constraints.

Using (10), the problem (3) is now equivalent to the unconstrained minimization problem

$$\underset{\bar{\xi}\in\mathbb{R}^q}{\text{minimize}} \quad f_0(T_u\bar{\xi} + u_0(w)) + g_0(T_z\bar{\xi} + z_0(w)) \tag{11}$$

with stationarity condition

$$0 = T_u^\mathsf{T} \nabla f_0(T_u\bar{\xi} + u_0(w)) + T_z^\mathsf{T} \nabla g_0(T_z\bar{\xi} + z_0(w)). \tag{12}$$

By now re-inserting the real-time signals $u(t)$ and $z(t)$ in place of $\bar{u}$ and $\bar{z}$, we obtain the optimality model

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} T_u^\mathsf{T} \nabla f_0(u) + T_z^\mathsf{T} \nabla g_0(z) \\ H_z z + H_u u + H_w w \end{bmatrix}. \tag{13}$$

In contrast with (7), (13) does not contain a dynamic state corresponding to any dual variable; in [1], a slightly different construction of (13) was referred to as the *feasible subspace* optimality model. Regulation of the error signal $e_1 \in \mathbb{R}^q$ to zero ensures stationarity, while regulation of $e_2 \in \mathbb{R}^{n_c}$ to zero ensures primal feasibility.

If T is selected to have full column rank, then $q = m - n_c$, and the total number of error signals in (13) is $m - n_c + n_c = m$, the number of plant inputs. Moreover, in this case, T_z and T_u enjoy several rank properties.

Lemma 3.1: If $T = \left[\begin{smallmatrix} T_z \\ T_u \end{smallmatrix}\right] \in \mathbb{R}^{(r+m)\times q}$ is a matrix of full column rank satisfying (9), then the following hold:

(i) $T_u \in \mathbb{R}^{m\times q}$ has full column rank;

(ii) $T_z \in \mathbb{R}^{r\times q}$ has full column rank if and only if

$$\mathrm{range}(T_u) \subseteq \mathrm{range}(G_u^\mathsf{T}) = \mathrm{range}(G_u^\dagger); \tag{14}$$

(iii) T_z has full column rank if G_u has full column rank;

(iv) if G_u has full row rank, then T_z may be chosen to have full column rank if and only if there exists a full column rank matrix X such that $(H_z + H_u G_u^\dagger)X = 0$, in which case one choice is $T_z = X$ and $T_u = G_u^\dagger X$.

Proof of Lemma 3.1: (i): Suppose by contradiction that $T_u\xi = 0$ for some ξ. From (9), we have that $T_z = G_u T_u$. Multiplying this by ξ, we find that $T_z\xi = 0$, so $\xi \in \mathrm{null}(T)$, which contradicts the fact that T has full column rank.

(ii): Beginning from $T_z = G_u T_u$, since T_u has full column rank, T_z has full column rank if and only if $\mathrm{range}(T_u) \subseteq \mathrm{null}(G_u)^\perp = \mathrm{range}(G_u^\mathsf{T})$. The final statement follows from the fact that $\mathrm{range}(G_u^\dagger) = \mathrm{range}(G_u^\mathsf{T})$ for any pseudoinverse $G_u^\dagger$ of G.

(iii): If G has full column rank then, $\mathrm{null}(G) = \{0\}$, and thus $\mathrm{range}(G_u^\mathsf{T}) = \mathbb{R}^m$, and the result follows from (ii).

(iv): First note that if G_u has full row rank, then $G_u^\dagger$ is given explicitly as $G_u^\dagger = G_u^\mathsf{T}(G_u G_u^\mathsf{T})^{-1}$, satisfying $G_u G_u^\dagger = I_r$, and $G_u^\dagger$ has full column rank. From (ii), we can then say that T_z has full column rank if and only if $T_u = G_u^\dagger X$ for some full column rank matrix $X \in \mathbb{R}^{r\times k}$. In this case, $T_z = G_u G_u^\dagger X = X$. Substituting into the second block equation in (9), X must satisfy

$$0 = (H_z + H_u G_u^\dagger)X$$

which shows the desired result. □

C. Optimality Model #3

The idea behind the third and final optimality model we consider is to exploit any available complementarity between the error signals in (13); the following is an extension of [1, Prop. 3.5].

*Proposition 3.2 (***Reduced Error***):* Suppose that there exists $\ell \in \mathbb{Z}$ such that $0 < \ell \leq \min\{q, n_c\}$ and matrices

$$\begin{aligned} C_1 &\in \mathbb{R}^{(q-\ell)\times q} & C_2 &\in \mathbb{R}^{(n_c-\ell)\times n_c} \\ C_1' &\in \mathbb{R}^{\ell\times q} & C_2' &\in \mathbb{R}^{\ell\times n_c} \end{aligned} \qquad \mathcal{C} := \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \\ C_1' & C_2' \end{bmatrix}$$

such that

(a) $\mathcal{C}$ has full row rank,
(b) $\mathrm{col}(C_1, 0, C_1')$ has full column rank,
(c) $\mathrm{col}(0, C_2, C_2')$ has full column rank, and
(d) $\mathrm{range}(C_1' T_u^\mathsf{T}) \cap \mathrm{range}(C_2'(H_z G_u + H_u)) = \{0\}$.

With $\mathrm{col}(e_1, e_2) \in \mathbb{R}^{q+n_c}$ given by (13), define the reduced set of $q + n_c - \ell$ error signals $e' = \mathrm{col}(e_1', e_2', e_3') = \mathcal{C}\mathrm{col}(e_1, e_2)$, or explicitly

$$\begin{bmatrix} e_1' \\ e_2' \\ e_3' \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \\ C_1' & C_2' \end{bmatrix} \begin{bmatrix} T_u^\mathsf{T} \nabla f_0(u) + T_z^\mathsf{T} \nabla g_0(z) \\ H_z z + H_u u + H_w w \end{bmatrix}. \tag{15}$$

Then $\bar{e}' = 0$ if and only if $\bar{e} = 0$.

