Distributionally Robust Stochastic Data-Driven Predictive Control with Optimized Feedback Gain

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***Abstract*— We consider the problem of direct data-driven predictive control for unknown stochastic linear time-invariant (LTI) systems with partial state observation. Building upon our previous research on data-driven stochastic control, this paper (i) relaxes the assumption of Gaussian process and measurement noise, and (ii) enables optimization of the gain matrix within the affine feedback policy. Output safety constraints are modelled using conditional value-at-risk, and enforced in a distributionally robust sense. Under idealized assumptions, we prove that our proposed data-driven control method yields control inputs identical to those produced by an equivalent model-based stochastic predictive controller. A simulation study illustrates the enhanced performance of our approach over previous designs.**

I. Introduction

Model predictive control (MPC) is a widely used technique for multivariate control [1], adept at handling constraints on inputs, states, and outputs while optimizing complex performance objectives. Constraints typically model actuator limits, or encode safety constraints in safety-critical applications, and MPC employs a system model to predict how inputs influence state evolution. Both deterministic and stochastic frameworks have been developed to account for plant uncertainty in MPC. While *Robust MPC* [2] approaches model uncertainty in a worst-case deterministic sense, work on *Stochastic MPC (SMPC)* [3] has focused on describing model uncertainty probabilistically. SMPC methods optimize over feedback control policies rather than control actions, resulting in performance benefits when compared to the naïve use of deterministic MPC [4], and SMPC frameworks can accommodate probabilistic and risk-aware constraints.

The system model required by MPC (and SMPC) must be obtained either from first-principles modelling or from identification, making MPC an *indirect* design method, since one goes from data to a controller through an intermediate modelling step [5]. In contrast, *direct* methods, or data-driven methods, seek to compute controllers directly from input-output data. Data-driven methods show promise for complex or difficult-to-model systems [6]. Early work on data-driven methods did not adequately account for constraints on inputs and outputs (see examples in [6]), leading to the development of *Data-Driven Predictive Control (DDPC)* as data-driven control methods addressing such constraints. Two well-known DDPC methods are Data-Enabled Predictive Control (DeePC) [7]–[9] and Subspace Predictive Control (SPC) [10], both of which have been applied in multiple experiments [11]–[14]. On the theoretical side, for *deterministic* LTI systems, both DeePC and SPC produce equivalent control actions to their model-based MPC counterparts [7], [10].

Real-world systems often deviate from idealized deterministic LTI models, exhibiting stochastic and non-linear behavior, with noise-corrupted data. To address these challenges, data-driven methods must account for noisy data and measurements. For instance, in SPC applications, the required predictor matrices are often computed using denoising techniques such as prediction error methods [13], [14]. Variants of DeePC were also developed for stochastic systems, including norm-based regularized DeePC [7], [8] and distributionally robust DeePC [8], [9]. Unlike in the deterministic case however, these stochastic adaptations of DeePC and SPC lack theoretical equivalence to any model-based MPC method.

Recognizing this gap, some recent advancements in DDPC have aimed to establish equivalence with MPC methods for stochastic systems. The work in [15], [16] proposed a data-driven control framework for stochastic systems with full state observation, and their method performs equivalently to full-observation SMPC if stochastic signals can be exactly represented by their Polynomial Chaos Expansion. In [17], a stochastic data-driven control method was developed by estimating the innovation sequence, yielding equivalent control performance to deterministic MPC when the innovation data is exact. This paper builds in particular on our previous work [18], where we proposed a data-driven control method for stochastic systems with partial state observation, and established that the method has equivalent control performance to partial-observation SMPC when offline data is noise-free.

Contribution: This paper contributes towards the continued development of high-performance data-driven predictive control methods for stochastic systems. Specifically, in this paper we develop a data-driven stochastic predictive control strategy utilizing distributionally robust conditional value-at-risk constraints, providing an improved safety constraint description when compared to our prior work in [18], and providing robustness against non-Gaussian (i.e., possibly heavy-tailed) process and measurement noise. Additionally, in contrast with the fixed feedback gain in [18], we consider control policies where feedback gains are decision variables in the optimization, giving a more flexible parameterization of control policies. As theoretical support for the approach, under technical conditions, we establish equivalence between our proposed design and a corresponding SMPC. Finally, a

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simulation case study compares and contrasts our design with other recent stochastic and data-driven control strategies.

