On the Role of Dual Sylvester and Invariance Equations in Systems and Control

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Abstract: Sylvester equations, their duals, and their nonlinear generalizations arise in a wide variety of systems and control problems. Here we provide an overview of classical and recent applications of these equations in output regulation, stabilization of cascaded systems, observer design, and model order reduction, for both linear and nonlinear systems. Similarities and differences between the linear and nonlinear cases are highlighted, with an emphasis on how design formulations and associated dual designs for linear systems may generalize to the nonlinear setting.

Keywords: Sylvester equation, output regulation, cascade stabilization, model order reduction, observers, nonlinear systems

1. INTRODUCTION

Equations and inequalities with matrix variables arise frequently in systems and control analysis and design problems. One of the simplest and most important examples is the *Sylvester equation*, which is the linear matrix equation

$$\mathbf{A}\boldsymbol{\Pi} - \boldsymbol{\Pi}\mathbf{B} = \mathbf{C}, \qquad (1)$$

where $\boldsymbol{\Pi} \in \mathbb{R}^{n\times m}$ is the unknown and $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times m}$, and $\mathbf{C} \in \mathbb{R}^{n\times m}$ are given matrices. Perhaps the most well-known instance of (1) is as the *Lyapunov* equation characterizing Hurwitz stability of a matrix $\mathbf{A}$, in which $\mathbf{B} = -\mathbf{A}^\top$, $\mathbf{C}$ is symmetric negative definite, and $\boldsymbol{\Pi}$ is symmetric positive definite.

An incomplete list of system-theoretic applications of the general equation (1) would include observer design [Luenberger, 1964], output regulation [Francis and Wonham, 1975], pole assignment [Shafai and Bhattacharyya, 1988], disturbance decoupling [Syrmos, 1994], cascade stabilization [Astolfi et al., 2022], and model order reduction [Gallivan et al., 2004]. In these contexts, the solution $\boldsymbol{\Pi}$ of (1) may be an intermediate variable, or may itself be a gain within a feedback design. In the context of nonlinear systems applications, (1) often generalizes into a (nonlinear) partial differential equation in an unknown function $\boldsymbol{\pi} : \mathbb{R}^m \to \mathbb{R}^n$, often expressing *invariance* of some manifold within the state space, and we therefore refer to such nonlinear Sylvester-like equations as *invariance* equations.

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As we will describe, in many of the above problem settings, a closely related *dual* Sylvester equation of the form

$$\boldsymbol{\Upsilon}\mathbf{A} - \mathbf{B}\boldsymbol{\Upsilon} = \mathbf{D} \qquad (2)$$

in the unknown $\boldsymbol{\Upsilon} \in \mathbb{R}^{m\times n}$ occurs as an alternative to (1), with $\mathbf{D} \in \mathbb{R}^{m\times n}$ given data. The dual equation (2) may lead to a complementary design procedure, or may be used alongside (1) as part of a joint procedure. Again in the nonlinear setting, (2) would generalize into a dual partial differential equation in an unknown function $\boldsymbol{\upsilon} : \mathbb{R}^n \to \mathbb{R}^m$.

Despite the widespread importance of (1), (2) and their nonlinear generalizations, it is difficult to locate in the literature a concise reference which describes and contrasts the different design procedures arising from (1) and (2), respectively, with a reasonably unified viewpoint across application areas. Moreover, it appears that design procedures for some problems based on (1) do not yet have corresponding dual design counterparts based on (2), in both the linear and nonlinear settings. As such, our objective in this paper is to provide such an overview from a unified point of view, and to begin the process of identifying possible gaps in generalization of design procedures across several application areas.

Contributions. We provide a concise overview of the applications of the Sylvester equations (1), (2) and their nonlinear generalizations in the areas of output regulation, cascade stabiliation, observer design, and model order reduction. We outline a range of classical and recent design procedures and dual design procedures for linear systems based on Sylvester equations, before describing corresponding extensions to nonlinear systems based on invariance equations. No novelty is claimed in the technical results; our contribution is organizational, highlighting the parallels and distinctions between the linear and nonlinear cases, and attempting to identify unexplored design pathways in both settings for future research.

Notation. The symbol $\sigma(A)$ denotes the spectrum of the matrix $A \in \mathbb{R}^{n\times n}$. The superscripts $\top$ and $*$ denote the transposition and conjugate transposition operators, respectively. The real part of $z \in \mathbb{C}$ is $\mathrm{Re}(z)$, and j denotes the imaginary unit. Given two mappings, $f : Y \to Z$ and $g : X \to Y$, with $f \circ g : X \to Z$ we denote the composite function that maps all $x \in X$ to $f(g(x)) \in Z$. The Lie derivative of the smooth scalar function $h(x)$ along the vector field $f(x)$ is denoted by $\mathcal{L}_f\, h(x) = \frac{\partial h}{\partial x}(x)f(x)$. We use the recursive notation $\mathcal{L}_f^k\, h(x) = \mathcal{L}_f\, \mathcal{L}_f^{k-1}\, h(x)$, with $\mathcal{L}_f^0\, h(x) = h(x)$. The space of continuously differentiable functions is indicated by C^1.

2. LINEAR SYSTEMS

We begin by describing how (1) arises in the problems of output regulation, stabilization, observer design, and model reduction for finite-dimensional linear time-invariant systems. As a preliminary result of importance, the left-hand side of (1) defines a linear mapping $\mathbf{\Pi} \mapsto \mathcal{S}(\mathbf{\Pi}) := \mathbf{A\Pi} - \mathbf{\Pi B}$ from $\mathbb{R}^{n\times m}$ into itself, and a fundamental result is that $\mathcal{S}$ is invertible – and thus, (1) possess a unique solution for each $\mathbf{C}$ – if and only if $\sigma(\mathbf{A})\cap\sigma(\mathbf{B}) = \emptyset$. Further general theoretical insights into (1) relevant to systems and control will be discussed in Section 2.5.

