Data-Driven Model Predictive Control for Linear Time-Periodic Systems

Ruiqi Li, John W. Simpson-Porco, and Stephen L. Smith

***Abstract*— We consider the problem of data-driven predictive control for an unknown discrete-time linear time-periodic (LTP) system of known period. Our proposed strategy generalizes both Data-enabled Predictive Control (DeePC) and Subspace Predictive Control (SPC), which are established data-driven control techniques for linear time-invariant (LTI) systems. The approach is supported by an extensive theoretical development of behavioral systems theory for LTP systems, culminating in a generalization of Willems' fundamental lemma. Our algorithm produces results identical to standard Model Predictive Control (MPC) for deterministic LTP systems. Robustness of the algorithm to noisy data is illustrated via simulation of a regularized version of the algorithm applied to a stochastic multi-input multi-output LTP system.**

I. Introduction

Control design methods can broadly be classified into *model-based* methods and *data-driven* methods. Model-based design methods rely on an accurate parametric representation of the system, which may come from first-principles modeling or from system-identification. Data-driven control, on the other hand, produces a control strategy directly from recorded historical data. As modern systems of interest become increasingly complex and difficult to identify, data-driven control techniques become increasingly preferable, and have attracted significant research interest in recent years. A comprehensive survey of early data-driven methods can be found in [1].

Model Predictive Control (MPC) is a particular model-based design method which has been widely used in industrial applications, such as autonomous driving [2] and mobile robots [3]. MPC is applicable to time-varying systems, and can incorporate input and output constraints, which typically model actuator saturation and safety constraints, respectively. Despite these benefits, MPC requires a parametric system model, and the modeling process (e.g., system identification) can be expensive [4].

Originating from the work by J. C. Willems, *behavioral systems theory* provides an alternative to the now-standard state-space framework [5]. In the behavioral approach, a system is characterized as a set of possible input-output trajectories (the *behavior*). Of particular note, the behavior of a finite-dimensional discrete-time linear time-invariant (LTI) system over a finite time interval can be expressed using collected historical data, a result now known as the *Fundamental Lemma* [6]. See [7] for an introduction of the behavioral approach with relevant algorithms, and [8] for perspectives of developing the theory. Currently, many results in the behavioral framework are restricted to LTI systems.

By leveraging the fundamental lemma, system outputs can be predicted without a parametric system model. From this observation, *data-driven MPC (DDMPC)* methods have been developed wherein the need for a parametric system model is eliminated [9]–[16]. A particular DDMPC algorithm named **D**ata-**e**nabl**e**d **P**redictive **C**ontrol (DeePC) [12]–[16] has been successfully applied to control problems in power systems [17], [18], motor drives [19] and quad-copters [20]. While the methods [9]–[16] focus on LTI systems, some extensions have been developed for linear parameter-varying systems [21] and for specific types of nonlinear systems [22]. In the spirit of these extensions, the extension to linear time-varying (LTV) systems is also of interest, and our focus here will be on developing analogous theory and control techniques for *linear time-periodic (LTP)* systems — a particular class of LTV systems. LTP systems can arise from linearization of nonlinear systems around periodic trajectories, such as in models of helicopters [23] and wind turbines [24].

Contributions: This paper develops behavioral systems theory and associated DDMPC results for LTP systems of known periods. Our key insight is that an established lifting technique (see [25], [26] and Section II) which transforms an LTP system into an LTI system can be leveraged to extend the behavioral theory of LTI systems to LTP systems. Based on this, in Section III we develop the behavioral theory for LTP systems — generalizing notions such as order and lag — and culminating in a natural extension of the fundamental lemma [6]. A benefit of this development is that all theory is stated directly in terms of the data and trajectories of the original LTP system, as opposed to the associated collection of lifted LTI systems. Leveraging this theory, in Section IV we put forward a DDMPC algorithm for LTP systems, generalizing the established DeePC and Subspace Predictive Control (SPC) [17]–[19] methods for LTI systems. We provide a performance guarantee that for deterministic LTP systems, our algorithm gives the same control policy as obtained from MPC. Finally, we illustrate the effectiveness of our approach via a simulation study in Section V. Due to space limitations, the majority of the proofs are omitted, but can be found in the extended version [27].

Notation: Let $[M_1; \dots; M_k] := [M_1^\top, \dots, M_k^\top]^\top$ denote the column concatenation of matrices $M_1, \dots, M_k$. Given a $\mathbb{R}^q$-valued discrete-time signal z with an integer index, for integers t_1, t_2 with $t_1 \leq t_2$, let $z_{[t_1,t_2]}$ (resp. $z_{[t_1,\infty)}$) denote either the sequence $\{z_t\}_{t=t_1}^{t_2}$ (resp. $\{z_t\}_{t=t_1}^{\infty}$) or the

This research is supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC).

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concatenated vector $[z_{t_1};\ldots;z_{t_2}] \in \mathbb{R}^{q(t_2-t_1+1)}$ (resp. the semi-infinite vector $[z_{t_1}; z_{t_1+1}; \ldots]$). Similarly, for integers $t_1 < t_2$, let $z_{[t_1,t_2)} := z_{[t_1,t_2-1]}$. Let $M^\dagger$ denote the pseudo inverse of a matrix M.

II. Linear Time-Periodic Systems and the Lifting Technique

In this section we review some classical notions for linear time-periodic systems. Consider a discrete-time linear time-varying (LTV) system

$$\mathcal{S} : \begin{cases} x_{t+1} = A_t x_t + B_t u_t \\ \quad y_t = C_t x_t + D_t u_t \end{cases} \tag{1}$$

with initial time $t_0 \in \mathbb{Z}$ and initial state x_{t_0}, where $t \in \mathbb{Z}$ is the time and $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, and $y_t \in \mathbb{R}^p$ are the state, input, and output of the system. The system (1) is said to be *linear time-periodic (LTP)* if there exists $T \in \mathbb{N}$ (a *period*) such that $A_{t+T} = A_t$ (and similarly for B_t, C_t, D_t) for all $t \in \mathbb{Z}$. The smallest T satisfying this condition is the *fundamental period*; without loss of generality, we assume going forward that T is the fundamental period. Note that when $T = 1$, the system (1) is linear time-invariant (LTI). A discrete-time LTP model may arise naturally in discrete time, or may have been obtained via appropriate sampling of a continuous-time LTP system.

For the LTV system (1) and integers t_1, t_2 with $t_1 \le t_2$, the state-transition matrix $\Phi_{t_1}^{t_2} \in \mathbb{R}^{n\times n}$ and impulse response matrix $G_{t_1}^{t_2} \in \mathbb{R}^{p\times m}$ from step t_1 to t_2 are defined as

$$\Phi_{t_1}^{t_2} := \begin{cases} I, & \text{if } t_2 = t_1 \\ A_{t_2-1}A_{t_2-2}\cdots A_{t_1}, & \text{if } t_2 > t_1, \end{cases} \tag{2a}$$

$$G_{t_1}^{t_2} := \begin{cases} D_{t_1}, & \text{if } t_2 = t_1 \\ C_{t_2}\Phi_{t_1+1}^{t_2} B_{t_1}, & \text{if } t_2 > t_1. \end{cases} \tag{2b}$$

Similarly, the associated (reversed) extended controllability matrix $\mathscr{C}_{t_1}^{t_2} \in \mathbb{R}^{n\times(t_2-t_1+1)m}$, the extended observability matrix $\mathscr{O}_{t_1}^{t_2} \in \mathbb{R}^{(t_2-t_1+1)p\times n}$, and the block matrix $\mathscr{I}_{t_1}^{t_2}$ of impulse-response coefficients are defined as

$$\mathscr{C}_{t_1}^{t_2} := \big[\Phi_{t_1+1}^{t_2+1}B_{t_1},\ \Phi_{t_1+2}^{t_2+1}B_{t_1+1},\ \ldots,\ \Phi_{t_2+1}^{t_2+1}B_{t_2}\big], \tag{2c}$$

$$\mathscr{O}_{t_1}^{t_2} := \big[C_{t_1}\Phi_{t_1}^{t_1};\ C_{t_1+1}\Phi_{t_1}^{t_1+1};\ \ldots;\ C_{t_2}\Phi_{t_1}^{t_2}\big], \tag{2d}$$

$$\mathscr{I}_{t_1}^{t_2} := \begin{bmatrix} G_{t_1}^{t_1} & & & \\ G_{t_1}^{t_1+1} & G_{t_1+1}^{t_1+1} & & \\ \vdots & \vdots & \ddots & \\ G_{t_1}^{t_2} & G_{t_1+1}^{t_2} & \cdots & G_{t_2}^{t_2} \end{bmatrix}. \tag{2e}$$

With this notation, the unique solution of (1) with initial condition x_{t_1} at time $t = t_1$ can be expressed as

$$x_{t_2} = \Phi_{t_1}^{t_2} x_{t_1} + \mathscr{C}_{t_1}^{t_2-1} u_{[t_1,t_2)}, \tag{2f}$$

$$y_{[t_1,t_2]} = \mathscr{O}_{t_1}^{t_2} x_{t_1} + \mathscr{I}_{t_1}^{t_2} u_{[t_1,t_2]}, \tag{2g}$$

for any $t_2 > t_1$ in (2f) and any $t_2 \ge t_1$ in (2g).