Notation: Let $M^\dagger$ be the pseudo-inverse of a matrix M. Let $\otimes$ denote the Kronecker product. Let $\mathbb{S}^q_+$ and $\mathbb{S}^q_{++}$ be the sets of $q \times q$ positive semi-definite and positive definite matrices, respectively. Let $\mathrm{col}(M_1, \dots, M_k)$ denote the column concatenation, and $\mathrm{Diag}(M_1, \dots, M_k)$ the block diagonal concatenation, of matrices/vectors $M_1, \dots, M_k$. Let $\mathbb{Z}_{[a,b]} := [a, b] \cap \mathbb{Z}$ denote a set of consecutive integers from a to b; let $\mathbb{Z}_{[a,b)} := \mathbb{Z}_{[a,b-1]}$. For a $\mathbb{R}^q$-valued discrete-time signal z_t with integer index t, let $z_{[t_1,t_2]}$ denote either a sequence $\{z_t\}_{t=t_1}^{t_2}$ or a concatenated vector $\mathrm{col}(z_{t_1}, \dots, z_{t_2}) \in \mathbb{R}^{q(t_2-t_1+1)}$ where the usage is clear from the context; similarly, let $z_{[t_1,t_2)} := z_{[t_1,t_2-1]}$. A matrix sequence $\{M_t\}_{t=t_1}^{t_2}$ and a function sequence $\{\pi_t(\cdot)\}_{t=t_1}^{t_2}$ are denoted by $M_{[t_1,t_2]}$ and $\pi_{[t_1,t_2]}$ respectively.

II. Problem Setup

We consider a stochastic linear time-invariant (LTI) system

$$x_{t+1} = Ax_t + Bu_t + w_t \tag{1a}$$

$$y_t = Cx_t + v_t \tag{1b}$$

with input $u_t \in \mathbb{R}^m$, state $x_t \in \mathbb{R}^n$, output $y_t \in \mathbb{R}^p$, process noise $w_t \in \mathbb{R}^n$, and measurement noise $v_t \in \mathbb{R}^p$. The system (A, B, C) is assumed to be a minimal realization, but the matrices themselves are *unknown* and the state x_t is *unmeasured*; we have access only to the input u_t and output y_t in (1). The probability distributions of w_t and v_t are *unknown*, but we assume that w_t and v_t have zero mean and zero auto-correlation (white noise), are uncorrelated, and their variances $\Sigma^{\mathsf{w}} \in \mathbb{S}^n_+$ and $\Sigma^{\mathsf{v}} \in \mathbb{S}^p_{++}$ are known. The initial state x_0 has known mean μ^{x}_0 and variance Σ^{x}_0 and is uncorrelated with the noise. We record these conditions as

$$\mathbb{E}\Big[\begin{bmatrix} w_t \\ v_t \end{bmatrix}\Big] = 0, \quad \mathbb{E}\Big[\begin{bmatrix} w_t \\ v_t \end{bmatrix}\begin{bmatrix} w_s \\ v_s \end{bmatrix}^{\mathsf{T}}\Big] = \begin{bmatrix} \delta_{ts}\Sigma^{\mathsf{w}} & 0 \\ 0 & \delta_{ts}\Sigma^{\mathsf{v}} \end{bmatrix}, \tag{2}$$

$$\mathbb{E}[x_0] = \mu^{\mathsf{x}}_0, \quad \mathrm{Var}[x_0] = \Sigma^{\mathsf{x}}_0, \quad \mathbb{E}\Big[x_0 \begin{bmatrix} w_t \\ v_t \end{bmatrix}^{\mathsf{T}}\Big] = 0, \tag{3}$$

where δ_{ts} denotes the Kronecker delta.

In a reference tracking control problem for (1), the objective is for the output y_t to follow a specified reference signal $r_t \in \mathbb{R}^p$. The trade-off between tracking error and control effort may be encoded in an instantaneous cost

$$J(u_t, y_t) := \|y_t - r_t\|^2_Q + \|u_t\|^2_R \tag{4}$$

to be minimized over a time horizon, with user-selected parameters $Q \in \mathbb{S}^p_+$ and $R \in \mathbb{S}^m_{++}$. This tracking should be achieved subject to constraints on the inputs and outputs. We consider here polytopic constraints, which in a deterministic setting would take the form $E \begin{bmatrix} u_t \\ y_t \end{bmatrix} \leq f$ for all $t \in \mathbb{Z}_{\geq 0}$, and for some fixed matrix $E \in \mathbb{R}^{q \times (m+p)}$ and vector $f \in \mathbb{R}^q$. We can equivalently express these constraints as the single constraint $h(u_t, y_t) \leq 0$, where