2.1 Output Regulation

In linear output regulation, see [Francis and Wonham, 1975, Davison, 1976], one studies systems of the form

$$\dot{w} = Sw, \tag{3a}$$
$$\dot{x} = Ax + Bu + Pw, \tag{3b}$$
$$e = Cx + Qw, \tag{3c}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state of the plant, $u(t) \in \mathbb{R}^{n_u}$ is the control input, and $e(t) \in \mathbb{R}^{n_e}$ is the error output to be regulated to zero. The triple (A, B, C) is assumed to be stabilizable and detectable. The signal w, with $w(t) \in \mathbb{R}^{n_w}$, models disturbances to be rejected or references to be tracked, and is generated by the *exosystem* (3a). The eigenvalues of S are typically assumed to lie on the imaginary axis, and hence w would consist of, *e.g.*, polynomials and sinusoids of different frequencies. The design objective is to construct a feedback control law guaranteeing (i) asymptotic stability of the origin when no disturbance is present ($w \equiv 0$), and (ii) asymptotic convergence of $e(t)$ to 0 when $w \neq 0$, irrespective of the particular w generated by (3a).

In the so-called *full information* design problem, both signals x and w are available for feedback design, and a feedback law of the form $u = K_x x + K_w w$ is proposed with K_x chosen such that $A + BK_x$ is Hurwitz. This leads to the block triangular closed-loop system

$$\begin{bmatrix}\dot{w}\\ \dot{x}\end{bmatrix} = \begin{bmatrix} S & 0\\ P + BK_w & A + BK_x\end{bmatrix}\begin{bmatrix} w\\ x\end{bmatrix}, \tag{4}$$

in which the w-subsystem drives the internally stable x-subsystem. The system (4) has stable eigenspace $\mathrm{Im}\left[\begin{smallmatrix}0\\ I\end{smallmatrix}\right]$ and complementary center eigenspace $\mathrm{Im}\left[\begin{smallmatrix}I\\ \Pi_{\mathrm{r}}\end{smallmatrix}\right]$, where Π_{r} is the unique solution of the Sylvester equation

$$\Pi_{\mathrm{r}} S = (A + BK_x)\Pi_{\mathrm{r}} + (P + BK_w).$$

It follows that any trajectory $(w(t), x(t))$ of (4) converges exponentially towards a steady-state trajectory $(w(t), x_{\mathrm{ss}}(t))$ satisfying $x_{\mathrm{ss}}(t) = \Pi_{\mathrm{r}} w(t)$. Zeroing of the corresponding steady-state error $e_{\mathrm{ss}}(t) = Cx_{\mathrm{ss}}(t) + Qw(t) = (C\Pi_{\mathrm{r}} + Q)w(t)$ therefore requires additionally that $C\Pi_{\mathrm{r}} + Q = 0$. The previous two requirements are commonly expressed as the *regulator equations*

$$\Pi_{\mathrm{r}} S = A\Pi_{\mathrm{r}} + B\Psi_{\mathrm{r}} + P, \tag{5a}$$
$$0 = C\Pi_{\mathrm{r}} + Q, \tag{5b}$$

where $\Psi_{\mathrm{r}} = K_w + K_x\Pi_{\mathrm{r}}$. The Sylvester equation (5a) guarantees the existence of an invariant subspace, with (5b) ensuring the error e is held at zero on that subspace.

Solvability of (5) is both necessary and sufficient for solvability of the full-information regulation problem. Furthermore, the equations (5) are solvable for *all* possible (P, Q) if and only if the so-called *non-resonance condition*

$$\mathrm{rank}\begin{pmatrix} A - \lambda I & B\\ C & 0\end{pmatrix} = n_x + n_e, \qquad \lambda \in \sigma(S), \tag{6}$$

holds. In fact, (6) is precisely the statement that the transmission zeros of the plant are disjoint from the eigenvalues of S, and when A is Hurwitz, it reduces to $\mathrm{rank}\, W(\lambda) = n_e$ for all $\lambda \in \sigma(S)$, where

$$W(s) = C(sI - A)^{-1}B \tag{7}$$

is the transfer matrix from the input u to the output e.

The non-resonance condition (6) is intimately connected to properties of solutions of two auxiliary Sylvester equations. Indeed, let M, N be such that $\sigma(A + MC) \cap \sigma(S) = \emptyset$ and $\sigma(A + BN) \cap \sigma(S) = \emptyset$, and consider the equations

$$\Pi\Phi = (A + MC)\Pi + BL \tag{8a}$$
$$\Upsilon(A + BN) = \Phi\Upsilon + RC \tag{8b}$$

in the variables Π and Υ, with $\sigma(\Phi) = \sigma(S)$ and R, L given matrices.

Theorem 1. Consider the Sylvester equations (8). *If* (Φ, R) *is a controllable pair then the following three statements are equivalent:*

(i) (5) *is solvable in* $(\Pi_{\mathrm{r}}, \Psi_{\mathrm{r}})$ *for all* (P, Q)*;*
(ii) the non-resonance condition (6) *holds;*
(iii) $(\Phi, \Upsilon B)$ *is controllable.*

If furthermore $n_e = n_u$ *and* (L, Φ) *is an observable pair, then a fourth equivalent statement is that* $(C\Pi, \Phi)$ *is observable.*

The equivalence (i) ⇔ (ii) is classical, see, *e.g.*, [Trentelman et al., 2001, Theorem 9.6]. The equivalence to (iii) is shown in [Astolfi et al., 2022, Proposition 2], and equivalence to (iv) can be proved by analogous arguments. The matrices ΥB and $C\Pi$ will reappear several times in the sequel.