Throughout the paper, we let $w_{[t_1,t_2]} := [u_{[t_1,t_2]}; y_{[t_1,t_2]}]$ denote a trajectory of the system (1).

A. Lifting an LTP System to an LTI System

We now recall a classical technique for "lifting" an LTP system into an LTI system [25].

Definition 1 (Lift of an LTP System)**.** For an LTP system $\mathcal{S}$ as in (1) of period T and an initial time $t_0 \in \mathbb{Z}$, the associated *lifted system $\mathcal{S}_{\mathsf{L}}(t_0)$ of $\mathcal{S}$ with initial time t_0* is the LTI system

$$\mathcal{S}_{\mathsf{L}}(t_0) : \begin{cases} \mathbf{x}_{\tau+1} = \mathfrak{A}\mathbf{x}_\tau + \mathfrak{B}\mathbf{u}_\tau \\ \quad \mathbf{y}_\tau = \mathfrak{C}\mathbf{x}_\tau + \mathfrak{D}\mathbf{u}_\tau \end{cases} \tag{3a}$$

with state $\mathbf{x}_\tau \in \mathbb{R}^n$, input $\mathbf{u}_\tau \in \mathbb{R}^{mT}$, output $\mathbf{y}_\tau \in \mathbb{R}^{pT}$, and time $\tau \in \mathbb{Z}$, where

$$\begin{aligned} \mathfrak{A} &:= \Phi_{t_0}^{t_0+T}, & \mathfrak{B} &:= \mathscr{C}_{t_0}^{t_0+T-1}, \\ \mathfrak{C} &:= \mathscr{O}_{t_0}^{t_0+T-1}, & \mathfrak{D} &:= \mathscr{I}_{t_0}^{t_0+T-1}. \end{aligned} \tag{3b}$$

The idea behind lifting is that each time step τ of the lifted system $\mathcal{S}_{\mathsf{L}}(t_0)$ corresponds to T successive time steps of the original LTP system $\mathcal{S}$. The state/input/output of $\mathcal{S}_{\mathsf{L}}(t_0)$ are related to the state/input/output of $\mathcal{S}$ via

$$\begin{aligned} \mathbf{x}_\tau &= x_{t_0+\tau T}, \\ \mathbf{u}_\tau &= u_{[t_0+\tau T,\, t_0+(\tau+1)T)}, \\ \mathbf{y}_\tau &= y_{[t_0+\tau T,\, t_0+(\tau+1)T)}. \end{aligned}$$

Each input vector $\mathbf{u}_\tau$ (or output vector $\mathbf{y}_\tau$) of $\mathcal{S}_{\mathsf{L}}(t_0)$ stacks the inputs (or outputs) of $\mathcal{S}$ over one period, and the state vector $\mathbf{x}_\tau$ is the state of $\mathcal{S}$ at the "beginning" of this period, as specified by the initial time t_0; see Fig. 1. Note from (3b) that the matrices $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ depend on the initial time t_0. Nonetheless, some properties of the lifted system — such as the eigenvalues of $\mathfrak{A}$ — are invariant under the choice of the initial time t_0. See [26] for more information on lifting and properties of the lifted system.

III. Behavioral Systems Theory for Linear Time-Periodic Systems

In this section we develop a set of results on behavioral systems theory for linear time-periodic systems.

A. Behavioural Representation of LTV Systems

In the framework of behavioral systems theory, the input-output trajectories of the system (1) are described independent of the state representation through the *behavior*.

Definition 2 (Behavior)**.** For the LTV system $\mathcal{S}$ in (1) and an integer t_1, the *behavior* $\mathscr{B}_{[t_1,\infty)}^{\mathcal{S}}$ of $\mathcal{S}$ on the time interval $[t_1,\infty)\cap\mathbb{Z}$ is the set

$$\mathscr{B}_{[t_1,\infty)}^{\mathcal{S}} := \left\{ \begin{bmatrix} u_{[t_1,\infty)} \\ y_{[t_1,\infty)} \end{bmatrix} \;\middle|\; \exists\, x_{t_1} \text{ s.t. (1) holds for all } t \ge t_1 \right\}.$$

Given $\mathscr{B}_{[t_1,\infty)}^{\mathcal{S}}$ and an integer $t_2 \ge t_1$, we let $\mathscr{B}_{[t_1,t_2]}^{\mathcal{S}}$ denote the restriction of $\mathscr{B}_{[t_1,\infty)}^{\mathcal{S}}$ to the interval $[t_1,t_2]$, and in the case $t_2 > t_1$, we let $\mathscr{B}_{[t_1,t_2)}^{\mathcal{S}} := \mathscr{B}_{[t_1,t_2-1]}^{\mathcal{S}}$. The behavior defines a subspace of the vector space of semi-infinite sequences, and contains all possible input-output trajectories of the system. Going forward, we focus primarily on the restricted behavior.

Lemma 3. *The restricted behavior $\mathscr{B}_{[t_1,t_2]}^{\mathcal{S}}$ of the LTV system $\mathcal{S}$ in* (1) *is a finite-dimensional vector space and*

$$\mathscr{B}_{[t_1,t_2]}^{\mathcal{S}} = \operatorname{ColSpan}\begin{bmatrix} 0 & I \\ \mathscr{O}_{t_1}^{t_2} & \mathscr{I}_{t_1}^{t_2} \end{bmatrix}.$$

Corollary 3.1. $\dim \mathscr{B}_{[t_1,t_2]}^{\mathcal{S}} = \operatorname{rank}(\mathscr{O}_{t_1}^{t_2}) + m(t_2 - t_1 + 1)$.

When $\mathcal{S}$ is an LTI system, the behavior is invariant under

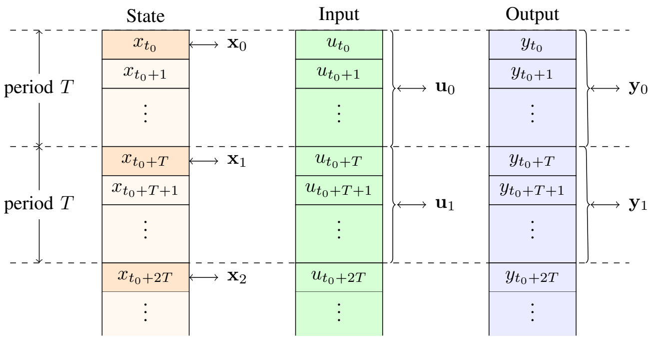

Fig. 1. Relationship between the state x_t, input u_t, output y_t of an LTP system $\mathcal{S}$ and the state $\mathbf{x}_\tau$, input $\mathbf{u}_\tau$, output $\mathbf{y}_\tau$ of its lifted system $\mathcal{S}_{\mathsf{L}}(t_0)$.

shift of the time interval, meaning that

$$\mathscr{B}^{\mathcal{S}}_{[t_1,t_2]} = \mathscr{B}^{\mathcal{S}}_{[t_1+s,t_2+s]} \quad \text{for all } s \in \mathbb{Z}.$$

This follows from Definition 2, due to shift-invariance of the system matrices. A similar result holds when $\mathcal{S}$ is an LTP system of period T, namely that

$$\mathscr{B}^{\mathcal{S}}_{[t_1,t_2]} = \mathscr{B}^{\mathcal{S}}_{[t_1+sT,t_2+sT]} \quad \text{for all } s \in \mathbb{Z}.$$

Unlike the case of LTI or LTP systems, the behavior of a general LTV system is not a shift-invariant subspace. However, given the behavior over an interval, the behavior on the first several steps of the interval can be easily constructed.