$$h(u_t, y_t) := \max_{i \in \{1,\dots,q\}} e_i^{\mathsf{T}} \begin{bmatrix} u_t \\ y_t \end{bmatrix} - f_i, \tag{5}$$

with $e_i \in \mathbb{R}^{m+p}$ the transposed i-th row of E and $f_i \in \mathbb{R}$ the i-th entry of f. For the system (1) which is subject to (possibly unbounded) stochastic disturbances, the deterministic constraint $h(u_t, y_t) \leq 0$ must be relaxed. Beyond a traditional chance constraint $\mathbb{P}[h(u_t, y_t) \leq 0] \geq 1 - \alpha$ with a violation probability $\alpha \in (0, 1)$, a *conditional value-at-risk (CVaR)* constraint is more conservative; the CVaR at level α of $h(u_t, y_t)$ is the expected value of $h(u_t, y_t)$ in the $\alpha\cdot$ 100% worst cases, and takes extreme violations into account. As the noise distributions are unknown, we must further guarantee satisfaction of the CVaR constraint for all possible distributions under consideration. Let $\mathbb{D}$ denote a joint distribution of all random variables in (1) satisfying (2) and (3), and let the *ambiguity set* $\mathcal{D}$ be the set of all such distributions. The *distributionally robust CVaR (DR-CVaR)* constraint [19], [20] is then

$$\sup_{\mathbb{D} \in \mathcal{D}} \mathbb{D}\text{-}\mathrm{CVaR}_\alpha\big[h(u_t, y_t)\big] \leq 0, \tag{6}$$

where $\mathbb{D}\text{-}\mathrm{CVaR}_\alpha[z]$ is the CVaR value of a random variable $z \in \mathbb{R}$ at level α, given distribution $\mathbb{D}$.

If the system matrices A, B, C were known, this constrained tracking control problem subject to (6) can be approached using SMPC, as described in Section III-A. Our objective is to develop a data-driven control method that produces equivalent control inputs as produced by SMPC.

III. Stochastic Model-Based and Data-Driven Predictive Control

We introduce a model-based SMPC framework in Section III-A and propose a data-driven control method in Section III-B, with their theoretical equivalence in Section III-C.

A. A framework of Stochastic Model Predictive Control

We focus here on output-feedback SMPC [21]–[23], which is typically approached by enforcing a separation principle which separates the steps of state estimation and feedback control. The formulation here broadly follows our prior work [18], but we now consider a DR-CVaR constraint in place of chance constraints, and we will allow optimization over the feedback gain in the affine policy. This SMPC scheme merges the established works on DR constrained control [19], [20] and output-error feedback [24], while the combined framework is part of our contribution. The SMPC method integrates state estimation, affine feedback-policy parameterization, and reformulation of DR-CVaR constraint.

1) State Estimation: SMPC follows a receding-horizon strategy and makes decisions for N upcoming steps at each *control step*. At control step $t = k$, we begin with prior information of the mean and variance of the state x_k, namely

$$\mathbb{E}[x_k] = \mu^{\mathsf{x}}_k, \qquad \mathrm{Var}[x_k] = \Sigma^{\mathsf{x}}_k, \tag{7}$$

which is computed from a Kalman filter, to be described next; at the initial time $k = 0$, μ^{x}_0 and Σ^{x}_0 are given parameters as in (3). Estimates of the future states over the desired horizon are computed through the Kalman filter,

$$\hat{x}_t := \hat{x}^-_t + L_t(y_t - C\hat{x}^-_t), \qquad t \in \mathbb{Z}_{[k,k+N)} \tag{8a}$$

$$\hat{x}^-_{t+1} := A\hat{x}_t + Bu_t, \qquad t \in \mathbb{Z}_{[k,k+N)} \tag{8b}$$

$$\hat{x}^-_k := \mu^{\mathsf{x}}_k \tag{8c}$$

where $\hat{x}_t$ and $\hat{x}_t^-$ denote the posterior and prior estimates of x_t, respectively, and the Kalman gain $L_t \in \mathbb{R}^{n\times p}$ in (8a) is obtained via the recursion

$$L_t := P_t^- C^\mathsf{T}(CP_t^- C^\mathsf{T} + \Sigma^\mathsf{v})^{-1}, \quad t \in \mathbb{Z}_{[k,k+N)} \tag{9a}$$
$$P_t := (I_n - L_t C)P_t^-, \quad t \in \mathbb{Z}_{[k,k+N)} \tag{9b}$$
$$P_{t+1}^- := AP_t A^\mathsf{T} + \Sigma^\mathsf{w}, \quad t \in \mathbb{Z}_{[k,k+N)} \tag{9c}$$
$$P_k^- := \Sigma_k^\mathsf{x}. \tag{9d}$$