2.2 Cascade Stabilization

Feedback stabilization of cascade systems of the form

$$\begin{aligned}\dot{x} &= Ax + Bu\\ \dot{z} &= \Phi z + RCx\end{aligned} \tag{9}$$

appears in many control problems. For example, in *error-feedback* output regulation, the z-dynamics represents the internal model regulator, in which Φ is designed to have the same spectrum of S and (Φ, R) is a controllable pair. In other contexts, the x-dynamics may represent actuators, or the z-dynamics may represent sensors, *e.g.*, [Kokotović et al., 1999]. State-feedback design for the stabilization of the cascade system (9) can be approached using either of the Sylvester equations in (8), *e.g.*, [Syrmos and Lewis, 1993], [Astolfi et al., 2022], [Chen and Simpson-Porco, 2023]. For concreteness and to illustrate the ideas, we describe here two design procedures arising in error-feedback output regulation, wherein the x-dynamics is stabilizable, Φ has eigenvalues on the imaginary axis, (Φ, R) is controllable, and (6) holds.

The first design is based on the Sylvester equation (8b), and is known in the literature as the *forwarding* approach, *e.g.*, [Astolfi et al., 2022, Section 3]. The main idea is to pre-stabilize the x dynamics with a feedback $u = Nx + v$, with v as a new input, and then to look for an invariant subspace arising from the stable x-dynamics driving the z-dynamics. When $v = 0$, such an invariant subspace $\{(x, z) \mid z = \Upsilon x\}$ is determined by the solution Υ to (8b). With this, setting $u = Nx + v$ and performing the change of coordinates

$$z \mapsto \zeta := z - \Upsilon x$$

one obtains the system

$$\begin{aligned} \dot{x} &= (A + BN)x + Bv, \\ \dot{\zeta} &= \Phi\zeta - \Upsilon Bv, \end{aligned} \tag{10}$$

which is now *decoupled*. Since (Φ, R) is controllable, so is $(\Phi, \Upsilon B)$, and one may design a feedback of the form $v = K_\zeta \zeta$ to obtain a system in block-triangular form. Stability of $A+BN$ and of $\Phi - \Upsilon BK_\zeta$ then implies stability of the closed-loop system. In the case in which Φ is skew-symmetric, one simple choice is given by $K_\zeta = B^\top \Upsilon^\top$.

An alternative approach proposed in [Chen and Simpson-Porco, 2023] is based instead on the Sylvester equation (8a). Consider the state feedback design $u = Nx + Lz$, for (9) with N such that $A + BN$ is Hurwitz. If $\mathcal{S}(\Pi) = \Pi\Phi - (A + BN)\Pi$ is the Sylvester operator, then the solution Π may be expressed as $\Pi = \mathcal{S}^{-1}(BL)$, which can now be viewed as a linear function of the remaining design variable L. With the change of coordinates

$$x \mapsto \xi := x - \Pi z$$

one obtains the system

$$\begin{aligned} \dot{\xi} &= (A + BN - \Pi RC)\xi - \Pi RC\Pi z, \\ \dot{z} &= (\Phi + RC\Pi)z + RC\xi. \end{aligned} \tag{11}$$

Since (Φ, R) is controllable, one may design a matrix Z and $\epsilon > 0$ sufficiently small such that $\Phi + \epsilon RZ$ is Hurwitz, and then solve the linear operator equation $\epsilon Z = C\Pi = C\mathcal{S}^{-1}(BL)$ to obtain the feedback gain L. Since Π will also be [1] $\mathcal{O}(\epsilon)$, one will have $A + BN - \Pi RC$ Hurwitz and $\Pi RC\Pi$ of order $\mathcal{O}(\epsilon^2)$, and straightforward arguments then establish that (11) is stable for sufficiently small ϵ.

2.3 Observer Design

In the original work by Luenberger [Luenberger, 1964], state observer design was based on the solution of a Sylvester equation. In particular, the idea was to look for an invertible linear change of coordinates $z = \Upsilon_o x$ transforming the plant dynamics

$$\dot{x} = Ax\,, \qquad y = Cx \tag{12}$$

with state x, where $x(t) \in \mathbb{R}^{n_x}$, and scalar output y, where $y \in \mathbb{R}$, into the form

$$\dot{z} = Fz + Gy, \tag{13}$$

where F is Hurwitz and (F, G) is controllable. For this stable transformed system, a trivial observer is obtained by copying the dynamics as

$$\dot{\hat{z}} = F\hat{z} + Gy,$$

and $\hat{x}$ is then obtained from $\hat{z}$ by inverting the transformation. The existence of Υ_o transforming the dynamics (12) into the form (13) is equivalent to solvability of the Sylvester equation

$$\Upsilon_o A = F\Upsilon_o + GC. \tag{14}$$

It was established that if (A, C) is observable, then for any Hurwitz $F \in \mathbb{R}^{n_x \times n_x}$ satisfying $\sigma(A) \cap \sigma(F) = \emptyset$, and for any vector G in $\mathbb{R}^{n_x}$ such that the pair (F, G) is controllable, the Sylvester equation (14) admits a unique solution $\Upsilon_o \in \mathbb{R}^{n_x \times n_x}$ which is itself an invertible matrix. The desired state estimate $\hat{x}$ is then given by $\hat{x} = \Upsilon^{-1}\hat{z}$.