Lemma 4. *For the LTV system $\mathcal{S}$ in* (1) *and integers $t_0 \le t_1 \le t_2$, if*

$$\mathscr{B}^{\mathcal{S}}_{[t_0,t_2]} = \operatorname{ColSpan}[U_{t_0};\dots;U_{t_2};Y_{t_0};\dots;Y_{t_2}]$$

for some matrices $U_{t_0},\dots,U_{t_2} \in \mathbb{R}^{m\times h}$ and $Y_{t_0},\dots,Y_{t_2} \in \mathbb{R}^{p\times h}$ with some $h \in \mathbb{N}$, then

$$\mathscr{B}^{\mathcal{S}}_{[t_0,t_1]} = \operatorname{ColSpan}[U_{t_0};\dots;U_{t_1};Y_{t_0};\dots;Y_{t_1}].$$

1) Controllability: In the behavioral framework, controllability is defined in a trajectory-based sense, as opposed to the more classical notion of state-controllability.

Definition 5 (Controllability [7], [28])**.** An LTV system $\mathcal{S}$ is *controllable* if for any $t_0 \in \mathbb{Z}$, any two trajectories $w^{\mathrm{I}}_{[t_0,\infty)}, w^{\mathrm{II}}_{[t_0,\infty)} \in \mathscr{B}^{\mathcal{S}}_{[t_0,\infty)}$, and any time $t_1 \ge t_0$, there exists a time $t_2 \ge t_1$ and a trajectory $w^{\diamond}_{[t_0,\infty)} \in \mathscr{B}^{\mathcal{S}}_{[t_0,\infty)}$ such that

$$w^{\diamond}_{[t_0,t_1)} = w^{\mathrm{I}}_{[t_0,t_1)}, \qquad w^{\diamond}_{[t_2,\infty)} = w^{\mathrm{II}}_{[t_2,\infty)}. \tag{4}$$

Put differently, an LTV system is controllable if we can "drive" from one trajectory to any other trajectory in a finite number of time steps. When $\mathcal{S}$ is an LTI system, the second equality in (4) is sometimes replaced by $w^{\diamond}_{[t_2,\infty)} = w^{\mathrm{II}}_{[t_0,\infty)}$ in the literature (e.g. [6], [12]). This alternative definition is equivalent to Definition 5 for LTI systems [28, Remark 4(i)].

B. A Definition of Order and Lag for LTV Systems

In behavioral systems theory, the *order* and *lag* are so-called integer invariants of an LTI system, and can be expressed using a minimal state representation of the behavior. In this subsection, we generalize those notions to LTV systems, and do so in a manner that avoids introducing notions of minimality for LTV models.

First, we review the LTI definitions of order and lag from the literature (e.g., [7], [29]). For an LTI state-space model $\mathcal{S} : (A,B,C,D)$, a *minimal representation* $\bar{\mathcal{S}}$ of its behavior $\mathscr{B}^{\mathcal{S}}_{[0,\infty)}$ is a state-space model $\bar{\mathcal{S}} : (\bar{A},\bar{B},\bar{C},\bar{D})$, having the minimal possible state dimension and sharing the same behavior as of $\mathcal{S}$, i.e., $\mathscr{B}^{\mathcal{S}}_{[0,\infty)} = \mathscr{B}^{\bar{\mathcal{S}}}_{[0,\infty)}$. The order $\mathbf{n}(\mathcal{S})$ of $\mathcal{S}$ is equal to the state dimension of $\bar{\mathcal{S}}$. Define matrix $\mathscr{O}^{t_2}_{t_1}$ (resp. $\bar{\mathscr{O}}^{t_2}_{t_1}$) from (2d) for system $\mathcal{S}$ (resp. $\bar{\mathcal{S}}$). The extended observability matrix $\bar{\mathscr{O}}^{s-1}_0$ of $\bar{\mathcal{S}}$ reaches full column rank equal to $\mathbf{n}(\mathcal{S})$ when s is sufficiently large, and the lag $\mathbf{l}(\mathcal{S})$ of $\mathcal{S}$ is the smallest integer s that $\bar{\mathscr{O}}^{s-1}_0$ has full column rank. We can express both $\mathbf{n}(\mathcal{S})$ and $\mathbf{l}(\mathcal{S})$ in terms of the model $\mathcal{S}$, which is not necessarily a minimal representation, as

$$\begin{aligned} \mathbf{n}(\mathcal{S}) &= \lim_{s\to\infty} \operatorname{rank}(\bar{\mathscr{O}}^{s-1}_0) = \lim_{s\to\infty} \operatorname{rank}(\mathscr{O}^{s-1}_0), \\ \mathbf{l}(\mathcal{S}) &= \min\{s \in \mathbb{N} : \operatorname{rank}(\bar{\mathscr{O}}^{s-1}_0) = \mathbf{n}(\mathcal{S})\} \\ &= \min\{s \in \mathbb{N} : \operatorname{rank}(\mathscr{O}^{s-1}_0) = \mathbf{n}(\mathcal{S})\}, \end{aligned} \tag{5a}$$

where we used the equality $\operatorname{rank}(\mathscr{O}^{s-1}_0) = \operatorname{rank}(\bar{\mathscr{O}}^{s-1}_0)$ which follows from $\mathscr{B}^{\mathcal{S}}_{[0,s)} = \mathscr{B}^{\bar{\mathcal{S}}}_{[0,s)}$ and Corollary 3.1.

Motivated by the above considerations, we introduce the following definition.

Definition 6 (**Order and Lag**)**.** For the LTV system $\mathcal{S}$ in (1), the *order* $\mathbf{n}(\mathcal{S},t)$ at time t and *lag* $\mathbf{l}(\mathcal{S},t)$ at time t are[1]

$$\begin{aligned} \mathbf{n}(\mathcal{S},t) &:= \lim_{s\to\infty} \operatorname{rank}(\mathscr{O}^{t+s-1}_t), \\ \mathbf{l}(\mathcal{S},t) &:= \min\{s \in \mathbb{N} : \operatorname{rank}(\mathscr{O}^{t+s-1}_t) = \mathbf{n}(\mathcal{S},t)\}. \end{aligned} \tag{5b}$$

When $\mathcal{S}$ is an LTI system, we write its order and lag in the sense of (5b) as $\mathbf{n}(\mathcal{S})$ and $\mathbf{l}(\mathcal{S})$ respectively, since they are time-independent. The definitions in (5b) are consistent with (5a) in the LTI case. Moreover, via Corollary 3.1,

$$\dim \mathscr{B}^{\mathcal{S}}_{[t,t+L)} = \mathbf{n}(\mathcal{S},t) + mL \qquad \forall t \in \mathbb{Z} \tag{5c}$$

for all integers $L \ge \mathbf{l}(\mathcal{S},t)$, which coincides with an established result [29, Cor. 5] or [7, Eq. (1)] in the LTI case.

The lag specifies a sufficient length of a trajectory such that, with any subsequent input, the resulting output after the trajectory is uniquely determined, as captured in Lemma 7(ii). This result generalizes [30, Lemma 1] or [7, Lemma 1] which is for the LTI case. Lemma 7(iii) gives an expression for the unique output, generalizing [31, Lemma 2] as the LTI case.

Lemma 7 (Uniqueness of Future Output)**.** *Consider the LTV system $\mathcal{S}$ in* (1), *a time step $t \in \mathbb{Z}$, and positive integers L, N. The following statements hold:*

(i) For any trajectory $w_{[t-L,t)} \in \mathscr{B}^{\mathcal{S}}_{[t-L,t)}$ and any input $u^*_{[t,t+N)}$, there exists an output $y^*_{[t,t+N)}$ satisfying

$$[u_{[t-L,t)}; u^*_{[t,t+N)}; y_{[t-L,t)}; y^*_{[t,t+N)}] \in \mathscr{B}^{\mathcal{S}}_{[t-L,t+N)}.$$

(ii) If $L \ge \mathbf{l}(\mathcal{S}, t-L)$, the output $y^*_{[t,t+N)}$ from (i) is unique.