While the noise here is potentially non-Gaussian, (8),(9) is the best affine state estimator in the mean-squared-error sense, regardless of the distributions of x_k, w_t, v_t once their means and variances are specified as in (2) and (7) [25, Sec. 3.1].

2) Feedback Control Policies: Stochastic feedback control requires the determination of (causal) feedback policies π_t which map the observation history into control actions. Our prior work [18] was based on an affine feedback policy $u_t = u_t^\mathsf{nom} + K(\hat{x}_t - x_t^\mathsf{nom})$ using the state estimate $\hat{x}_t$ from (8a), wherein the gain K was fixed, the *nominal input* u_t^nom was a decision variable, and the *nominal state* x_t^nom was obtained through simulation of the noise-free system

$$x_{t+1}^\mathsf{nom} := Ax_t^\mathsf{nom} + Bu_t^\mathsf{nom}, \quad t \in \mathbb{Z}_{[k,k+N)} \tag{10a}$$
$$y_t^\mathsf{nom} := Cx_t^\mathsf{nom}, \quad t \in \mathbb{Z}_{[k,k+N)} \tag{10b}$$
$$x_k^\mathsf{nom} := \mu_k^\mathsf{x} \tag{10c}$$

with *nominal output* y_t^nom. Here we investigate control policies where the feedback gain K is also a time-varying decision variable to be optimized. However, the naive parameterization

$$u_t \leftarrow u_t^\mathsf{nom} + K_t(\hat{x}_t - x_t^\mathsf{nom}) \tag{11}$$

leads to non-convex bilinear terms involving the decision variables u_t^nom and K_t. Here, we instead leverage an *output error feedback* control policy [24]

$$u_t \leftarrow \pi_t\big(\hat{x}_{[k,t]}^-\big) := u_t^\mathsf{nom} + \textstyle\sum_{s=k}^{t} M_t^s(y_s - C\hat{x}_s^-) \tag{12}$$

with decision variables u_t^nom and $M_t^s \in \mathbb{R}^{m\times p}$; note that $y_s - C\hat{x}_s^-$ is the innovation. The policy parameterization (12) contains within it the policy (11) as a special case: indeed, if $\{K_\tau\}_{\tau=s}^t$ is a sequence of gain matrices, then the selection

$$M_t^s \leftarrow K_t(A + BK_{t-1})(A + BK_{t-2}) \cdots (A + BK_s)L_s$$

reduces (12) to (11). Crucially, (12) also leads to a jointly convex optimization problem in the gains and nominal inputs.

With the state estimator (8) and control policy (12), both the input u_t and output y_t of (1) can be written as affine functions of the decision variables. Specifically, let

$$\eta_k := \mathrm{col}(x_k - \mu_k^\mathsf{x}, w_{[k,k+N)}, v_{[k,k+N)}) \in \mathbb{R}^{n_\eta},$$

which is a vector of uncorrelated zero-mean random variables, of dimension $n_\eta := n + nN + pN$, and of variance

$$\Sigma_k^\eta := \mathrm{Diag}(\Sigma_k^\mathsf{x},\, I_N \otimes \Sigma^\mathsf{w},\, I_N \otimes \Sigma^\mathsf{v}) \tag{13}$$

considering (2) and (7). Through direct calculation using (1), (8), and (12), one can now establish that

$$\begin{bmatrix} u_t \\ y_t \end{bmatrix} = \begin{bmatrix} u_t^\mathsf{nom} \\ y_t^\mathsf{nom} \end{bmatrix} + \Lambda_t\, \eta_k, \quad t \in \mathbb{Z}_{[k,k+N)}, \tag{14}$$

where y_t^nom is obtained by (10), and where $\Lambda_t \in \mathbb{R}^{(m+p)\times n_\eta}$ is dependent on the gains M_t^s through the relations