We conclude by noting that the problem of *disturbance observation* is dual to the problem of output regulation discussed in Section 2.1, and thus many of the results described therein translate to the problem of estimating external disturbances, although such results appear to be difficult to locate in the literature. Moreover, we note that a dual Sylvester approach to the observer design problem appears to be absent in the literature.

2.4 Model Order Reduction

Consider a linear continuous-time system described by

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{15}$$

with $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^{n_u}$, $y(t) \in \mathbb{R}^{n_y}$, and transfer matrix $W(s)$ as in (7). The idea behind the *moment matching* approach to model order reduction is to determine a reduced-order model that interpolates the transfer function of the full-order model (15) at special points of interest [Antoulas, 2005]. We formalise this idea for the case $n_u = 1$ and $n_y = 1$, but similar results holds for the general case. For $s_i \in \mathbb{C} \setminus \sigma(A)$, the 0-*moment of* (15) *at* s_i is defined as the complex number $\eta_0(s_i) = W(s_i)$, and the k-*moment of* (15) *at* s_i is defined as the complex number $\eta_k(s_i) = \frac{(-1)^k}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}s^k}W(s)\right]_{s=s_i}$, with $k \geq 1$ integer.

Given a set of *interpolation points* $\{s_j\} \subset \mathbb{C}$, the problem of model order reduction is to construct a new model

$$\dot{x}_r = A_r x_r + B_r u, \qquad y_r = C_r x_r, \tag{16}$$

with $x_r(t) \in \mathbb{R}^{n_r}$, $n_r < n_x$, and $y_r(t) \in \mathbb{R}$ such that the moments of (16) match the moments of (15) at the interpolation points. The idea is that if enough points are matched, the transfer function of the reduced-order model resembles that of the full-order model. In [Gallivan et al., 2004, 2006], it is observed that the moment matching problem can be reformulated using dual Sylvester equations.

[1] We use $\mathcal{O}(\epsilon^k)$ for standard big-O notation, namely $f \in \mathcal{O}(\epsilon^k)$ if $\lim_{\epsilon \to 0} f(\epsilon^k)/\epsilon^{k-1} = 0$.

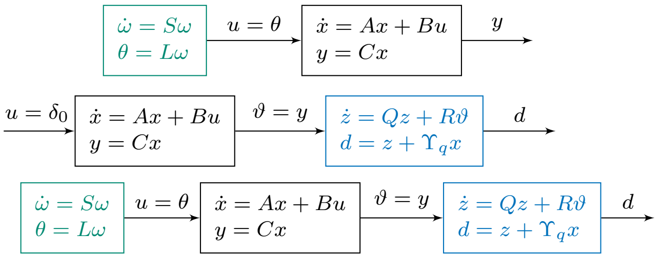

Fig. 1. Diagrammatic illustration of the direct (**top**), the swapped (**middle**), and the two-sided (**bottom**) interconnections.

This is explained next following the formulation in [Scarciotti and Astolfi, 2024]. Consider two sets of interpolation points

$$\mathcal{I}_1 = \{s_1, s_2, \ldots, s_{n_r}\} \subset \mathbb{C} \setminus \sigma(A)$$
$$\mathcal{I}_2 = \{s_{n_r+1}, s_{n_r+2}, \ldots, s_{2n_r}\} \subset \mathbb{C} \setminus \sigma(A)$$

with $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$, and consider two matrices $S \in \mathbb{R}^{n_r \times n_r}$ and $Q \in \mathbb{R}^{n_r \times n_r}$ such that $\sigma(S) = \mathcal{I}_1$ and $\sigma(Q) = \mathcal{I}_2$, and two vectors $L \in \mathbb{R}^{1 \times n_r}$ and $R \in \mathbb{R}^{n_r \times 1}$ such that the pair (S, L) is observable and the pair (Q, R) is controllable. One can establish [2] that there exist invertible matrices T and $\widetilde{T}$ such that $C\Pi_s = [\,\eta_0(s_1) \ \ldots \ \eta_0(s_{n_r})\,]\,T$ and $\Upsilon_q B = \widetilde{T}\,[\,\eta_0(s_{n_r+1}) \ \ldots \ \eta_0(s_{2n_r})\,]^\top$, where $\Pi \in \mathbb{R}^{n \times n_r}$ and $\Upsilon \in \mathbb{R}^{n_r \times n}$ are the unique solutions of the Sylvester equations

$$\Pi_s S = A\Pi_s + BL, \tag{17a}$$
$$Q\Upsilon_q = \Upsilon_q A + RC. \tag{17b}$$

Since moments are coordinate invariant, [Astolfi, 2010] noted that the quantities $C\Pi_s$ and $\Upsilon_q B$ can be viewed as *equivalent definitions* of moments. In particular, a reduced-order model that preserves $C\Pi_s$ or $\Upsilon_q B$ is a reduced-order model by moment matching at $\mathcal{I}_1$ or at $\mathcal{I}_2$.