Proof: The "if" direction is immediate. To establish "only if", note first that by feasibility of (3) in Assumption 2.1, for each w there exists a pair $(u(w), z(w))$ such that

$$\begin{aligned} 0 &= H_z z(w) + H_u u(w) + H_w w \\ z(w) &= G_u u(w) + G_w w, \end{aligned}$$

which implies that

$$0 = (H_z G_u + H_u) u(w) + (H_z G_w + H_w) w. \quad (16)$$

With this, any equilibrium $(\bar{z}, \bar{u})$ can be expressed as

$$(\bar{z}, \bar{u}) = (z(w) + G_u v, u(w) + v). \quad (17)$$

for some $v \in \mathbb{R}^m$. The equilibrium values of the error signals (e_1', e_2', e_3') from (15) can now be computed by substituting (17) into the definition and simplifying using (16), which yields the expressions

$$\begin{bmatrix} \bar{e}_1' \\ \bar{e}_2' \\ \bar{e}_3' \end{bmatrix} = \begin{bmatrix} C_1 \bar{\zeta} \\ C_2 (H_z G_u + H_u) v \\ C_1' \bar{\zeta} + C_2' (H_z G_u + H_u) v \end{bmatrix}$$

where for compactness $\bar{\zeta} = T_u^{\mathsf{T}} \nabla f_0(\bar{u}) + T_z^{\mathsf{T}} \nabla g_0(\bar{z})$. Since $T_z = G_u T_u$, it follows that $\mathrm{range}(T_z^{\mathsf{T}}) \subseteq \mathrm{range}(T_u^{\mathsf{T}})$, so $\bar{\zeta} \in \mathrm{range}(T_u^{\mathsf{T}})$, and therefore there exists ξ such that $\bar{\zeta} = T_u^{\mathsf{T}} \xi$. Obviously, $\bar{\zeta} = 0$ if and only if $\xi \in \mathrm{null}(T_u^{\mathsf{T}})$. With this, we can express these equilibrium values as

$$\bar{e}' = \begin{bmatrix} \bar{e}_1' \\ \bar{e}_2' \\ \bar{e}_3' \end{bmatrix} = \begin{bmatrix} C_1 \\ 0 \\ C_1' \end{bmatrix} T_u^{\mathsf{T}} \xi + \begin{bmatrix} 0 \\ C_2 \\ C_2' \end{bmatrix} (H_z G_u + H_u) v \quad (18)$$

If $\bar{e}' = 0$, then properties (a), (b), and (c) above immediately imply that $T_u^{\mathsf{T}} \xi = 0$ and $(H_z G_u + H_u) v = 0$. The first equality immediately implies that $0 = \bar{\zeta} = T_u^{\mathsf{T}} \nabla f_0(\bar{u}) + T_z^{\mathsf{T}} \nabla g_0(\bar{z})$, so $\bar{e}_1 = 0$. After again using (16), the second equality implies that $\bar{e}_2 = 0$, which completes the proof. ■

In summary, under the conditions (a),(b), and (c) above, steady-state zeroing of the $q + n_c - \ell$ error signals e' in (15) is equivalent to steady-state zeroing of the $q + n_c$ error signals e in (13). Zeroing e' has the obvious advantage of reducing the dimension of the bank of integrators (5) required in the final controller implementation.

IV. Stabilizers for Low-Gain OSS Control

This section details stabilizer designs to accompany the optimality models described in Section III. We follow a low-gain integral control approach [19]–[21], wherein the controller is tuned to operate slowly compared to the stable plant dynamics (1). Taken together, the optimality model (4), integral controller (5), and stabilizer (to be designed) will have the nonlinear state-space form

$$\tau \dot{x}_c = f_c(x_c, z, u), \qquad u = k_c(x_c),$$

with concatenated state x_c and time constant $\tau > 0$. The maps f_c and k_c will be (at least) locally Lipschitz continuous on the domain of interest. When τ is large, x_c will change slowly, as will u, and the plant (1) will quickly converge to the quasi-steady-state output $\bar{z}(u, w) := G_u u + G_w w$. It follows from singular perturbation arguments [22, Chp. 11] that if the reduced system

$$\dot{x}_c = f_c(x_c, \bar{z}(u, w), u), \qquad u = k_c(x_c).$$

possesses a unique locally exponentially stable equilibrium point, then there will exist $\tau^\star > 0$ such that for all $\tau \in (0, \tau^\star)$, the closed-loop system will possess a unique locally exponentially stable equilibrium point, and the controller will solve Problem 2.1. In what follows, we therefore jump immediately to analyses of the reduced dynamics arising from our different stabilizer designs, and stability results are stated with the understanding that a full singular perturbation argument can indeed be made.

A. Optimality Model #1

For the optimality model (7), we present two approaches for the design of accompanying stabilizers.

1) Primal-Dual Stabilizer: Integrating the error from (7) as $\tau \dot{\eta} = -e$, and selecting the simple stabilizer $u = \eta$ leads immediately to the *primal-dual*-type algorithm

$$\begin{aligned} \tau_p \dot{u} &= -\nabla f_0(u) - G_u^{\mathsf{T}} \nabla g_0(z) - (H_z G_u + H_u)^{\mathsf{T}} \mu \\ \tau_d \dot{\mu} &= H_z z + H_u u + H_w w \end{aligned} \quad (19)$$

with time constants $\tau_p, \tau_d > 0$. As the stability of closely related schemes has already been examined in the literature (e.g., [7]) we simply state the key stability result.

Proposition 4.1 (**Primal-Dual**): If $u \mapsto f_0(u) + g_0(G_u u)$ is strongly convex on $\mathcal{U}$, and if $H_z G_u + H_u$ has full row rank, then there exists $\tau^\star > 0$ such that for all $(\tau_p, \tau_d) \in (\tau^\star, \infty)^2$, the controller (19) solves Problem 2.1.