$$\Lambda_t := \begin{bmatrix} \Delta_{t-k}^\mathsf{U} \\ \Delta_{t-k}^\mathsf{Y} \end{bmatrix} \mathcal{M}\, \Delta_k^\mathsf{M} + \begin{bmatrix} 0_{m\times n_\eta} \\ \Delta_{t-k}^\mathsf{A} \end{bmatrix}, \quad t \in \mathbb{Z}_{[k,k+N)}, \tag{15}$$

where $\mathcal{M} \in \mathbb{R}^{mN\times pN}$ is a concatenation of M_t^s as

$$\mathcal{M} := \begin{bmatrix} M_k^k & & & \\ M_{k+1}^k & M_{k+1}^{k+1} & & \\ \vdots & \vdots & \ddots & \\ M_{k+N-1}^k & M_{k+N-1}^{k+1} & \cdots & M_{k+N-1}^{k+N-1} \end{bmatrix}, \tag{16}$$

and where $\Delta_{t-k}^\mathsf{U} \in \mathbb{R}^{m\times mN}$, $\Delta_{t-k}^\mathsf{Y} \in \mathbb{R}^{p\times mN}$, $\Delta_{t-k}^\mathsf{A} \in \mathbb{R}^{p\times n_\eta}$ and $\Delta_k^\mathsf{M} \in \mathbb{R}^{pN\times n_\eta}$ are independent of both decision variables u^nom and M_t^s, with expressions available in Appendix A.

3) Tractable Formulation of DR-CVaR Constraint: Given (14) and the mean and variance of η_k, the DR-CVaR constraint (6) can be equivalently written as a second-order cone (SOC) constraint in terms of the decision variables u^nom and M_t^s.

Lemma 1 (SOC Expression of DR-CVaR Constraint)**.** *With $h(u_t, y_t)$ as in (5), for $t \in \mathbb{Z}_{[k,k+N)}$, (6) holds if and only if*

$$2\sqrt{\tfrac{1-\alpha}{\alpha}}\, \Big\|\Sigma_k^{\eta\,\frac{1}{2}} \Lambda_t^\mathsf{T} e_i\Big\|_2 \le -e_i^\mathsf{T} \begin{bmatrix} u_t^\mathsf{nom} \\ y_t^\mathsf{nom} \end{bmatrix} + f_i,\ i \in \mathbb{Z}_{[1,q]}. \tag{17}$$

Proof. Substituting (14) into (5), $h(u_t, y_t)$ can be written as

$$h(u_t, y_t) = \max_{i\in\{1,\ldots,q\}} e_i^\mathsf{T}\Lambda_t\eta_k + e_i^\mathsf{T}\mathrm{col}(u_t^\mathsf{nom}, y_t^\mathsf{nom}) - f_i,$$

where the random variable η_k has zero mean and variance Σ_k^η. According to [20, Thm. 3.3], (6) holds if and only if there exist $\theta_t \in \mathbb{R}$ and $\Theta_t \in \mathbb{S}_+^{n_\eta+1}$ satisfying the LMIs

$$0 \ge \alpha\theta_t + \mathrm{Trace}\big[\Theta_t\, \mathrm{Diag}(\Sigma_k^\eta,\, 1)\big]$$
$$\Theta_t \succeq \begin{bmatrix} 0_{n_\eta\times n_\eta} & \Lambda_t^\mathsf{T} e_i \\ e_i^\mathsf{T}\Lambda_t & e_i^\mathsf{T}\mathrm{col}(u_t^\mathsf{nom}, y_t^\mathsf{nom}) - f_i - \theta_t \end{bmatrix},\ i \in \mathbb{Z}_{[1,q]}.$$

From [26, Thm. 1], these LMIs are feasible in (θ_t, Θ_t) if and only if (17) holds, which completes the proof. ■

4) Optimization Problem: The SMPC optimization problem at control step $t = k$ is formulated as follows, with an expected cost summing (4) over N future steps as

$$\begin{aligned} \underset{u^\mathsf{nom}, M_t^s}{\text{minimize}} \quad & \mathbb{E}\big[\textstyle\sum_{t=k}^{k+N-1} J(u_t, y_t)\big] \\ \text{subject to} \quad & (1), (2), (6), (12) \text{ for } t \in \mathbb{Z}_{[k,k+N)}, \\ & \text{and } (7), (8) \end{aligned} \tag{18}$$