(iii) Moreover, if the behavior $\mathscr{B}^{\mathcal{S}}_{[t-L,t+N)}$ can be expressed as

$$\mathscr{B}^{\mathcal{S}}_{[t-L,t+N)} = \operatorname{ColSpan}[U_{\mathrm{p}}; U_{\mathrm{f}}; Y_{\mathrm{p}}; Y_{\mathrm{f}}]$$

for some matrices $U_{\mathrm{p}} \in \mathbb{R}^{mL\times h}, U_{\mathrm{f}} \in \mathbb{R}^{mN\times h}, Y_{\mathrm{p}} \in \mathbb{R}^{pL\times h}, Y_{\mathrm{f}} \in \mathbb{R}^{pN\times h}$ with some $h \in \mathbb{N}$, then the unique

[1] $\mathbf{n}(\mathcal{S},t)$ is well-defined in (5b), since $\operatorname{rank}(\mathscr{O}^{t+s-1}_t)$ is bounded by the state dimension n and is non-decreasing as we increase s because $\mathscr{O}^{t+s-1}_t$ is augmented with extra rows. Thus, $\mathbf{l}(\mathcal{S},t)$ is also well-defined in (5b).

output $y^*_{[t,t+N)}$ from (ii) is given as[2]

$$y^*_{[t,t+N)} = Y_{\mathrm{f}} \begin{bmatrix} U_{\mathrm{p}} \\ U_{\mathrm{f}} \\ Y_{\mathrm{p}} \end{bmatrix}^{\dagger} \begin{bmatrix} u_{[t-L,t)} \\ u^*_{[t,t+N)} \\ y_{[t-L,t)} \end{bmatrix}.$$

C. Behavioral Systems Theory for LTP Systems

Now we limit our discussion to LTP systems. We first establish the relationship between the behavior of an LTP system and the behavior of any corresponding lifted system.

Lemma 8. *For an LTP system $\mathcal{S}$ of period T and its lifted system $\mathcal{S}_{\mathsf{L}}(t_0)$ with initial step $t_0 \in \mathbb{Z}$, it holds that*

$$\mathscr{B}^{\mathcal{S}}_{[t_0,\, t_0+sT)} = \mathscr{B}^{\mathcal{S}_{\mathsf{L}}(t_0)}_{[0,s)} \qquad \forall s \in \mathbb{N}.$$

Remark 9 (**Dependence on Initial Step** t_0)**.** The lifted system and its behavior depend on the initial step t_0. For instance, consider the following single-state SISO LTP system $\mathcal{S}$ of period $T = 2$.

$$\mathcal{S} : \begin{cases} x_{t+1} = x_t + (-1)^t u_t \\ \quad y_t = x_t \end{cases}$$

The corresponding lifted system $\mathcal{S}_{\mathsf{L}}(t_0)$ for $t_0 \in \mathbb{Z}$.

$$\mathcal{S}_{\mathsf{L}}(t_0) : \begin{cases} \mathbf{x}_{\tau+1} = \mathbf{x}_\tau + (-1)^{t_0} \begin{bmatrix} 1 & -1 \end{bmatrix} \mathbf{u}_\tau \\ \quad \mathbf{y}_\tau = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbf{x}_\tau + (-1)^{t_0} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{u}_\tau \end{cases}$$

It follows (via Lemma 3) that for $t_0 \in \mathbb{Z}$ the restricted behavior of $\mathcal{S}_{\mathsf{L}}(t_0)$ on interval $[0,0]$ is

$$\mathscr{B}^{\mathcal{S}_{\mathsf{L}}(t_0)}_{[0,0]} = \mathrm{ColSpan} \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 1 & (-1)^{t_0} & 0 \end{array} \right].$$

One can now observe that $\mathscr{B}^{\mathcal{S}_{\mathsf{L}}(0)}_{[0,0]}$ and $\mathscr{B}^{\mathcal{S}_{\mathsf{L}}(1)}_{[0,0]}$ are different subspaces. Hence, it is necessary to specify the initial time t_0 when introducing the lifted system. □

1) Order and Lag: Notions of order and lag for LTV systems have been introduced in Definition 6. The next result relates the order and lag of an LTP system to the order and lag of its lifted system.

Lemma 10. *For an LTP system $\mathcal{S}$ of period T, we have (i) $\mathbf{n}(\mathcal{S}_{\mathsf{L}}(t)) = \mathbf{n}(\mathcal{S}, t)$, and (ii) $\mathbf{l}(\mathcal{S}_{\mathsf{L}}(t)) = \lceil \mathbf{l}(\mathcal{S}, t)/T \rceil$.*

For unknown LTP systems with known periods and state dimensions, we can establish bounds of their orders and lags.

Corollary 10.1. *For an LTP system $\mathcal{S}$ as in* (1) *of period T, we have (i) $\mathbf{n}(\mathcal{S}, t) \le n$, and (ii) $\mathbf{l}(\mathcal{S}, t) \le nT$.*

2) Controllability: The controllability of an LTP system is equivalent to the controllability of its lifted systems.

Lemma 11. *An LTP system $\mathcal{S}$ is controllable if, and only if, its lifted systems $\mathcal{S}_{\mathsf{L}}(t_0)$ are controllable for all $t_0 \in \mathbb{Z}$.*

D. A Fundamental Lemma for LTP Systems

According to the so-called Fundamental Lemma [6, Thm. 1], under technical conditions, the restricted behavior of an LTI system can be completely described via recorded historical data. This result is reviewed as Lemma 13 below. We first review the notion of persistent excitation.

[2]Note that $y^*_{[t,t+N)}$ is unique even though the matrices $U_{\mathrm{p}}, U_{\mathrm{f}}, Y_{\mathrm{p}}, Y_{\mathrm{f}}$ may not be unique.

Definition 12 (Persistent Excitation)**.** A sequence $z_{[t_1,t_2]}$ is *persistently exciting (p.e.) of order K*, for positive integer $K \le t_2 - t_1 + 1$, if the associated block-Hankel matrix of depth K

$$\mathcal{H}_K(z_{[t_1,t_2]}) := \begin{bmatrix} z_{t_1} & z_{t_1+1} & \cdots & z_{t_2-K+1} \\ z_{t_1+1} & z_{t_1+2} & \cdots & z_{t_2-K+2} \\ \vdots & \vdots & \ddots & \vdots \\ z_{t_1+K-1} & z_{t_1+K} & \cdots & z_{t_2} \end{bmatrix}$$

has full row rank.

Lemma 13 (Fundamental Lemma [6])**.** *Let $\mathcal{S}$ be an LTI system, and let $w^{\mathrm{d}}_{[t_1,t_2]}$ be a trajectory of $\mathcal{S}$. For $K \in \mathbb{N}$, if*

(i) $\mathcal{S}$ is controllable, and
(ii) $u^{\mathrm{d}}_{[t_1,t_2]}$ is p.e. of order $K + \mathbf{n}(\mathcal{S})$,

then

$$\mathrm{ColSpan}\big(\mathcal{H}_K(w^{\mathrm{d}}_{[t_1,t_2]})\big) = \mathscr{B}^{\mathcal{S}}_{[0,K)},$$

where $\mathcal{H}_K(w^{\mathrm{d}}_{[t_1,t_2]}) := \big[\mathcal{H}_K(u^{\mathrm{d}}_{[t_1,t_2]}); \mathcal{H}_K(y^{\mathrm{d}}_{[t_1,t_2]})\big]$.

Based on the lifting operation, we now define a natural extension of persistent excitation for LTP systems, and present a corresponding version of the fundamental lemma.

Definition 14 (Periodic Persistent Excitation)**.** A sequence $z_{[t_1,t_2]}$ is *T-periodically persistently exciting (T-p.p.e.) of order K*, for $K, T \in \mathbb{N}$ satisfying $K \le t_2 - t_1 + 1$, if

$$\mathcal{H}^T_K(z_{[t_1,t_2]}) := \begin{bmatrix} z_{t_1} & z_{t_1+T} & \cdots & z_{t_1+PT} \\ z_{t_1+1} & z_{t_1+T+1} & \cdots & z_{t_1+PT+1} \\ \vdots & \vdots & \ddots & \vdots \\ z_{t_1+K-1} & z_{t_1+T+K-1} & \cdots & z_{t_1+PT+K-1} \end{bmatrix}$$

has full row rank, where $P := \lfloor (t_2 - t_1 - K + 1)/T \rfloor$.[3]

Lemma 15 (Fundamental Lemma for LTP Systems)**.** *Let $\mathcal{S}$ be an LTP system of period T, and let $w^{\mathrm{d}}_{[t_1,t_2]}$ be a trajectory of $\mathcal{S}$ on interval $[t_1, t_2]$. For $K \in \mathbb{N}$, if*

(i) $\mathcal{S}$ is controllable, and
(ii) $u^{\mathrm{d}}_{[t_1,t_2]}$ is T-p.p.e. of order $(\lceil K/T \rceil + \mathbf{n}(\mathcal{S}, t_1))T$,

then

$$\mathrm{ColSpan}\big(\mathcal{H}^T_K(w^{\mathrm{d}}_{[t_1,t_2]})\big) = \mathscr{B}^{\mathcal{S}}_{[t_1,t_1+K)}, \tag{6}$$

where $\mathcal{H}^T_K(w^{\mathrm{d}}_{[t_1,t_2]}) := \big[\mathcal{H}^T_K(u^{\mathrm{d}}_{[t_1,t_2]}); \mathcal{H}^T_K(y^{\mathrm{d}}_{[t_1,t_2]})\big]$.