2) Inversion-Based Stabilizer: Our second design departs slightly from the architecture in Figure 1 by omitting the bank of integrators. If ∇f is an invertible mapping, then one may explicitly solve (7b) for u to obtain the controller

$$\tau \dot{\mu} = H_z z + H_u u + H_w w \quad (20a)$$
$$u = (\nabla f_0)^{-1} (-G_u^{\mathsf{T}} \nabla g_0(z) - (H_z G_u + H_u)^{\mathsf{T}} \mu), \quad (20b)$$

consisting of an integration on the constraint violation (20a) followed by a nonlinear static output feedback (20b). The following result characterizes a case of interest where this is well-posed and leads to closed-loop stability.

Theorem 4.2 (**Inversion-Based OSS Control**): Consider the LTI system (1) with the inversion-based controller (20). Assume that $H_z G_u + H_u$ has full row rank, and that f_0 is strongly convex and essentially smooth[1] on $\mathcal{U}$. Then there exists $\tau^\star > 0$ such that for all $\tau \in (0, \tau^\star)$, the controller (20) solves Problem 2.1.

Proof: The reduced dynamics of the closed-loop (1), (20) are obtained by substituting (2) into (20), yielding

$$\dot{\mu} = N u + \tilde{N} w \quad (21a)$$
$$u = (\nabla f_0)^{-1} (-G_u^{\mathsf{T}} \nabla g_0(G_u u + G_w w) - N^{\mathsf{T}} \mu) \quad (21b)$$

[1]A continuously differentiable convex function $f_0 : \mathcal{U} \to \mathbb{R}$ is essentially smooth on $\mathcal{U}$ if $f(x) \to \infty$ as x tends to the boundary of $\mathcal{U}$.

where for compactness we have set $N := H_z G_u + H_u$ and $\tilde{N} := H_z G_w + H_w$. By the strong convexity and rank assumptions, for each $w \in \mathcal{W}$, (3) possesses a unique primal-dual optimal solution $(u^\star, \mu^\star)$, characterized by (6), and corresponding to the unique equilibrium point of (21). Set $\xi^\star := -G_u^\mathsf{T} \nabla g_0(G_u u^\star + G_w w) - N^\mathsf{T}\mu^\star$. Strong convexity and essential smoothness of f_0 on $\mathcal{U}$ imply that $(\nabla f_0)^{-1} : \mathbb{R}^m \to \mathcal{U}$ is well-defined [23, Sec. 26] and globally Lipschitz continuous. Further, since ∇f_0 is locally Lipschitz continuous, it is Lipschitz continuous on a compact set containing $(\nabla f_0)^{-1}(\xi^\star)$, and it then follows by duality results in [24] that $(\nabla f_0)^{-1}$ will be strongly monotone and Lipschitz continuous on a compact set containing $\xi^\star$. Now define

$$J_w(u) := f_0(u) + g_0(G_u u + G_w w).$$

With this, (21b) can be expressed as

$$\dot{\mu} = Nu + \tilde{N}w, \qquad \nabla J_w(u) = -N^\mathsf{T}\mu.$$

Making the change of variables $\tilde{\mu} = \mu^\star - \mu$ and $\tilde{u} = u - u^\star$ and using that $(u^\star, \mu^\star)$ satisfy (6), we obtain the error dynamics

$$\dot{\tilde{\mu}} = -N\tilde{u}, \quad \Psi_w(\tilde{u}) := \nabla J_w(\tilde{u} + u^\star) - \nabla J_w(u^\star) = N^\mathsf{T}\tilde{\mu}$$

Since g_0 is convex with locally Lipschitz continuous gradient, Ψ_w inherits the same properties as ∇f_0. Thus, Ψ_w^{-1} is well-defined, and is strongly monotone and locally Lipschtiz on a compact set containing the origin. Eliminating $\tilde{u}$, the reduced dynamics now simplify to

$$\dot{\tilde{\mu}} = -N\Psi_w^{-1}(N^\mathsf{T}\tilde{\mu})$$

with equilibrium point at $\tilde{\mu} = 0$. With Lyapunov candidate $V(\tilde{\mu}) = \frac{1}{2}\|\tilde{\mu}\|_2^2$ we compute that

$$\dot{V}(\tilde{\mu}) = -(N^\mathsf{T}\tilde{\mu})^\mathsf{T}\Psi_w^{-1}(N^\mathsf{T}\tilde{\mu}) \leq -m_\Psi \|N^\mathsf{T}\tilde{\mu}\|_2^2$$

for some $m > 0$ and all $\tilde{\mu}$ such that $\|\tilde{\mu}\|_2$ is sufficiently small. Since N has full row rank, we further have $\dot{V}(\tilde{\mu}) \leq -cV(\tilde{\mu})$ for some $c > 0$, which establishes local exponential stability of the reduced dynamics. ∎

B. Optimality Model #2

For the optimality model (13), we present two approaches for the design of accompanying stabilizers.

1) Explicit Two-Loop Stabilizer Design: Integrating both error signals from the optimality model (13) with two time constants $\tau_1, \tau_2 > 0$, we look for an integral feedback design of the form

$$\begin{aligned} \tau_1 \dot{\eta}_1 &= -e_1 = -T_u^\mathsf{T}\nabla f_0(u) - T_z^\mathsf{T}\nabla g_0(z) \\ \tau_2 \dot{\eta}_2 &= -e_2 = -(H_z z + H_u u + H_w w) \\ u &= K_1\eta_1 + K_2\eta_2 \end{aligned} \tag{22}$$

for matrices K_1, K_2 to be designed. Inspired by [20], the idea we will pursue is the sequential closing of these two integral control loops. First, the loop involving η_2 will be closed and tuned assuming η_1 is absent, then the η_1 loop will be closed around the η_2 loop, as show in Figure 2.

The following result provides tuning criteria.

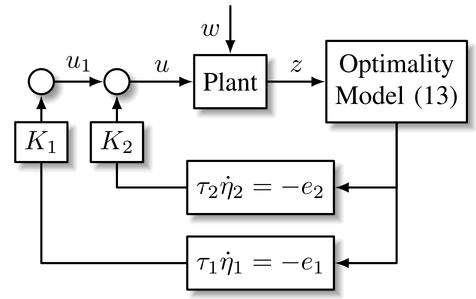

Fig. 2: Two-loop stabilizer design for OM #2.