The expected quadratic cost in (18) can be computed to be a deterministic quadratic function of u^nom and M_t^s. In particular, since $\mathbb{E}[\mathrm{col}(u_t, y_t)] = \mathrm{col}(u_t^\mathsf{nom}, y_t^\mathsf{nom})$ and $\mathrm{Var}[\mathrm{col}(u_t, y_t)] = \Lambda_t\Sigma_k^\eta\Lambda_t^\mathsf{T}$ via (14) with Σ_k^η in (13), the expected cost in (18) evaluates to

$$\textstyle\sum_{t=k}^{k+N-1} \Big[J(u_t^\mathsf{nom}, y_t^\mathsf{nom}) + \Big\| \big[\begin{smallmatrix} R & \\ & Q \end{smallmatrix}\big]^{\frac{1}{2}} \Lambda_t (\Sigma_k^\eta)^{\frac{1}{2}} \Big\|_\mathsf{F}^2\Big], \tag{19}$$

where we used that $\mathbb{E}\big[\|z\|_S^2\big] = \big\|\mathbb{E}[z]\big\|_S^2 + \big\|S^{\frac{1}{2}}\mathrm{Var}[z]^{\frac{1}{2}}\big\|_\mathsf{F}^2$ for any random vector z and fixed matrix $S \succeq 0$; $\|\cdot\|_\mathsf{F}$ denotes the Frobenius norm. Using (19) and the reformulation (17) of (6), the stochastic problem (18) is equivalent to the deterministic second-order cone problem (SOCP)

$$\begin{aligned} \underset{u^\mathsf{nom}, M_t^s}{\text{minimize}} \quad & (19) \\ \text{subject to} \quad & (15), (17) \text{ for } t \in \mathbb{Z}_{[k,k+N)}, \text{ and } (10) \end{aligned} \tag{20}$$

Problem (20) has a unique optimal solution whenever feasible, since (19) is jointly strongly convex[1] in u^{nom} and M_t^s.

5) Online SMPC Algorithm: The nominal inputs u^{nom} and the feedback gains M_t^s determined from (20) complete the parameterization of the control policies $\pi_{[k,k+N)}$ in (12). The upcoming N_c control inputs $u_{[k,k+N_\mathrm{c})}$ are decided by the first N_c policies $\pi_{[k,k+N_\mathrm{c})}$ respectively, with parameter $N_\mathrm{c} \in \mathbb{Z}_{[1,N]}$. Then, the next control step is set as $t = k + N_\mathrm{c}$. At the new control step, the initial condition (7) is iterated as the prior mean and prior variance of the state x_{k+N_c},

$$\mu^{\mathsf{x}}_{k+N_\mathrm{c}} = \hat{x}^{-}_{k+N_\mathrm{c}}, \quad \Sigma^{\mathsf{x}}_{k+N_\mathrm{c}} = P^{-}_{k+N_\mathrm{c}}, \tag{21}$$

obtained through the Kalman filter (8), (9). The entire SMPC control process is shown in Algorithm 1.

Algorithm 1 Distributionally Robust Optimized-Gain SMPC (DR/O-SMPC)

Input: horizon lengths N, N_c, system matrices A, B, C, noise variances $\Sigma^{\mathsf{w}}, \Sigma^{\mathsf{v}}$, initial-state mean μ^{x}_0 and variance Σ^{x}_0, cost matrices Q, R, constraint coefficients E and f, and CVaR level α.

Compute $\Delta^{\mathsf{U}}_{[0,N)}$, $\Delta^{\mathsf{Y}}_{[0,N)}$, $\Delta^{\mathsf{A}}_{[0,N)}$ through Appendix A.
Initialize the control step $k \leftarrow 0$ and the initial condition $\mu^{\mathsf{x}}_k \leftarrow \mu^{\mathsf{x}}_0$, $\Sigma^{\mathsf{x}}_k \leftarrow \Sigma^{\mathsf{x}}_0$.
while `true` **do**
  Compute Kalman gains $L_{[k,k+N)}$ via (9).
  Compute matrix Δ^{M}_k through Appendix A.
  Solve nominal inputs $u^{\mathsf{nom}}_{[k,k+N)}$ and feedback-gain matrices M_t^s from problem (20), and thus formulate control policies $\pi_{[k,k+N)}$ via (12).
  for t **from** k **to** $k + N_\mathrm{c} - 1$ **do**
    Compute $\hat{x}^{-}_t$ via (8).
    Measure y_t from the system (1).
    Input $u_t \leftarrow \pi_t(\hat{x}^{-}_{[k,t]})$ to the system (1).
  $\mu^{\mathsf{x}}_{k+N_\mathrm{c}} \leftarrow \hat{x}^{-}_{k+N_\mathrm{c}}$ and $\Sigma^{\mathsf{x}}_{k+N_\mathrm{c}} \leftarrow P^{-}_{k+N_\mathrm{c}}$ as in (21).
  Set $k \leftarrow k + N_\mathrm{c}$.