Proof. We first prove the case when K is a multiple of T, i.e., where $K = K_1 T$ for some $K_1 \in \mathbb{N}$. Let

$$\mathbf{w}_{[\tau_1,\tau_2]} := [\mathbf{u}_{[\tau_1,\tau_2]}; \mathbf{y}_{[\tau_1,\tau_2]}]$$

denote a trajectory of the lifted system $\mathcal{S}_{\mathsf{L}}(t_1)$. From Lemma 8, we know that $\mathscr{B}^{\mathcal{S}_{\mathsf{L}}(t_1)}_{[0,s)} = \mathscr{B}^{\mathcal{S}}_{[t_1,t_1+sT)}$ for all $s \in \mathbb{N}$, so we can establish such a trajectory $\mathbf{w}^{\mathrm{d}}_{[0,P)} \in \mathscr{B}^{\mathcal{S}_{\mathsf{L}}(t_1)}_{[0,P)}$ that

$$\mathbf{w}^{\mathrm{d}}_{[0,P)} := w^{\mathrm{d}}_{[t_1,t_1+PT)}$$

with $P := \lfloor (t_2 - t_1 + 1)/T \rfloor$, i.e., P is the number of whole periods in the interval $[t_1, t_2]$. With abuse of notation, we let $\mathbf{n} := \mathbf{n}(\mathcal{S}, t_1)$ and $\mathbf{n}_{\mathsf{L}} := \mathbf{n}(\mathcal{S}_{\mathsf{L}}(t_1))$ in this proof, then we have $\mathbf{n} = \mathbf{n}_{\mathsf{L}}$ via Lemma 10(i). By direct substitution, one can verify that[4]

$$\mathcal{H}_{K_1+\mathbf{n}_{\mathsf{L}}}(\mathbf{u}^{\mathrm{d}}_{[0,P)}) = \mathcal{H}^T_{(K_1+\mathbf{n})T}(u^{\mathrm{d}}_{[t_1,t_2]}), \tag{7a}$$

$$\mathcal{H}_{K_1}(\mathbf{w}^{\mathrm{d}}_{[0,P)}) = \mathcal{H}^T_K(w^{\mathrm{d}}_{[t_1,t_2]}). \tag{7b}$$

[3]One can observe that $\mathcal{H}^T_K(z_{[t_1,t_2]})$ is obtained by retaining every T-th column of $\mathcal{H}_K(z_{[t_1,t_2]})$.

[4]Note that in (7a), the block rows on the left-hand side are of size mT, where m is the input dimension of $\mathcal{S}$, while on the right-hand side, the block rows are of size m. Similarly for (7b).

Since $u^{\mathrm{d}}_{[t_1,t_2]}$ is T-p.p.e. of order $(K_1+\mathbf{n})T$ (i.e., the right-hand side of (7a) has full row rank), we know that $\mathbf{u}^{\mathrm{d}}_{[0,P)}$ is p.e. of order $K_1+\mathbf{n}_{\mathsf{L}}$ (as the left-hand side of (7a) has full row rank). We also know via Lemma 11 that $\mathcal{S}_{\mathsf{L}}(t_1)$ is controllable because $\mathcal{S}$ is controllable. Thus by Lemma 13 we have

$$\mathrm{ColSpan}(\mathcal{H}_{K_1}(\mathbf{w}^{\mathrm{d}}_{[0,P)})) = \mathscr{B}^{\mathcal{S}_{\mathsf{L}}(t_1)}_{[0,K_1)}. \tag{7c}$$

Substitute (7b) into the left-hand side of (7c), and replace the right-hand side of (7c) using $\mathscr{B}^{\mathcal{S}_{\mathsf{L}}(t_1)}_{[0,K_1)} = \mathscr{B}^{\mathcal{S}}_{[t_1,t_1+K)}$ (via Lemma 8), and then we obtain the result (6).

Next, we show the result for all $K\in\mathbb{N}$. Let $K_1 := \lceil K/T\rceil$ and $\widehat{K} := K_1 T$, i.e., $\widehat{K}$ is the smallest multiple of T greater than or equal to K. Since $\lceil K/T\rceil = \lceil \widehat{K}/T\rceil$, $u^{\mathrm{d}}_{[t_1,t_2]}$ is T-p.p.e. of order $(\lceil \widehat{K}/T\rceil + \mathbf{n}(\mathcal{S},t_1))T$, so we have the condition (ii) of this lemma for the case $K\leftarrow\widehat{K}$ (as a multiple of T), which case we have already proved. We therefore have

$$\mathrm{ColSpan}(\mathcal{H}^T_{\widehat{K}}(w^{\mathrm{d}}_{[t_1,t_2]})) = \mathscr{B}^{\mathcal{S}}_{[t_1,t_1+\widehat{K})}. \tag{7d}$$

Define $\mathcal{H}_\triangle := \mathcal{H}^T_K(w^{\mathrm{d}}_{[t_1,t_2-(\widehat{K}-K)]})$. One can verify that $\mathcal{H}^T_K(u^{\mathrm{d}}_{[t_1,t_2-(\widehat{K}-K)]})$ consists of the first K block rows of $\mathcal{H}^T_{\widehat{K}}(u^{\mathrm{d}}_{[t_1,t_2]})$, and similarly for y^{d}. Hence, applying Lemma 4 for (7d) we have

$$\mathrm{ColSpan}(\mathcal{H}_\triangle) = \mathscr{B}^{\mathcal{S}}_{[t_1,t_1+K)}. \tag{7e}$$

However, note that

$$\mathrm{ColSpan}(\mathcal{H}_\triangle) \subseteq \mathrm{ColSpan}(\mathcal{H}^T_K(w^{\mathrm{d}}_{[t_1,t_2]})) \subseteq \mathscr{B}^{\mathcal{S}}_{[t_1,t_1+K)}, \tag{7f}$$

where the first inclusion ($\subseteq$) above is because $\mathcal{H}_\triangle$ is a sub-matrix of $\mathcal{H}^T_K(w^{\mathrm{d}}_{[t_1,t_2]})$ with the same column size, and the second inclusion above is because each column of $\mathcal{H}^T_K(w^{\mathrm{d}}_{[t_1,t_2]})$ is a vector in the behavior set $\mathscr{B}^{\mathcal{S}}_{[t_1,t_1+K)}$. The equality (6) now follows by combining (7e) and (7f). ☐

When $\mathbf{n}(\mathcal{S},t_1)$ is unknown but bounded by some $n\in\mathbb{Z}$, we may obtain (ii) in Lemma 15 by requiring the input $u^{\mathrm{d}}_{[t_1,t_2]}$ to be T-p.p.e. of a sufficient order $(\lceil K/T\rceil + n)T$. This is because by definition a signal being T-p.p.e of order K' is also T-p.p.e. of any smaller order $K''\le K'$.

IV. Data-Driven Model Predictive Control for Linear Time-Periodic Systems

Based on our previous results extending behavioral systems theory to LTP systems, in this section, we develop a DDMPC algorithm for LTP systems $\mathcal{S}$ as in (1) of known period T.

A. Prediction, Control, and Initial Horizons

We consider a receding-horizon control strategy, in which at time t the control signal u on interval $[t,t+N_{\mathrm{c}})\cap\mathbb{Z}$ (the *control horizon*) is computed by minimizing an appropriate cost function of the predicted trajectory over a finite horizon $[t,t+N)\cap\mathbb{Z}$ (the *prediction horizon*), where $N_{\mathrm{c}},N\in\mathbb{N}$ are design parameters with $N_{\mathrm{c}}\le N$.