A family of reduced-order models that match the moments $C\Pi_s$ of (15) at $\mathcal{I}_1$ is given by

$$\dot x_r = (S - B_r L)x_r + B_r u, \qquad y_r = C\Pi_s x_r, \tag{18}$$

for any B_r such that $\sigma(S) \cap \sigma(S - B_r L) = \emptyset$. Similarly, a family of reduced-order models that match the moments $\Upsilon_q B$ of (15) at $\mathcal{I}_2$ is given by

$$\dot x_r = (Q - RC_r)x_r + \Upsilon_q B u, \qquad y_r = C_r x_r, \tag{19}$$

for any R such that $\sigma(Q) \cap \sigma(Q - RC_r) = \emptyset$. Finally, for the family (18) ((19), respectively), we can select B_r (C_r, respectively) such that also the moments at $\mathcal{I}_2$ ($\mathcal{I}_1$, respectively) are matched. This is achieved for the family (18) ((19), respectively) with the selection

$$B_r = (\Upsilon_q \Pi_s)^{-1}\Upsilon_q B, \ \ (C_r = C\Pi_s(\Upsilon_q\Pi_s)^{-1}, \text{ resp.}) \tag{20}$$

Furthermore, [Astolfi, 2010, Scarciotti and Astolfi, 2024] observed that, under additional assumptions, the moments are also in a one-to-one relation with the steady-state responses (provided they exist) of interconnections between the system and certain "signal generators". To see this, first consider the signal generator

$$\dot w = Sw, \qquad \theta = Lw, \tag{21}$$

with $w(t) \in \mathbb{R}^{n_r}$ and $\theta(t) \in \mathbb{R}$, and the interconnection (with $u = \theta$) between this generator and system (15), namely

$$\dot w = Sw, \qquad \dot x = Ax + BLw, \qquad y = Cx. \tag{22}$$

[2] This result holds also for higher-order moments.

This interconnection (see Fig. 1, top diagram) is called the *direct interconnection*. One can easily show that the output of this interconnection is

$$y(t) = C\Pi_s w(t) + Ce^{At}(x(0) - \Pi_s w(0)).$$

If w is a bounded signal and A is Hurwitz, one has that the moments at $\mathcal{I}_1$ are in a one-to-one relation with the steady-state output response $y_{ss} = C\Pi_s w$ of (22).

A similar connection can be established also with the Sylvester equation (17b). Consider the filter

$$\dot z = Qz + R\vartheta, \qquad d = z + \Upsilon_q x, \tag{23}$$

with $z(t) \in \mathbb{R}^{n_r}$, $\vartheta(t) \in \mathbb{R}$, and $d(t) \in \mathbb{R}^{n_r}$, and the interconnection (with $\vartheta = y$) between this filter and system (15), namely

$$\dot x = Ax + Bu, \quad \dot z = Qz + RCx, \quad d = z + \Upsilon_q x, \tag{24}$$

with $u = \delta_0$, where δ_0 indicates the Dirac-delta generalised function. This interconnection (see Fig. 1, middle diagram) is called the *swapped interconnection*. If A is Hurwitz, $x(0) = 0$ and $z(0) = 0$, one can show that

$$z_{ss}(t) = d(t) = e^{Qt}\Upsilon_q B, \tag{25}$$

that is, the moments at $\mathcal{I}_2$ are in a one-to-one relation with the steady-state response z_{ss} in (24).

Finally, consider the signal generator (21), the filter (23), and the interconnection between these and system (15), yielding the system

$$\begin{aligned} \dot w &= Sw, & \dot x &= Ax + BLw, \\ \dot z &= Qz + RCx, & d &= z + \Upsilon_q x. \end{aligned} \tag{26}$$

This interconnection (see Fig. 1, bottom diagram) is called the *two-sided interconnection*. One can easily show that the signals in this interconnection satisfy the equation

$$d(t) - z(t) = \Upsilon_q \Pi_s w(t) + \Upsilon_q e^{At}(x(0) - \Pi_s w(0)), \tag{27}$$

and that if w is a bounded signal and A is Hurwitz

$$d_{ss}(t) - z_{ss}(t) = \Upsilon_q \Pi_s w(t). \tag{28}$$

This third key matrix $\Upsilon_q\Pi_s$ is defined by the Sylvester equation

$$Q\Upsilon_q\Pi_s - \Upsilon_q\Pi_s S = RC\Pi_s - \Upsilon_q BL, \tag{29}$$

and also characterizes a steady-state response.

The significance of this series of observations is that the matrices $C\Pi_s$, $\Upsilon_q B$, and $\Upsilon_q\Pi_s$, which can be used to construct reduced-order models at $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_1 \cup \mathcal{I}_2$, respectively, are all uniquely characterized by steady-state signals in the interconnections in Fig. 1. Since these steady states exist even when the linear elements are replaced by nonlinear elements, this interpretation allows for a natural extension of the moment matching problem to nonlinear systems.

2.5 Further Comments on Sylvester Equations

We have seen that the Sylvester equations (1), (2) play crucial roles in several different control-theoretic contexts. As such, computing the solutions and understanding their properties [de Souza and Bhattacharyya, 1981] is of fundamental importance. Several solution approaches are discussed in [Antoulas, 2005, Chapter 6]. Numerical methods include Kronecker vectorization, eigenvector/characteristic polynomial analysis, the matrix sign function, and the Bartels-Stewart method. In contrast, analytic solution methods enable further system-theoretic insights, and pave the way for nonlinear extensions.