*Theorem 4.3 (***Two-Loop Stabilizer***):* Consider the plant (1) with controller (22), and assume that $T = \left[\begin{smallmatrix} T_z \\ T_u \end{smallmatrix}\right]$ has been selected to have full column rank. Suppose that $N := H_z + G_u H_u$ has full row rank and that $\xi \mapsto f_0(T_u\xi + u_0(w)) + g_0(T_z\xi + z_0(w))$ is strongly convex on $\Xi_w = \{\xi \mid T_u\xi + u_0(w) \in \mathcal{U}\}$ for each $w \in \mathcal{W}$. Select K_2 such that $-NK_2$ is Hurwitz, define the projection matrix

$$\Pi_\mathrm{c} := I_m - K_2[NK_2]^{-1}N, \tag{23}$$

and finally, for any $P \succ 0$, select K_1 of full column rank such that $\Pi_\mathrm{c} K_1 = T_u P$. Then the controller (22) solves Problem 2.1 for all sufficiently large tunings of the form $\tau_1 \gg \tau_2 \gg 0$.

The convexity assumptions required in Theorem 4.3 are slightly weaker than those in the results of Section IV-A. Specifically, the present convexity condition is a statement about the behaviour of the objective functions on the affine solution set to (8), as opposed to, e.g., f_0 being strongly convex on its entire domain $\mathcal{U}$.

Proof of Theorem 4.3: The proof is based on two sequential applications of [21, Thm 3.1]; the constructions for the controller gains K_1, K_2 will be shown to be well-posed along the way. First, consider the controller

$$\tau_2\dot{\eta}_2 = -(H_z z + H_u u + H_w w), \quad u = K_2\eta_2 + u_1 \tag{24}$$

with *constant* auxiliary input u_1. With the plant (1) interconnected with (24), the reduced dynamics will have the form

$$\dot{\eta}_2 = -Nu - (H_z G_w + H_w)w, \quad u = K_2\eta_2 + u_1. \tag{25}$$

Since N has full row rank, there exists K_2 such that $-NK_2$ is Hurwitz; for instance, take $K_2 = N^\dagger = N^\mathsf{T}(NN^\mathsf{T})^{-1}$. It follows that the reduced dynamics (25) are globally exponentially stable, and we conclude from [21, Thm 3.1] that we may select $\tau_2 > 0$ sufficiently large such that the (LTI) closed-loop system (1) with controller (24) is internally exponentially stable and achieves $e_2(t) \to 0$ as $t \to \infty$ for all asymptotically constant exogenous inputs w and u_1.

We now consider the plant defined by (1) and (24) with state (x, η_2), inputs (u_1, w) and nonlinear output e_1 defined in (13). By the previous constructions, the state dynamics are LTI and internally exponentially stable. We seek to compute the steady-state input-output relation of this system, which is characterized by the equations

$$\begin{aligned} \bar{z} &= G_u\bar{u} + G_w w, & 0 &= N\bar{u} + (H_z G_w + H_w)w \\ \bar{u} &= K_2\bar{\eta}_2 + \bar{u}_1, & \bar{e}_1 &= T_u^\mathsf{T}\nabla f_0(\bar{u}) + T_z^\mathsf{T}\nabla g_0(\bar{z}). \end{aligned}$$

Solving, one finds that

$$\bar{\eta}_2 = -[NK_2]^{-1}N\bar{u}_1 - \mathcal{D}w$$

where $\mathcal{D} = [NK_2]^{-1}(H_zG_w + H_w)$, and therefore that

$$\begin{aligned}\bar{u} &= \bar{u}_1 - K_2[NK_2]^{-1}N\bar{u}_1 - K_2\mathcal{D}w \\ &= \Pi_{\mathrm{c}}\bar{u}_1 - K_2\mathcal{D}w.\end{aligned}$$

where Π_{c} is as in (23). A straightforward calculation shows that $\Pi_{\mathrm{c}}^2 = \Pi_{\mathrm{c}}$, so Π_{c} is an oblique projection matrix. Moreover, for later use, note that $\mathrm{null}(N) \subseteq \mathrm{range}(\Pi_{\mathrm{c}})$. Substituting our expression for $\bar{u}$ back into the expression for $\bar{e}_1$, we obtain the steady-state input-output relationship

$$\begin{aligned}\bar{e}_1 = T_u^{\mathsf{T}}\nabla f_0(\Pi_{\mathrm{c}}\bar{u}_1 + u_0(w)) \\ + T_z^{\mathsf{T}}\nabla g_0(G_u\Pi_{\mathrm{c}}\bar{u}_1 + z_0(w))\end{aligned} \tag{26}$$

where $(u_0(w), z_0(w))$ are as in (10). We now consider the feedback controller

$$\tau_1\dot{\eta}_1 = -e_1, \qquad u_1 = K_1\eta_1, \tag{27}$$

where K_1 is selected such that $\Pi_{\mathrm{c}}K_1 = T_uP$ with $P \succ 0$. From Lemma 3.1 (i) we know that T_u has full column rank. Moreover, by construction from (9), we have that $(H_zG_u + H_u)T_u = 0$, meaning that $\mathrm{range}(T_u) \subseteq \mathrm{null}(N) \subseteq \mathrm{range}(\Pi_{\mathrm{c}})$. Thus, the equation $\Pi_{\mathrm{c}}K_1 = T_uP$ is always solvable for a matrix K_1 of full column rank. To complete the proof, we verify that the reduced dynamics, given by

$$\dot{\eta}_1 = -T_u^{\mathsf{T}}\nabla f_0(\Pi_{\mathrm{c}}K_1\eta_1 + u_0) - T_z^{\mathsf{T}}\nabla g_0(G_u\Pi_{\mathrm{c}}K_1\eta_1 + z_0)$$

are exponentially stable. Substituting for $\Pi_{\mathrm{c}}K_1$ and using from (9) that $T_z = G_uT_u$, we can simplify this to obtain

$$\begin{aligned}\dot{\eta}_1 &= -T_u^{\mathsf{T}}\nabla f_0(T_uP\eta_1 + u_0) - T_z^{\mathsf{T}}\nabla g_0(G_uT_uP\eta_1 + z_0) \\ &= -T_u^{\mathsf{T}}\nabla f_0(T_uP\eta_1 + u_0) - T_z^{\mathsf{T}}\nabla g_0(T_zP\eta_1 + z_0) \\ &:= -\Phi(P\eta_1)\end{aligned}$$