In the present data-driven scenario, the initial condition of the system at time t is specified by the recent trajectory in a past interval $[t-L,t)\cap\mathbb{Z}$ called the *initial horizon*, with parameter $L\in\mathbb{N}$. According to Lemma 7, if $L\ge \mathbf{l}(\mathcal{S},t-L)$, we can uniquely predict the future output, given any future input. Notice via Corollary 10.1 that the lag $\mathbf{l}(\mathcal{S},t-L)$ is bounded by nT, so the output prediction is always unique when we select $L\ge nT$. We call the union $[t-L,t+N)\cap\mathbb{Z}$ of the initial and prediction horizons as the *total horizon*; see Fig. 2.

B. Offline Data Collection

The restricted behavior $\mathscr{B}^{\mathcal{S}}_{[t-L,t+N)}$ on the total horizon must be known for us to predict future trajectories and compute control actions in the DDMPC framework. In previous work on DDMPC for LTI systems [9]–[16], the behavior $\mathscr{B}^{\mathcal{S}}_{[t-L,t+N)}$ can be represented using recorded offline data. We may extend this strategy to the case where $\mathcal{S}$ is an LTP system. However, since the system is periodic, its behavior $\mathscr{B}^{\mathcal{S}}_{[t-L,t+N)}$ can equal one of T different possible subspaces, depending on the time t. Fortunately, all T possibilities for the behavior $\mathscr{B}^{\mathcal{S}}_{[t-L,t+N)}$ can be covered using collected data.

1) **Offline Data:** Let $w^{\mathrm{d}}_{[t_{\mathrm{d1}},t_{\mathrm{d2}}]}$ be offline data collected from the system $\mathcal{S}$ on the interval $[t_{\mathrm{d1}},t_{\mathrm{d2}}]$, where we require that the input signal $u^{\mathrm{d}}_{[t_{\mathrm{d1}},t_{\mathrm{d2}}]}$ is T-p.p.e. of order $(\lceil K/T\rceil + n)T$, with $K := L+N+T-1$. Arrange the data into the "uncropped" data matrices $U^{\mathrm{d}}\in\mathbb{R}^{mK\times h}$ and $Y^{\mathrm{d}}\in\mathbb{R}^{pK\times h}$:

$$U^{\mathrm{d}} := \mathcal{H}^T_K(u^{\mathrm{d}}_{[t_{\mathrm{d1}},t_{\mathrm{d2}}]}), \qquad Y^{\mathrm{d}} := \mathcal{H}^T_K(y^{\mathrm{d}}_{[t_{\mathrm{d1}},t_{\mathrm{d2}}]}),$$

where h denotes the common width of U^{d} and Y^{d}, given by $h := \lfloor (t_{\mathrm{d2}}-t_{\mathrm{d1}}-K+1)/T\rfloor + 1$. We extract from U^{d} and Y^{d} the T sets of *data matrices* $U^\theta_{\mathrm{p}}\in\mathbb{R}^{mL\times h}$, $U^\theta_{\mathrm{f}}\in\mathbb{R}^{mN\times h}$, $Y^\theta_{\mathrm{p}}\in\mathbb{R}^{pL\times h}$ and $Y^\theta_{\mathrm{f}}\in\mathbb{R}^{pN\times h}$, defined as

$$\begin{aligned} U^\theta_{\mathrm{p}} &:= U^{\mathrm{d}}_{[\theta,\theta+L-1]}, & U^\theta_{\mathrm{f}} &:= U^{\mathrm{d}}_{[\theta+L,\theta+L+N-1]}, \\ Y^\theta_{\mathrm{p}} &:= Y^{\mathrm{d}}_{[\theta,\theta+L-1]}, & Y^\theta_{\mathrm{f}} &:= Y^{\mathrm{d}}_{[\theta+L,\theta+L+N-1]}, \end{aligned} \tag{8a}$$

where each set has an exclusive index $\theta\in\{1,\dots,T\}$. In (8a), we let $U^{\mathrm{d}}_{[r_1,r_2]}\in\mathbb{R}^{m(r_2-r_1+1)\times h}$ denote the sub-matrix consisting of the r_1-th, ..., r_2-th block rows of U^{d}, and similarly for $Y^{\mathrm{d}}_{[r_1,r_2]}$, with abuse of notation.

2) **Representation of Behavior:** The matrices U^θ_{p}, U^θ_{f}, Y^θ_{p}, Y^θ_{f} built from offline data can represent the behavior on the total horizon at time $t^\theta := t_{\mathrm{d1}}+\theta+L-1$, that is,

$$\mathrm{ColSpan}\big[U^\theta_{\mathrm{p}};U^\theta_{\mathrm{f}};Y^\theta_{\mathrm{p}};Y^\theta_{\mathrm{f}}\big] = \mathscr{B}^{\mathcal{S}}_{[t^\theta-L,t^\theta+N)}. \tag{8b}$$

under assumption that $\mathcal{S}$ is controllable and the input $u^{\mathrm{d}}_{[t_{\mathrm{d1}},t_{\mathrm{d2}}]}$ is T-p.p.e of order $(\lceil K/T\rceil + \mathbf{n}(\mathcal{S},t_{\mathrm{d1}}))T$; see Fig. 3. A proof of (8b) can be found in the extended version [27].

Since $\{t^\theta\}_{\theta=1}^T$ are consecutive time steps in one period, by periodicity of $\mathcal{S}$, the subspaces $\mathscr{B}^{\mathcal{S}}_{[t^\theta-L,t^\theta+N)}$ with different selections of the index $\theta\in\{1,\dots,T\}$ cover all T possibilities of the behavior $\mathscr{B}^{\mathcal{S}}_{[t-L,t+N)}$ for different time steps t. Define the *proper index* $\Theta(t)$ at time t.

$$\Theta(t) := 1 + (t - t_{\mathrm{d1}} - L \mod T) \tag{8c}$$

Thus, $\theta=\Theta(t)$ is the "correct" index θ such that the data matrices U^θ_{p}, U^θ_{f}, Y^θ_{p}, Y^θ_{f} represent the behavior on the total horizon at time t, i.e.,

$$\mathrm{ColSpan}\big[U^{\Theta(t)}_{\mathrm{p}};U^{\Theta(t)}_{\mathrm{f}};Y^{\Theta(t)}_{\mathrm{p}};Y^{\Theta(t)}_{\mathrm{f}}\big] = \mathscr{B}^{\mathcal{S}}_{[t-L,t+N)}, \tag{8d}$$

because of (8b), periodicity of $\mathcal{S}$ and the fact that $t-t^{\Theta(t)}$ is a multiple of T.

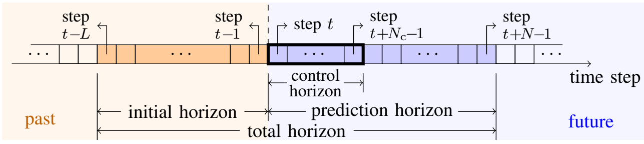

Fig. 2. Time horizons at time t. The initial horizon is a past interval up to time $t-1$. The prediction horizon is a future interval starting at time t. The total horizon is their union.

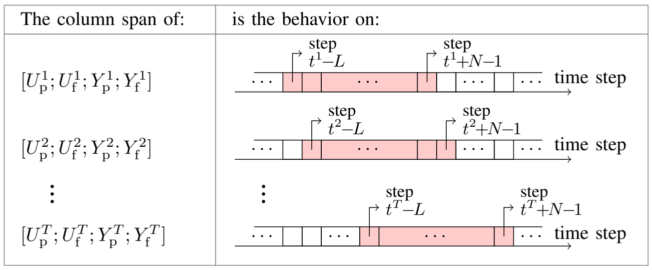

Fig. 3. The column span of the matrix $[U_\mathrm{p}^\theta; U_\mathrm{f}^\theta; Y_\mathrm{p}^\theta; Y_\mathrm{f}^\theta]$ is the behavior on the interval $[t^\theta - L, t^\theta + N) \cap \mathbb{Z}$. Note that $\{t^\theta\}_{\theta=1}^T$ are consecutive time steps in one period.

C. Online Process

Now we introduce the online process of the algorithm. Suppose we have collected the offline data in Section IV-B.