With the notation of equation (17a) and renaming Π_s simply as Π, consider the case in which the matrix A is Hurwitz and the spectrum of S has the simple form $\sigma(S) = \{\omega_0, \pm j\omega_1, \ldots, \pm j\omega_\ell\}$ with $0 = \omega_0 < \omega_1 < \cdots < \omega_\ell$. Manipulation of (17a) and routine complex analysis yields the contour integral expression

$$\Pi = \frac{1}{2\pi j}\int_\gamma (sI - A)^{-1} BL(sI - S)^{-1}\,\mathrm{d}s,$$

where γ is any Cauchy contour enclosing the eigenvalues of S and excluding the eigenvalues of A, and thus

$$C\Pi = \frac{1}{2\pi j}\int_\gamma W(s)L(sI - S)^{-1}\,\mathrm{d}s.$$

Application of the residue theorem then establishes that

$$C\Pi = \eta_0(0)LX_0 + \mathrm{Re}\sum\nolimits_{k=0}^{\ell} \eta_0(j\omega_k)LX_k,$$

where X_0, X_k depend only on the eigendecomposition of S. This formula reveals that in some contexts, the quantity $C\Pi$ does not depend densely on the triplet (A, B, C), but only on the moments η_k (in this case, the 0-moments) at the eigenvalues of S; similar arguments can be applied to (17b) and the quantity ΥB. This opens the door to data-driven design approaches based on frequency response data, as exploited for stabilization in, *e.g.*, [Paunonen, 2016, Theorem 11, 13], or [Chen and Simpson-Porco, 2023] and for model order reduction in *e.g.* [Mayo and Antoulas, 2007], [Scarciotti and Astolfi, 2017a] and [Simard, 2023]. Alternatively, under the same assumptions the solution to (17a) can be expressed as

$$\Pi = \int_{-\infty}^{0} e^{-At} BLe^{St}\,\mathrm{d}t. \tag{30}$$

The expression (30) can be interpreted as an integral involving solutions of the underlying differential equations, paving the way for nonlinear extensions.

2.6 *Infinite-Dimensional Linear Systems*

The use of Sylvester equation (1) has been widely studied also in the case in which $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and $\mathbf{\Pi}$ are infinite-dimensional linear operators, see, *e.g.* [Phóng, 1991]. Similar sufficient conditions (disjoint spectra of $\mathbf{A}, \mathbf{B}$) extends the linear finite-dimensional case and similar applications in output regulation, stabilization and model reduction have been proposed. For instance, in the context of output regulation, we refer to [Paunonen et al., 2008, Paunonen, 2016, Vanspranghe and Brivadis, 2023] for linear abstract systems and [Astolfi et al., 2021] for the case of repetitive control. In the context of stabilization, the Sylvester equation has been used in [Natarajan, 2021] and in the context of forwarding approach in [Marx et al., 2022, 2021, Vanspranghe and Brivadis, 2023], leading to very similar results. Concerning the model reduction theory, we refer to [Ionescu and Iftime, 2012] for linear systems and [Scarciotti and Astolfi, 2016] for time-delay systems. Due to space limitations, we omit further discussion of infinite-dimensional linear systems and shift our focus to the nonlinear (finite-dimensional) extension in the next section.

3. NONLINEAR SYSTEMS

In the context of nonlinear systems the Sylvester equations (1) and (2) generalize into nonlinear partial differential equations (PDEs). Let us consider first the case of a cascaded nonlinear system of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(w, x), \end{aligned} \tag{31}$$

in which $x(t) \in \mathbb{R}^{n_x}$, $w(t) \in \mathbb{R}^{n_w}$, s and f are C^1 functions, and the w-subsystem is Poisson stable. Steady-state solutions to (31) can be described using ω-limit sets, and are characterized (on compact sets) by the existence of a compact set $\mathcal{W} \subset \mathbb{R}^{n_w}$ and an upper semicontinuous set-valued map $\pi : \mathcal{W} \rightrightarrows \mathbb{R}^{n_x}$ such that

$$\mathcal{A} = \{(x, w) \in \mathbb{R}^{n_x} \times \mathcal{W} : x \in \pi(w)\}$$

is compact and satisfies

$$\begin{pmatrix} f(w, x) \\ s(w) \end{pmatrix} \in T_{\mathcal{A}}(x, w),$$

where $T_{\mathcal{A}}(x, w)$ is the contingent cone to $\mathcal{A}$ at (x, w), see [Aubin, 1991]. The latter condition expresses *invariance* of the set $\mathcal{A}$. We refer to [Byrnes and Isidori, 2003, Isidori and Byrnes, 2008, Petit et al., 2018, Bin et al., 2023] for precise definitions, properties and statements on ω-limit sets of solutions to (31).

To obtain a more recognizable generalization of (1), additional assumptions are required. In particular, if π is a *single-valued* C^1 map, then invariance is equivalent to π satisfying the *invariance equation*

$$\mathcal{L}_s\pi(w) = f(w, \pi(w)). \tag{32}$$

To see that (32) genearlizes (1), note that if s and f in the above are linear vector fields, that is, if $s(w) := \mathbf{B}w$ and $f(w, x) := \mathbf{A}x - \mathbf{C}w$, then we recover $\pi(w) = \mathbf{\Pi}w$ where $\mathbf{\Pi}$ satisfies the Sylvester equation (1). An intermediate case of interest occurs when s is nonlinear and f takes the quasi-linear form

$$f(w, x) := Ax + Bl(w),$$

where A, B are matrices with A Hurwitz and $l : \mathbb{R}^{n_w} \to \mathbb{R}^{n_u}$ is an integrable nonlinear map. In this case, the unique solution to the invariance equation (32) can be explicitly expressed as

$$\pi(w) = \int_{-\infty}^{0} e^{-A\tau} Bl(\phi_w(\tau, w))\,\mathrm{d}\tau, \tag{33}$$

where $\phi_w(t, w^\circ)$ denotes the solution to $\dot{w} = s(w)$ at time t starting from w°, see, *e.g.* [Isidori and Byrnes, 2008, Section 4]. In the fully linear case in which $s(w) = Sw$ and $l(w) = Lw$, (33) reduces precisely to (30).