B. Stochastic Data-Driven Predictive Control (SDDPC)

We develop in this section a data-driven control method, whose performance will be shown in Section III-C to be equivalent to SMPC under certain tuning conditions. In the spirit of DeePC and SPC, our proposed control method consists of an offline process, where data is collected, and an online process, in which the collected data is used to make real-time control decisions.

1) Use of Offline Data: In data-driven control, sufficiently rich offline data must be collected to capture the internal dynamics of the system. We now describe how offline data is to be collected, and use the collected data to compute some quantities that are useful to formulate our control method in the rest of the section. We first develop results with noise-free data, then address the case of noisy data.

[1]Strong convexity in u^{nom} is clear from the first term; strong convexity in M_t^s can be shown by noting that a sub-matrix of $\mathrm{col}(\mathcal{J}_k, \dots, \mathcal{J}_{k+N-1})$ with $\mathcal{J}_t := \mathrm{Diag}(R,Q)^{1/2}\Lambda_t(\Sigma^\eta_k)^{1/2}$ is $\bar{\mathcal{J}}_\mathsf{L}\mathcal{M}\bar{\mathcal{J}}_\mathsf{R}$, where $\bar{\mathcal{J}}_\mathsf{L} := I_N \otimes R^{1/2}$ and $\bar{\mathcal{J}}_\mathsf{R} := (I_{pN} - \bar{\Delta}^\mathsf{R}\bar{\Delta}^\mathsf{L})(I_N \otimes (\Sigma^\mathsf{v})^{1/2})$ are non-singular matrices.

Consider a deterministic version of the system (1), reproduced for convenience as

$$x_{t+1} = Ax_t + Bu_t, \qquad y_t = Cx_t. \tag{22}$$

By assumption, (22) is minimal; let $L \in \mathbb{N}$ be such that the extended observability matrix $\mathcal{O} := \mathrm{col}(C, CA, \dots, CA^{L-1})$ has full column rank and the extended (reversed) controllability matrix $\mathcal{C} := [A^{L-1}B, \dots, AB, B]$ has full row rank. Let $u^{\mathrm{d}}_{[1,T_\mathrm{d}]}, y^{\mathrm{d}}_{[1,T_\mathrm{d}]}$ be a T_d-length trajectory of input-output data collected from (22). The input sequence $u^{\mathrm{d}}_{[1,T_\mathrm{d}]}$ is assumed to be *persistently exciting* of order $K_\mathrm{d} := L + 1 + n$, i.e., its associated K_d-depth block-Hankel matrix $\mathcal{H}_{K_\mathrm{d}}(u^{\mathrm{d}}_{[1,T_\mathrm{d}]}) \in \mathbb{R}^{mK_\mathrm{d} \times (T_\mathrm{d} - K_\mathrm{d} + 1)}$, defined as

$$\mathcal{H}_{K_\mathrm{d}}(u^{\mathrm{d}}_{[1,T_\mathrm{d}]}) := \begin{bmatrix} u^{\mathrm{d}}_1 & u^{\mathrm{d}}_2 & \cdots & u^{\mathrm{d}}_{T_\mathrm{d}-K_\mathrm{d}+1} \\ u^{\mathrm{d}}_2 & u^{\mathrm{d}}_3 & \cdots & u^{\mathrm{d}}_{T_\mathrm{d}-K_\mathrm{d}+2} \\ \vdots & \vdots & \ddots & \vdots \\ u^{\mathrm{d}}_{K_\mathrm{d}} & u^{\mathrm{d}}_{K_\mathrm{d}+1} & \cdots & u^{\mathrm{d}}_{T_\mathrm{d}} \end{bmatrix},$$

has full row rank. We formulate data matrices $U_1 \in \mathbb{R}^{mL\times h}$, $U_2 \in \mathbb{R}^{m \times h}$, $Y_1 \in \mathbb{R}^{pL \times h}$ and $Y_2 \in \mathbb{R}^{p\times h}$ of width $h := T_\mathrm{d} - L$ by partitioning associated Hankel matrices as

$$\begin{aligned} \mathrm{col}(U_1, U_2) &:= \mathcal{H}_{L+1}(u^{\mathrm{d}}_{[1,T_\mathrm{d}]}), \\ \mathrm{col}(Y_1, Y_2) &:= \mathcal{H}_{L+1}(y^{\mathrm{d}}_{[1,T_\mathrm{d}]}). \end{aligned} \tag{23}$$