1) **Warm-Up Process:** Before controlling the system, we require to know the proper index $\Theta(t)$ at current time t. However, this value will be unknown unless $t_{\mathrm{d}1}$ is known during the data collection process, and will generally be unknown in a practical implementation. To address this, we developed a heuristic algorithm to identify the value of $\Theta(t)$, which can be found in the extended version [27]. Let $\widehat{\Theta}(t)$ denote the estimate of $\Theta(t)$ at time t.

Once we obtain the estimate $\widehat{\Theta}(t)$ at time t, the value of $\widehat{\Theta}(t')$ for any future time $t' \geq t$ is derived according to

$$\widehat{\Theta}(t+1) := \begin{cases} \widehat{\Theta}(t) + 1, & \text{if } \widehat{\Theta}(t) \in \{1, \dots, T-1\}, \\ 1, & \text{if } \widehat{\Theta}(t) = T, \end{cases}$$

which is the same way that $\Theta(t)$ evolves with time t.

2) **Control Process:** With the proper index identified or known, we can start the control process. We provide for the LTP system $\mathcal{S}$ two alternative controllers, which generalize Data-enabled Predictive Control (DeePC) and Subspace Predictive Control (SPC) methods in the literature.

Let u^* and y^* denote the future input and predicted output respectively. At step t, we consider the quadratic cost

$$\textstyle\sum_{i=t}^{t+N-1} \|y_i^* - r_i\|_Q^2 + \|u_i^*\|_R^2 \tag{9a}$$

with cost matrices $Q \succeq 0$ and $R \succ 0$ as parameters, and constrain the future input-output signal

$$u_i^* \in \mathcal{U}, \quad y_i^* \in \mathcal{Y}, \qquad \forall i \in [t, t+N) \cap \mathbb{Z} \tag{9b}$$

with user-defined constraint sets $\mathcal{U} \subseteq \mathbb{R}^m$ and $\mathcal{Y} \subseteq \mathbb{R}^p$. The *periodic DeePC (P-DeePC) problem* at time t is

$$\underset{g, u^*, y^*}{\operatorname{minimize}} \;\; \text{(9a)} \;\; \text{s.t.} \;\; \text{(9c) and (9b)} \tag{P-DeePC}$$

with an auxiliary variable $g \in \mathbb{R}^h$, where (9c) is given as

$$\begin{bmatrix} U_\mathrm{p}^{\widehat{\Theta}(t)} \\ U_\mathrm{f}^{\widehat{\Theta}(t)} \\ Y_\mathrm{p}^{\widehat{\Theta}(t)} \\ Y_\mathrm{f}^{\widehat{\Theta}(t)} \end{bmatrix} g = \begin{bmatrix} u_{[t-L,t)} \\ u^*_{[t,t+N)} \\ y_{[t-L,t)} \\ y^*_{[t,t+N)} \end{bmatrix}. \tag{9c}$$

The *periodic SPC (P-SPC) problem* at time t

$$\underset{u^*, y^*}{\operatorname{minimize}} \;\; \text{(9a)} \;\; \text{s.t.} \;\; \text{(9d) and (9b)} \tag{P-SPC}$$

with (9d) given as

$$y^*_{[t,t+N)} = Y_\mathrm{f}^{\widehat{\Theta}(t)} \begin{bmatrix} U_\mathrm{p}^{\widehat{\Theta}(t)} \\ U_\mathrm{f}^{\widehat{\Theta}(t)} \\ Y_\mathrm{p}^{\widehat{\Theta}(t)} \end{bmatrix}^\dagger \begin{bmatrix} u_{[t-L,t)} \\ u^*_{[t,t+N)} \\ y_{[t-L,t)} \end{bmatrix}. \tag{9d}$$

After solving the optimal future trajectory $w^*_{[t,t+N)}$ from either (P-DeePC) or (P-SPC), we apply the first N_c inputs $u^*_{[t,t+N_\mathrm{c})}$ to the system $\mathcal{S}$. The whole control process is illustrated in Algorithm 1.

Algorithm 1 Control Process

Input: the time step t, the estimated proper index $\widehat{\Theta}(t)$, the reference signal r and the data matrices $U_\mathrm{p}^\theta, U_\mathrm{f}^\theta, Y_\mathrm{p}^\theta, Y_\mathrm{f}^\theta$ for $\theta \in \{1, \dots, T\}$.

1: **while** `true` **do**
2: Solve $w^*_{[t,t+N)}$ from (P-DeePC) or (P-SPC).
3: Input $u_{[t,t+N_\mathrm{c})} \leftarrow u^*_{[t,t+N_\mathrm{c})}$ to the system $\mathcal{S}$.
4: Set $t \leftarrow t + N_\mathrm{c}$ and update $\widehat{\Theta}(t)$ correspondingly.

---

3) **Performance Guarantee:** In the deterministic case, both P-DeePC and P-SPC produce the same control actions that one would obtain from traditional MPC applied to the LTP system. The MPC problem for (1) at time t,

$$\underset{x^*, u^*, y^*}{\operatorname{minimize}} \;\; \text{(9a)} \;\; \text{s.t.} \;\; \text{(9e) and (9b)} \tag{MPC}$$

where (9e) is given as follows.

$$\begin{aligned} x_{i+1}^* &= A_i x_i^* + B_i u_i^*, && \forall i \in [t, t+N) \cap \mathbb{Z} \\ y_i^* &= C_i x_i^* + D_i u_i^*, && \forall i \in [t, t+N) \cap \mathbb{Z} \\ x_t^* &= x_t \end{aligned} \tag{9e}$$

Proposition 16. *Consider an LTP system $\mathcal{S}$ as in* (1) *of period T. Let $w^\mathrm{d}_{[t_{\mathrm{d}1}, t_{\mathrm{d}2}]}$ be offline data from $\mathcal{S}$ on interval $[t_{\mathrm{d}1}, t_{\mathrm{d}2}]$. For time step $t \in \mathbb{Z}$ and $L, N \in \mathbb{N}$, assume that*

(i) $L \geq \mathbf{l}(\mathcal{S}, t-L)$,
(ii) *$\mathcal{S}$ is controllable in the sense of Definition 5,*
(iii) *$u^\mathrm{d}_{[t_{\mathrm{d}1}, t_{\mathrm{d}2}]}$ is T-p.p.e. of order $(\lceil K/T \rceil + \mathbf{n}(\mathcal{S}, t_{\mathrm{d}1}))T$, with $K := L + N + T - 1$, and*
(iv) $\widehat{\Theta}(t) = \Theta(t)$.

Suppose we know the state x_t and recent trajectory $w_{[t-L,t)}$ of $\mathcal{S}$. Then,

- *the unique optimal trajectory $w^*_{[t,t+N)}$ by* (P-DeePC)*,*
- *the unique optimal trajectory $w^*_{[t,t+N)}$ by* (P-SPC)*, and*
- *the unique optimal trajectory $w^*_{[t,t+N)}$ by* (MPC)

are all same.

This result generalizes [12, Cor. 5.1] and [31, Thm. 1], which results claim the equivalence of DeePC, SPC and MPC for LTI systems.

Remark 17. Our extension of DeePC and SPC to LTP systems is based on the insight that the data collected from an LTP system is equivalent to data collected from an appropriate LTI lifted system. In particular, after stacking LTP-system data into lifted-system data, we can apply the established LTI DDMPC methods and compute control signals for the lifted system, and thereby obtain control signals for the original LTP system. A benefit of our treatment here is that discussion of lifted systems can be entirely omitted once proper behavioral systems concepts are defined directly on the LTP system, as we have done in Section III. □

4) **Regularization:** To adapt our methods for stochastic LTP systems with noisy measurements, we may regularize both P-DeePC and P-SPC. Regularizing P-DeePC is similar as regularizing DeePC [13]–[16]. Here we exhibit quadratic regularization, where (P-DeePC) is modified as follows,

$$\underset{g,u^*,y^*,\sigma_{\mathrm{y}}}{\text{minimize}} \quad \text{(9a)} + \lambda_{\mathrm{y}}\|\sigma_{\mathrm{y}}\|_2^2 + \lambda_{\mathrm{g}}\|g\|_2^2 \ \text{ s.t. } \ \text{(9f) and (9b)}$$

with a slack variable $\sigma_{\mathrm{y}} \in \mathbb{R}^{pL}$, positive parameters $\lambda_{\mathrm{y}}, \lambda_{\mathrm{g}}$, and a modified constraint (9f) from (9c).

$$\begin{bmatrix} U_{\mathrm{p}}^{\widehat{\Theta}(t)} \\ U_{\mathrm{f}}^{\widehat{\Theta}(t)} \\ Y_{\mathrm{p}}^{\widehat{\Theta}(t)} \\ Y_{\mathrm{f}}^{\widehat{\Theta}(t)} \end{bmatrix} g = \begin{bmatrix} u_{[t-L,t)} \\ u^*_{[t,t+N)} \\ y_{[t-L,t)} \\ y^*_{[t,t+N)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sigma_{\mathrm{y}} \\ 0 \end{bmatrix} \tag{9f}$$

To regularize P-SPC, in the computation of the pseudo-inverse in (9d), we treat as zero the singular values smaller than a selected threshold σ_{SPC}; the remainder of the settings in regularized P-SPC are same as in P-SPC.