The main convexity assumption implies that Φ is strongly monotone on the feasible set. Similar to the proof of Theorem 4.2, a simple Lyapunov analysis with candidate $V(\eta_1) = \|\eta_1 - \eta_1^\star\|_P^2$ now completes the proof of local exponential stability of the reduced dynamics. □

2) Robust & Optimal Stabilizer Design: Theorem 4.3 provides an explicit design for an appropriate gain $K = \left[\begin{smallmatrix} K_1 & K_2 \end{smallmatrix}\right]$ for use in (22). The quality of this design can be significantly improved by leveraging techniques from robust and optimal control theory. With the plant in quasi steady-state as given by (2), the reduced dynamics associated with (22) will have the form

$$\dot{\eta}_1 = -T_u^{\mathsf{T}}\nabla f_0(u) - T_z^{\mathsf{T}}\nabla g_0(G_uu + G_ww) \tag{28a}$$

$$\dot{\eta}_2 = -Nu - \tilde{N}w \tag{28b}$$

$$u = K_1\eta_1 + K_2\eta_2. \tag{28c}$$

where, as before, we set $N := H_z + G_uH_u$ and $\tilde{N} := H_zG_u + H_w$. We now recognize the design of K_1, K_2 in (28) a *state-feedback* design problem. To the system (28), we associate the performance outputs

$$z_1 = \dot{\eta}_1, \qquad z_2 = \rho\dot{\eta}_2$$

corresponding to the stationarity and primal feasibility violations, respectively, with $\rho > 0$ a tuning parameter. To minimize conservatism in the design that follows, we will separate out any quadratic cost portions from f_0 and g_0, so that

$$\begin{aligned}\nabla f_0(u) &= Q_1u + \nabla\tilde{f}_0(u) \\ \nabla g_0(z) &= Q_2z + \nabla\tilde{g}_0(z)\end{aligned}$$

for matrices $Q_1, Q_2 \succeq 0$ and $\tilde{f}_0, \tilde{g}_0$ being convex. With this, the overall system can be represented in standard linear-fractional representation (LFR) form [25], [26] as

$$\begin{bmatrix}\dot{\eta}_1 \\ \dot{\eta}_2 \\ \hline q_1 \\ q_2 \\ \hline z_1 \\ z_2\end{bmatrix} = \left[\begin{array}{cc|c|cc|c} 0 & 0 & -Q & -T_u^{\mathsf{T}} & -T_z^{\mathsf{T}} & -T_z^{\mathsf{T}}Q_2G_w \\ 0 & 0 & -N & 0 & 0 & -\tilde{N} \\ \hline 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & G_u & 0 & 0 & G_w \\ \hline 0 & 0 & -Q & -T_u^{\mathsf{T}} & -T_z^{\mathsf{T}} & -T_z^{\mathsf{T}}Q_2G_w \\ 0 & 0 & -\rho N & 0 & 0 & -\rho\tilde{N}\end{array}\right]\begin{bmatrix}\eta_1 \\ \eta_2 \\ \hline u \\ \hline p_1 \\ \hline p_2 \\ \hline w\end{bmatrix}$$

$$:= \left[\begin{array}{c|c|c|c} \mathcal{A} & \mathcal{B} & \mathcal{B}_1 & \mathcal{B}_2 \\ \hline \mathcal{C}_1 & \mathcal{E}_1 & \mathcal{D}_1 & \mathcal{D}_{12} \\ \hline \mathcal{C}_2 & \mathcal{E}_2 & \mathcal{D}_{21} & \mathcal{D}_2 \end{array}\right]\mathrm{col}(\eta_1, \eta_2, u, p_1, p_2, w)$$

where $Q := T_u^{\mathsf{T}}Q_1 + T_z^{\mathsf{T}}Q_2G_u$, and with

$$\begin{aligned}p_1 &= \Delta_1(q_1) := \nabla\tilde{f}_0(q_1) \\ p_2 &= \Delta_2(q_2) := \nabla\tilde{g}_0(q_2).\end{aligned}$$

Since ∇f_0 and ∇g_0 are gradients of convex functions, the blocks Δ_i satisfy (incremental) pointwise sector constraints of the form [27]

$$\begin{bmatrix}\Delta_i(q_i) - \Delta_i(q_i') \\ q_i - q_i'\end{bmatrix}^{\mathsf{T}}\Theta_i\begin{bmatrix}\Delta(q_i) - \Delta(q_i') \\ q_i - q_i'\end{bmatrix} \geq 0 \tag{29}$$

for all arguments q_i, q_i' and for some symmetric block two-by-two matrix Θ_i. The specific form of Θ_i depends on whether the objective function is (i) convex, (ii) convex with Lipschitz continuous gradient, (iii) strongly convex, or (iv) strongly convex and with Lipschitz continuous gradient. For example, if f_0 is strongly convex with parameter m_f and its gradient has Lipschitz constant L_f, we may take [27]

$$\Theta_1 = \begin{bmatrix}-2 & m_f + L_f \\ m_f + L_f & -2m_fL_f\end{bmatrix} \otimes I_m.$$