3.1 *Output Regulation*

As discussed in the linear case of Section 2, invariance equations such as (32) play a a crucial role in many control-theoretic problems. In the context of nonlinear output regulation, one considers systems of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(w, x, u) \\ e &= h(w, x), \end{aligned}$$

generalizing (3), and the Byrnes-Isidori regulator equations [Byrnes and Isidori, 2003], given by

$$\mathcal{L}_s \pi(w) = f(w, \pi(w), \psi(w))$$
$$0 = h(w, \pi(w))$$

extend *de facto* the linear regulator equations (5). The design of nonlinear internal model regulators leverages observer theory, and relies again on the use of invariance-like equations, see, *e.g.* [Byrnes and Isidori, 2003, 2004, Marconi et al., 2007, Isidori and Byrnes, 2008, Bin et al., 2023]. While some nonlinear versions of the non-resonance condition (6) have appeared in the literature [Marconi et al., 2004, Wang et al., 2020], there appears to be no corresponding generalization of Theorem 1 to the nonlinear case.

3.2 Cascade Stabilization

For stabilization of nonlinear cascade systems, the *forwarding* approach described in Section 2.2 was first developed in the nonlinear context, see [Mazenc and Praly, 1996, Astolfi and Praly, 2017, Giaccagli et al., 2022]. Consider first the simple cascaded nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ \dot{z} &= h(x) \end{aligned} \tag{34}$$

in which the z-subsystem has no internal dynamics. The construction of a stabilizing feedback control law is based on the solution to the invariance equation

$$\mathcal{L}_f \upsilon(x) = h(x),$$

the unique solution of which is given by

$$\upsilon(x) = \int_0^\infty h(\phi_x(s, x))\, \mathrm{d}s, \tag{35}$$

where $\phi_x(t, x^\circ)$ denotes the solution to $\dot{x} = f(x)$ at time t starting from x°, see [Mazenc and Praly, 1996, Lemma IV.2]. Such a solution is indeed well-defined whenever the origin of the autonomous system $\dot{x} = f(x)$ is globally asympotically stable and locally exponentially stable. With $\upsilon(x)$ in hand, a stabilizing feedback is then $u = \ell(x)\mathcal{L}_g \upsilon(x) z$, where $\ell(x)$ is an additional design parameter; see [Astolfi and Praly, 2017] for further details.

In cases where $\upsilon(x)$ cannot be exactly computed, stabilizing feedbacks have also been developed based on first-order approximation of $\upsilon(x)$ at the origin. Specifically, the local approximation of the invariance equation (35) at the origin reduces to the simple Sylvester-like equation

$$\Upsilon A = C, \qquad \text{where} \quad A = \frac{\partial f}{\partial x}(0), \quad C = \frac{\partial h}{\partial x}(0), \tag{36}$$

with $\Upsilon = \frac{\partial \upsilon}{\partial x}(0)$; see [Mazenc and Praly, 1996, Astolfi and Praly, 2017] for further discussion. Thus, aspects of the linear theory may be leveraged for design in the nonlinear case. Forwarding has been also extended to more complex cascade forms, such as

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ \dot{z} &= \Phi z + Rh(x) \end{aligned}$$

in which Φ is a matrix with simple imaginary eigenvalues, see, *e.g.*, [Astolfi et al., 2022] and references therein. Nonlinear forwarding design based on dual invariance equations does not appear to have been explored.

3.3 Observer Design

In the context of observer design, the Luenberger approach of Section 2.3 has been generalized as the so-called KKL (Kazantis-Kravaris-Luenberger) observer, see, *e.g.*, [Andrieu and Praly, 2006, Brivadis et al., 2023]. Given the nonlinear system

$$\dot{x} = f(x), \quad y = h(x), \tag{37}$$

with $x(t) \in \mathbb{R}^{n_x}$ and $y(t) \in \mathbb{R}$, the idea is to obtain an injective state transformation $z = \upsilon_o(x)$ which evolves according to $\dot{z} = Fz + Gy$ with F Hurwitz, for which again a trivial observer is

$$\dot{\hat{z}} = F\hat{z} + Gy, \tag{38}$$

with $\hat{z}(t) \in \mathbb{R}^{n_z}$. This is achieved if $\upsilon_o : \mathbb{R}^{n_x} \to \mathbb{R}^{n_z}$ is a C^1 mapping satisfying the invariance equation

$$\mathcal{L}_f \upsilon_o(x) = F\upsilon_o(x) + Gh(x)\,. \tag{39}$$

Under so-called *backward indistinguishability* of solutions of the system (37), it can be shown that if $n_z \geq 2n_x + 1$, then for almost any controllable pair (F, G) with F being Hurwitz, the invariance equation (39) admits an *injective* solution $\upsilon_o(x)$, which can be expressed as

$$\upsilon_o(x) = \int_{-\infty}^0 e^{-Fs} G h(\phi_x(s, x))\, \mathrm{d}s,$$

where again $\phi_x(t, x^\circ)$ denotes the solution to $\dot{x} = f(x)$ at time t starting from x°. Thus, there exists a left-inverse ψ of υ_o, allowing one to recover a state estimate via $\hat{x} = \psi(\hat{z})$ with $\hat{z}$ being the solution to (38). One can then show that $\hat{x}$ converges asymptotically to the solution x of the observed plant (37). We refer to [Andrieu and Praly, 2006, Brivadis et al., 2023] for more details. It appears that KKL observers based on dual invariance equations have not been examined in the literature. Finally, we remark that the use of these type of invariance equations has also been employed for studying the asymptotic sensitivity properties to measurement noise of high-gain observers, see, *e.g.* [Astolfi et al., 2016].