V. Simulations

We illustrate the algorithm proposed in Section IV and its robustness to noisy data via numerical example. Consider the mass-spring-damper system in Fig. 4.

The control objective is reference tracking for the positions (x_1, x_2, x_3) of the three masses. There are three control inputs: the force F applied to the mass m_1, and the end positions x_4 and x_5 of the free ends of the springs k_4 and k_5. The stiffness and damping parameters k_i and c_i are periodic functions of time, given in Table I, and each has a period of 1 second. We discretize the system with a sampling time 0.2s, and thus the period of the discretized system is $T = 5$. A process noise $w_t \overset{\text{i.i.d}}{\sim} \mathcal{N}(0_{6\times1}, \sigma^2 I_6)$ and a measurement noise $v_t \overset{\text{i.i.d}}{\sim} \mathcal{N}(0_{3\times1}, \sigma^2 I_3)$ are added to the discrete-time model, with noise amplitude $\sigma^2 = 10^{-3}$. The control parameters are selected in Table II.

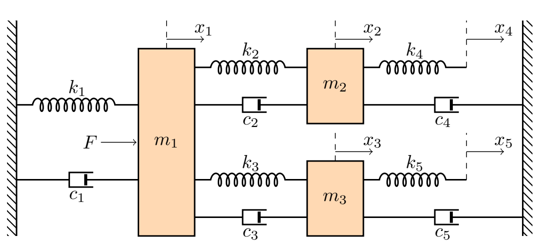

Fig. 4. A spring-mass-damper model for simulation.

For collection of offline data, we apply a random input signal $u_t^{\mathrm{d}} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0_{3\times1}, I_3)$ and measure the resulting positions (x_1, x_2, x_3). The online process starts at time $t = 0$, with the warm-up process terminating at time $t = 40$. In our simulation, the proper index $\Theta(t)$ was correctly identified.

We start control at time $t = 40$, and apply sequential changes in the reference signals given by $r_t = [0; 0; 0]$ for $40 \le t < 60$, $r_t = [5; 0; 0]$ for $60 \le t < 80$, $r_t = [5; 15; 0]$ for $80 \le t < 100$, and $r_t = [5; 15; -10]$ for $t \ge 100$. We evaluate the control performance via the one-step cost $\|y_t - r_t\|_Q^2 + \|u_t\|_R^2$, and the results are shown in Fig. 5.

For comparison purposes, we also plot the closed-loop responses under (i) MPC using a perfect system model with full-state measurements, and (ii) the regularized DeePC and regularized SPC methods for LTI systems. For the latter, the settings are the same as for P-DeePC (resp. P-SPC), except that we replace the matrices $U_{\mathrm{p}}^{\widehat{\Theta}(t)}, U_{\mathrm{f}}^{\widehat{\Theta}(t)}, Y_{\mathrm{p}}^{\widehat{\Theta}(t)}, Y_{\mathrm{f}}^{\widehat{\Theta}(t)}$ in (9f) (resp. (9d)) by $U_{\mathrm{p}}^1, U_{\mathrm{f}}^1, Y_{\mathrm{p}}^1, Y_{\mathrm{f}}^1$ respectively, i.e., we use a single set of data matrices at all time t. Around the step changes of the reference signal, all controllers have comparable performances with similar cost values. For the steady-state performance when the reference signal stays constant, the proposed regularized P-DeePC (resp. P-SPC) method outperforms the direct use of regularized DeePC (resp. SPC) of LTI systems. This significant difference indicates the necessity of using different sets of data matrices $U_{\mathrm{p}}^\theta, U_{\mathrm{f}}^\theta, Y_{\mathrm{p}}^\theta, Y_{\mathrm{f}}^\theta$ with different indices θ as in (9f) and (9d) for P-DeePC and P-SPC respectively at different time steps.

VI. Conclusion and Future Work

We proposed a DDMPC algorithm for unknown LTP systems with known periods. For deterministic LTP systems,

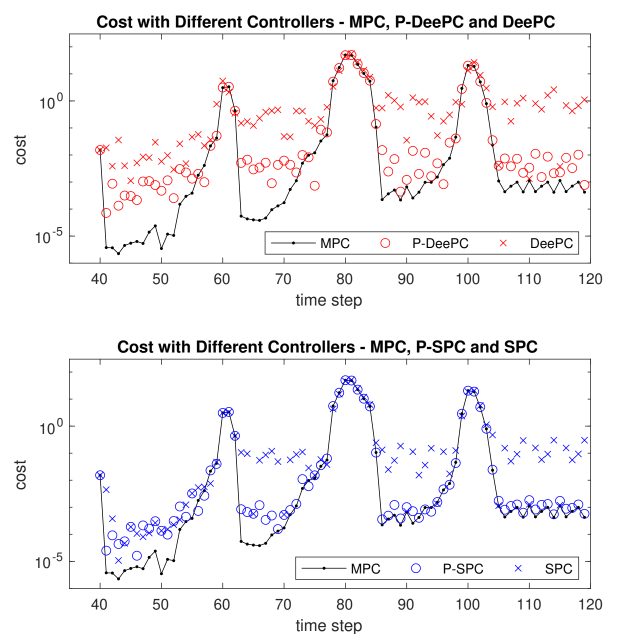

Fig. 5. The cost of the simulated LTP system, using MPC, regularized P-DeePC, regularized P-SPC, regularized DeePC of LTI systems and regularized SPC of LTI systems, respectively.

the method is equivalent to classical MPC, but without the requirement of a parametric model. The approach is supported by extensions of results from behavioral systems theory to LTV and LTP systems. Simulation results provide evidence that the approach is robust to measurement noise and stochasticity, and that it significantly outperforms a naive application of data-driven LTI control methods.

There are several open directions for future work. First, as our design requires a priori knowledge of the period T, relaxing this assumption is of interest, as is investigating the robustness of the approach to errors in the selected period. Second, we note that there remain open questions in the behavioral theory of LTP systems, such as what relationships can be established between the behaviors of the T different lifted systems arising from a given LTP system.

TABLE I
PHYSICAL QUANTITIES OF THE SIMULATED SYSTEM

Quantity (Unit)	**Symbol**	**Value**[a]
Mass (kg)	m_1, m_2, m_3	$6, 4, 3$ respectively
Spring Stretch ($\mathrm{N/m}$)	k_1	$10 - 4\sin(2\pi t) + 2\sin(4\pi t)$
	k_2, k_3	$7 - 3\cos(4\pi t)$
	k_4, k_5	$4 - 2\sin(4\pi t)$
Viscosity ($\mathrm{N \cdot s/m}$)	c_1	$9 + 3\sin(2\pi t)$
	c_2, c_3	$5 + 2\cos(2\pi t)$
	c_4, c_5	15

[a]Variable t is the time in seconds.

TABLE II
CONTROL PARAMETERS

Parameter	**Value**
initial horizon length L	30
prediction horizon length N	30
control horizon length N_{c}	1
tracking cost matrix Q in (9a)	$\mathrm{diag}(1, 1, 1)$
input cost matrix R in (9a)	$\mathrm{diag}(10^{-6}, 10^{-4}, 10^{-4})$
input constraint set $\mathcal{U}$ in (9b)	$[-8, 8] \times [-3, 3] \times [-3, 3]$
output constraint set $\mathcal{Y}$ in (9b)	$[-20, 20]^3$
regularization parameter λ_{y} for P-DeePC	10^6
regularization parameter λ_{g} for P-DeePC	10^{-3}
SV threshold σ_{SPC} in regularized P-SPC	0.5

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