For selected Θ_1, Θ_2, we set

$$\begin{aligned}\mathbf{\Theta} &:= \{\mathrm{daug}(\Theta_1/\theta_1, \Theta_2/\theta_2) \mid \theta_1, \theta_2 > 0\} \\ \Theta_\gamma &:= \mathrm{diag}(-I_{n_w}, \tfrac{1}{\gamma^2}I_{q+n_{\mathrm{c}}}).\end{aligned}$$

with $\gamma > 0$, and where daug denotes the diagonal augmentation operation [25]. By standard arguments involving the S-Procedure, the state feedback $u = K\left[\begin{smallmatrix}\eta_1 \\ \eta_2\end{smallmatrix}\right]$ will render the reduced dynamics (28) exponentially stable and achieve $\mathscr{L}_2$-performance on the $w \mapsto z$ channel strictly upper bounded by γ if there exists $P \succ 0$ and $\Theta \in \mathbf{\Theta}$ such that

$$(\star)^{\mathsf{T}} \left[\begin{array}{cc|c|c} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ \hline 0 & 0 & \Theta & 0 \\ \hline 0 & 0 & 0 & \Theta_\gamma \end{array}\right] \left[\begin{array}{ccc} I_{q+n_c} & 0 & 0 \\ \mathcal{A}+\mathcal{B}K & \mathcal{B}_1 & \mathcal{B}_2 \\ \hline 0 & I_{m+r} & 0 \\ \mathcal{C}_1+\mathcal{E}_1 K & \mathcal{D}_1 & \mathcal{D}_{12} \\ \hline 0 & 0 & I_{n_w} \\ \mathcal{C}_2+\mathcal{E}_2 K & \mathcal{D}_{21} & \mathcal{D}_2 \end{array}\right] \prec 0. \tag{30}$$

Defining $Y = P^{-1} \succ 0$, performing a congruence transformation on (30) using $\mathrm{diag}(Y, I, I)$, and applying the *Dualization Lemma* [26, Cor. 4.10], (30) is equivalent to the LMI problem (31) of finding $Y \succ 0$, $Z \in \mathbb{R}^{m\times(q+n_c)}$, and $\theta_1, \theta_2 > 0$ such that The resulting feedback gain is then recovered as $K = ZY^{-1}$. A design minimizing γ can then be obtained via semidefinite programming by minimizing γ^2 subject to (31). We summarize in the following result.

*Proposition 4.4 (***Robust Optimal Stabilizer):** Suppose that (31) is feasible, and set $K = ZY^{-1}$. Then the controller (22) solves Problem 2.1 for all sufficiently large tunings of the form $\tau_1 = \tau_2 = \tau \gg 0$.

The interested reader will have no issues extending the robust/optimal design approach above to other optimality models; the details are omitted. The only pieces of plant model information required for this LMI-based design procedure are the plant DC gain matrices G_u and G_w.

V. Examples

A. Simulation Example

We first illustrate our ideas via an academic example. The LTI system (1) is a randomly generated underactuated stable system with $x \in \mathbb{R}^{30}$, $u \in \mathbb{R}^4$, and $z \in \mathbb{R}^5$, and is such that G_u has full column rank. The problem of interest is asymptotic tracking of step reference signals r_1, r_2 for z_1, z_2, while constraining z_3, z_4, z_5 within specified limits. This should be achieved with minimum control effort, and subject to hard constraints on controls u. We formulate this as

$$\text{minimize} \quad \Big[\sum\nolimits_{k=1}^{m} \tfrac{1}{2}\bar{u}_k^2 + \gamma \mathsf{B}(\bar{u}_k)\Big] + c\,\mathsf{P}(\bar{z}) \tag{32a}$$

$$\text{subject to} \quad \bar{z} = G_u \bar{u} + G_w w \tag{32b}$$

$$0 = z_i - r_i, \qquad i \in \{1, 2\} \tag{32c}$$

where $\gamma, c > 0$ are design parameters for the barrier and penalty functions

$$\mathsf{B}(u_k) = -\log(u_k^{\max} - u_k) - \log(-u_k^{\min} + u_k)$$

$$\mathsf{P}(\bar{z}) = \tfrac{1}{2}\sum\nolimits_{k=3}^{5} \max(0, z_k^{\min} - \bar{z}_k, \bar{z}_k - z_k^{\max})^2$$

Note that the objective function f_0 is strongly convex and essentially smooth on $\mathcal{U} = \prod_k [u_k^{\min}, u_k^{\max}]$.

In the following tests we set $u_k^{\min} = -0.75$, $u_k^{\max} = 0.75$, $z_k^{\min} = -1$, $z_k^{\max} = 1$ for all k, $\gamma = 0.01$, and $c = 50$. The response of the closed-loop system to sequential step changes in the references $(r_1, r_2) = (2, -2)$ at $t = 10$s and $t = 40$s is simulated. We illustrate the performance of the inversion-based controller (20) in Figure 3, the two-loop controller (22) in Figure 4, and the optimal controller of Proposition 4.4 in Figure 5.[2] In all cases, the controller asymptotically tracks the desired reference signals while maintaining the input and output constraints and minimizing steady-state control effort. The performance in Figure 4 is slightly poorer than that in Figure 3 due to cross-coupling effects between the two dynamic loops. The robust/optimal controller of Figure 5 does not suffer from these cross-coupling effects, and even provides an improvement over the inversion-based controller.

B. Frequency Control in Power Systems

Dynamic models of high-voltage AC power systems have the property that they are internally stable, and that the steady-state frequency deviation in the system is proportional to the net imbalance between generation and load [28]. This leads to a steady-state model (2) of the form

$$\Delta\bar{\omega} = \tfrac{1}{\beta}\mathbb{1}_m\mathbb{1}_m^{\mathsf{T}}\Delta\bar{u} - \tfrac{1}{\beta}\mathbb{1}_m\bar{w}. \tag{33}$$

where $\Delta\bar{\omega} \in \mathbb{R}^m$ denotes frequency deviations at m buses of the system, $\Delta\bar{u} \in \mathbb{R}^m$ denotes steady-state generation change at those buses, $\bar{w}$ denotes the total loading change in the system, and $\beta > 0$ is a constant. The problem of interest is minimization of generation cost, subject to regulation of frequency, expressed as

$$\underset{\Delta\bar{u},\Delta\bar{\omega}}{\text{minimize}} \quad \sum\nolimits_{i=1}^{m} J_i(\Delta\bar{u}_i) \tag{34a}$$

$$\text{subject to} \quad (33) \text{ and } 0 = \beta\Delta\bar{\omega}_m \tag{34b}$$