3.4 Model Order Reduction

The generalization of moment-matching model order reduction methods to nonlinear systems is based on the system interconnection interpretation in Figure 1. Consider a nonlinear, minimal [3], single-input, single-output, system described by the equation

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x), \end{aligned} \tag{40}$$

where f, g and h are smooth mappings such that $f(0) = 0$, $g(0) = 0$, $h(0) = 0$. Next, define the signal generator

$$\dot{w} = s(w), \qquad \theta = l(w), \tag{41}$$

with s and l smooth mappings such that $s(0) = 0$ and $l(0) = 0$, and the interconnection (with $u = \theta$) between this generator and system (40), namely

$$\begin{aligned} \dot{w} &= s(w), \\ \dot{x} &= f(x) + g(x)l(w), \\ y &= h(x). \end{aligned} \tag{42}$$

Analogous to the linear interconnection described in (22), we can define the (direct) moment of the system (40) at (s, l) as the function $h \circ \pi_s$, where π is the unique solution of the invariance equation

$$\mathcal{L}_s \pi_s(w) = f(\pi_s(w)) + g(\pi_s(w))l(w), \tag{43}$$

which is a generalization of (17a). The signal generator (41) captures the requirement that one is interested in

[3] See [Scarciotti and Astolfi, 2017b, Definition 2.12].

studying the behaviour of system (40) only in specific circumstances, in particular, that a reduced-order model by moment matching is a model that matches the steady-state output response of the system for the same class of inputs of interest.

It is also possible to develop a dual theory of moment matching in the nonlinear case [Ionescu and Astolfi, 2016, Scarciotti and Astolfi, 2024]. Consider the nonlinear filter

$$\dot{z} = q(z) + r(z)\vartheta, \tag{44}$$

with q and r smooth mappings such that $q(0) = 0$ and $r(0) = 0$, and the interconnection (with $\vartheta = y$) between this filter and system (40), namely

$$\dot{x} = f(x) + g(x)u, \quad \dot{z} = q(z) + r(z)h(x). \tag{45}$$

Analogous to the swapped linear interconnection in (24), we can define the (swapped) moment of system (40) at (q, r) as $\mathcal{L}_g \upsilon_q$, where υ_q is the unique solution of the invariance equation

$$\mathcal{L}_f \upsilon_q(x) = -r(-\upsilon_q(x))h(x) - q(-\upsilon_q(x)), \tag{46}$$

which is a generalization of (17b).

Consider now a candidate reduced-order model, described by

$$\dot{x}_r = f_r(x_r) + g_r(x_r)u, \quad y_r = h_r(x_r). \tag{47}$$

where $x_r(t) \in \mathbb{R}^{n_r}$, $n_r < n_x$, $y_r(t) \in \mathbb{R}$, $f_r(0) = 0$, $g_r(0) = 0$, and $h_r(0) = 0$. The system (47) matches the moment $h \circ \pi_s$ at (s, l), if f_r, g_r and h_r satisfy

$$\mathcal{L}_s p(w) = f_r(p(w)) + g_r(p(w))l(w) \tag{48}$$

and

$$h_r(p(w)) = h(\pi_s(w)) \tag{49}$$

for all w and some mapping p. These equations are satisfied by the selection $f_r(x_r) = s(x_r) - g_r(x_r)l(x_r)$, $h_r(x_r) = h(\pi_s(x_r))$, where g_r is free [4], for the mapping $p(w) = w$. Thus, the model

$$\begin{aligned} \dot{x}_r &= s(x_r) - g_r(x_r)l(x_r) + g_r(x_r)u, \\ y_r &= h(\pi_s(x_r)), \end{aligned} \tag{50}$$

is a reduced-order model that matches the moment of system (40) at (s, l). In the fully linear case, this family reduces precisely to (18).

Similarly, regarding the two-sided interconnection, the model (50) matches also the moment $\mathcal{L}_g \upsilon_q$ at (q, r) if g_r is such that

$$\mathcal{L}_{f_r} \chi(x_r) = -q(-\chi(x_r)) - r(-\chi(x_r))h_r(x_r) \tag{51}$$

and

$$[\mathcal{L}_{g_r} \chi(x_r)]_{x_r = w} = [\mathcal{L}_g \upsilon_q(x)]_{x = \pi_s(w)}, \tag{52}$$

for all w and some mapping χ. This is achieved with the selection

$$g_r(x_r) = \left[\left(\frac{\partial v}{\partial x} \frac{\partial \pi}{\partial \xi} \right)^{-1} \mathcal{L}_g \upsilon_q(x) \right]_{x = \pi_s(x_r)} \tag{53}$$

which gives $\chi(x_r) = \upsilon_q(\pi_s(x_r))$. Thus, system (50) with (53) is a reduced-order model of system (40) matching the moments at (s, l) and (q, r), simultaneously. An analogous nonlinear enhancement of the family (19) can be obtained. This is omitted for reasons of space.

[4] As long as the solution of (48) is unique.

4. CONCLUSIONS AND PERSPECTIVES

We have provided here a unified perspective on the use of Sylvester equations, invariance equations, and associated dual equations in a selection of linear and nonlinear control problems. Perhaps our key observation is that across a variety of problem settings, design procedures based on both direct and dual Sylvester-like equations are natural, but that dual designs procedures have not been explored in a number of settings. In particular, future research will seek to provide a nonlinear counterpart to Theorem 1, and to examine the feasibility of dual design formulations for both linear and nonlinear observer design, linear and nonlinear disturbance observation, and nonlinear cascade stabilization.

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