where J_i captures the ith generation cost and embeds any associated unit limits. Under appropriate assumptions on J_i, our previous results immediately yield several provably stable controllers for the solution of this problem. The primal-dual controller (19) reduces to

$$\tau_{\mathrm{p}}\Delta\dot{u}_i = -\nabla J_i(\Delta u_i) - \mu, \qquad \tau_{\mathrm{d}}\dot{\mu} = \Delta\omega_m$$

which consists of a single frequency integrator and a decentralized update for u_i; the inversion-based controller (20) reduces to

$$\tau\dot{\mu} = \Delta\omega_m, \qquad \Delta u_i = \nabla J_i^*(\mu),$$

which precisely recovers the control scheme proposed in [29]. For the two-loop design of Section IV-B.1, we follow (9) and must compute

$$\mathrm{null}\begin{bmatrix} I_r & -G_u \\ H_z & H_u \end{bmatrix} = \mathrm{null}\begin{bmatrix} I_m & -\frac{1}{\beta}\mathbb{1}_m\mathbb{1}_m^{\mathsf{T}} \\ \beta e_m^{\mathsf{T}} & 0 \end{bmatrix},$$

where e_m is the mth unit vector of $\mathbb{R}^m$. This nullspace is spanned by vectors of the form $\mathrm{col}(0, \xi)$ where $\xi^{\mathsf{T}}\mathbb{1}_m = 0$. Let $L = L^{\mathsf{T}}$ denote the Laplacian matrix of an undirected, connected, and weighted graph over m nodes [30, Chp. 6–8]. In this case $\mathrm{null}(L) = \mathbb{1}_m$, and therefore $\mathrm{range}(L^{\mathsf{T}}) = \{\xi \mid \xi^{\mathsf{T}}\mathbb{1}_m = 0\}$. If we block partition L as

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

[2]The primal-dual controller (19) produces results similar to the inversion-based controller when the time constant τ_{p} is small compared to τ_{d}.

$$(\star)^{\mathsf{T}}\left[\begin{array}{cc|c|cc} 0 & -I_{q+n_c} & 0 & 0 & 0 \\ -I_{q+n_c} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\Theta^{-1} & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n_w} & 0 \\ 0 & 0 & 0 & 0 & -\gamma^2 I_{m+n_c} \end{array}\right]\left[\begin{array}{ccc} -(\mathcal{A}Y+\mathcal{B}Z)^{\mathsf{T}} & -(\mathcal{C}_1 Y+\mathcal{E}_1 Z)^{\mathsf{T}} & -(\mathcal{C}_2 Y+\mathcal{E}_2 Z)^{\mathsf{T}} \\ I_{q+n_c} & 0 & 0 \\ \hline -\mathcal{B}_1^{\mathsf{T}} & -\mathcal{D}_1^{\mathsf{T}} & -\mathcal{D}_{21}^{\mathsf{T}} \\ 0 & I_{m+r} & 0 \\ \hline -\mathcal{B}_2^{\mathsf{T}} & -\mathcal{D}_{12}^{\mathsf{T}} & -\mathcal{D}_2^{\mathsf{T}} \\ 0 & 0 & I_{q+n_c} \end{array}\right] \prec 0. \tag{31}$$

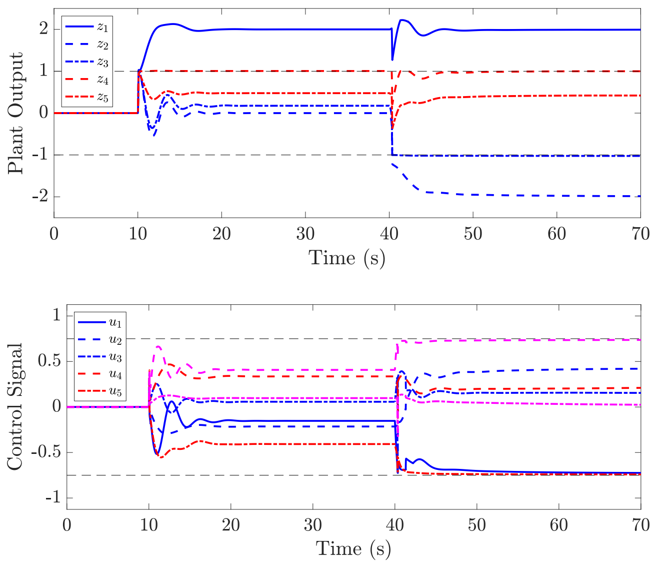

Fig. 3: System with inversion-based stabilizer; $\tau = 2$.

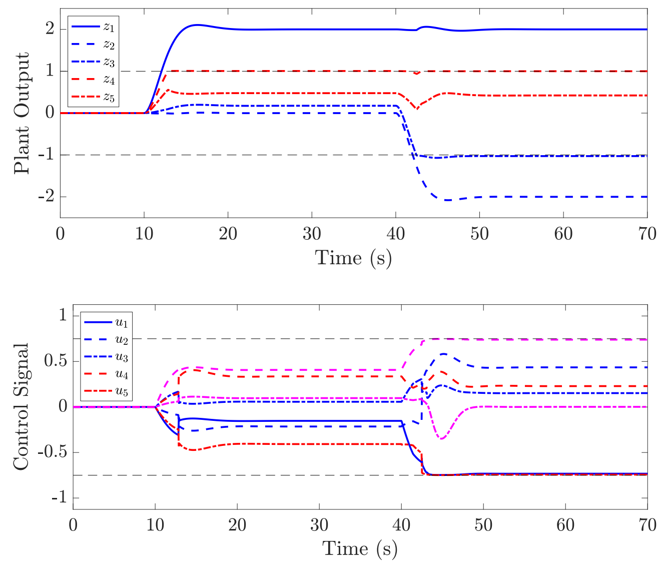

Fig. 5: System with optimal stabilizer; $\rho = 100, \tau = 0.02$.

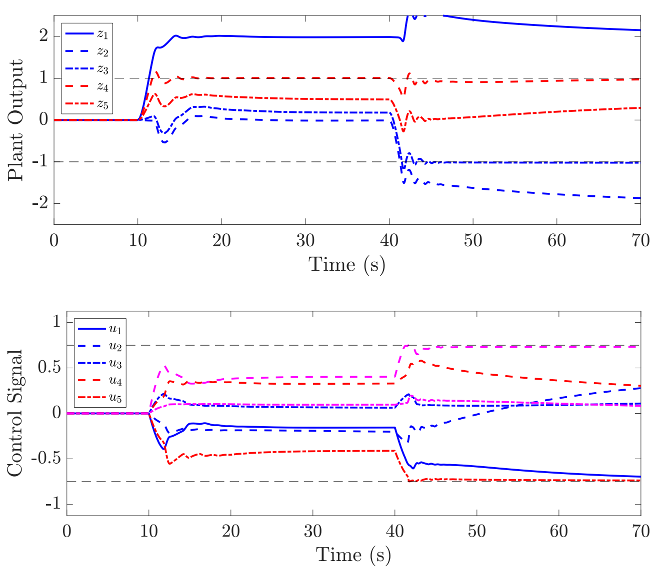

Fig. 4: System with two-loop stabilizer; $\tau_1 = 5, \tau_2 = 1$.

with $L_{11} \in \mathbb{R}^{(m-1)\times(m-1)}$, then it follows that an eligible selection of T is $T_z = 0$ and $T_u = \left[\begin{smallmatrix} L_{11}^{\mathsf{T}} \\ L_{12}^{\mathsf{T}} \end{smallmatrix}\right]$. Since $H_z G_u + H_u = \mathbb{1}_m^{\mathsf{T}}$, to satisfy Theorem 4.3 we may select $K_2 = e_m$. In this case, $\Pi_{\mathrm{c}} = I_m - e_m \mathbb{1}_m^{\mathsf{T}}$, and one may then verify that an eligible selection of K_1 satisfying $\Pi_{\mathrm{c}} K_1 = T_u P$ is $K_1 = \left[\begin{smallmatrix} L_{11}^{\mathsf{T}} \\ 0 \end{smallmatrix}\right] P$. Selecting $P = L_{11}^{-1} \succ 0$, we obtain the simpler choice $K_1 = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$. With these constructions, the stabilizing controller of Theorem 4.3 becomes

$$\begin{aligned} \tau_1 \dot{\eta}_{1,i} &= -\sum\nolimits_{j=1}^{m} a_{ij}(\nabla J_i(\Delta u_i) - \nabla J_j(\Delta u_j)) \\ \tau_2 \dot{\eta}_2 &= -\Delta\omega_m \\ \Delta u &= \mathrm{col}(\eta_1, \eta_2) \end{aligned}$$

for $i \in \{1, \ldots, m-1\}$, where $J(\Delta u) = \sum_{i=1}^{m} J_i(\Delta u_i)$. This novel controller has the interpretation that the mth generation unit implements integral control on the frequency deviation, while generators $\{1, \ldots, m-1\}$ implement a distributed averaging scheme to reach agreement on their marginal costs $\nabla J_i(\Delta u_i)$ of power production.

For the final design leveraging the results of Section III-C, it is convenient to rewrite (34) in the equivalent form

$$\underset{\Delta\bar{u},\Delta\bar{\omega}}{\text{minimize}} \quad \sum\nolimits_{i=1}^{m} J_i(\Delta\bar{u}_i) \tag{35a}$$

$$\text{subject to} \quad \text{(33) and } \mathbb{0}_m = \Delta\bar{\omega} \tag{35b}$$

where now the frequency constraint is vectorized to hold at all buses. We now compute that

$$\mathrm{null}\begin{bmatrix} I_r & -G_u \\ H_z & H_u \end{bmatrix} = \mathrm{null}\begin{bmatrix} I_m & -\frac{1}{\beta}\mathbb{1}_m\mathbb{1}_m^{\mathsf{T}} \\ I_m & 0 \end{bmatrix},$$

and following similar steps to before, we may selection $T_z = 0$ and $T_u = L^{\mathsf{T}}$; note that in this case, we have selected T_u

to not have full column rank. The error signals (13) now become

$$e_1 = L\nabla J(\Delta u), \qquad e_2 = \Delta\omega$$

In this case, we have $q = n_{\mathrm{c}} = m$. Following Proposition 3.2, consider the selection $\ell = m$, C_1 and C_2 being empty, and $C_1' = C_2' = I_m$. With this, we have

$$\begin{aligned}\mathrm{range}(C_1' T_u^{\mathsf{T}}) \cap \mathrm{range}(C_2'(H_z G_u + H_u)) \\ = \mathrm{range}(L) \cap \mathrm{range}(\mathbb{1}_m \mathbb{1}_m^{\mathsf{T}}) = \{0\}\end{aligned}$$

so conditions (a)–(c) in Proposition 3.2 are verified, and

$$e' = \Delta\omega + L\nabla J(\Delta u)$$

is an optimality model. Integrating and selecting the stabilizer $\Delta u = \eta$, we obtain the control scheme

$$\tau \Delta\dot{u} = -\Delta\omega - L\nabla J(\Delta u)$$

which is a distributed-averaging proportional integral (DAPI) frequency controller. While we have not developed here a general theory of low-gain integral control stabilizers associated with the optimality model of Section III-C, this particular frequency controller has been extensively studied and is known to be stable for sufficiently large τ; see [31] and the references therein.

Our results herein establish that the above controllers – and many others, which can be obtained by varying the above constructions – lead to provably stable closed-loop systems. These stability guarantees can even be pushed to nonlinear power system models by mirroring the arguments in, e.g., [29], [31].

VI. Conclusions

Several low-gain controller designs have been presented which can be used to drive a stable LTI system towards the solution of a linearly-constrained convex optimization problem. This renders the general OSS control framework presented in [1] constructive for a practically-relevant class of systems, and examples have been presented to illustrate the design procedure and the flexibility inherent within it. Open directions include the extension of this framework to nonlinear and discrete-time systems, along with continued exploration of applications for optimal steady-state control.

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