The data matrices in (23) will now be used to represent a quantity $\mathbf{\Gamma} \in \mathbb{R}^{p \times (mL + pL)}$ related to the system (22),

$$\mathbf{\Gamma} = \begin{bmatrix} \mathbf{\Gamma}_\mathsf{U} & \mathbf{\Gamma}_\mathsf{Y} \end{bmatrix} := \begin{bmatrix} \mathbf{H} & CA^L \end{bmatrix} \begin{bmatrix} I_{mL} & \\ \mathbf{G} & \mathcal{O} \end{bmatrix}^{\dagger}, \tag{24}$$

in which $\mathbf{G} \in \mathbb{R}^{pL\times mL}$ and $\mathbf{H} \in \mathbb{R}^{p \times mL}$ are the impulse response matrices.

$$\left[\begin{array}{c} \mathbf{G} \\ \hline \mathbf{H} \end{array}\right] := \left[\begin{array}{cccc} 0_{p\times m} & & & \\ CB & 0_{p\times m} & & \\ \vdots & \ddots & \ddots & \\ CA^{L-2}B & \cdots & CB & 0_{p\times m} \\ \hline CA^{L-1}B & \cdots & CAB & CB \end{array}\right]$$

Lemma 2 (Data Representation of $\mathbf{\Gamma}$)**.** *If* (22) *is controllable and the input data* $u^{\mathrm{d}}_{[1,T_\mathrm{d}]}$ *is persistently exciting of order* $L + 1 + n$*, then, given the data matrices in* (23)*, the matrix* $\mathbf{\Gamma}$ *defined in* (24) *can be expressed as*

$$[\mathbf{\Gamma}_\mathsf{U}, \mathbf{\Gamma}_\mathsf{Y}, 0_{p\times m}] = Y_2\, \mathrm{col}(U_1, Y_1, U_2)^{\dagger}.$$

Proof. This is a reduced version of [18, Lemma 4], where $U_2, Y_2, \mathbf{H}, \mathbf{\Gamma}$ were defined differently and had L block rows instead. The result of this lemma can be obtained by extracting the first block row of the result in [18, Lemma 4]. ∎

With Lemma 2, the matrix $\mathbf{\Gamma}$ is represented using offline data collected from system (22), and the matrix will be used as part of the construction for our data-driven control method.

In the case where the measured data is corrupted by noise, as will usually be the case, the pseudo-inverse computation in Lemma 2 is numerically fragile and does not recover the desired matrix $\mathbf{\Gamma}$. A standard technique to robustify this computation is to replace the pseudo-inverse $W^\dagger$ of $W := \mathrm{col}(U_1, Y_1, U_2)$ in Lemma 2 with its Tikhonov regularization $(W^\mathsf{T}W + \lambda I_h)^{-1}W^\mathsf{T}$ with a regularization parameter $\lambda > 0$.

2) Auxiliary State-Space Model: The SMPC approach of Section III-A uses as sub-components a state estimator, an affine feedback law and a DR-CVaR constraint. We now leverage the offline data as described in Section III-B-1 to directly design analogs of these components based on data, without knowledge of the system matrices.

We begin by constructing an auxiliary state-space model which has equivalent input-output behavior to (1), but is parameterized only by the recorded data sequences. Define auxiliary signals $\mathbf{x}_t, \mathbf{w}_t \in \mathbb{R}^{n_{\mathsf{aux}}}$ of dimension $n_{\mathsf{aux}} := mL + pL + pL^2$ for system (1) by

$$\mathbf{x}_t := \left[\begin{array}{c} u_{[t-L,t)} \\ \hline y^{\circ}_{[t-L,t)} \\ \hline \rho_{[t-L,t)} \end{array}\right], \quad \mathbf{w}_t := \left[\begin{array}{c} 0_{mL\times 1} \\ \hline 0_{pL\times 1} \\ \hline 0_{pL(L-1)\times 1} \\ \rho_t \end{array}\right] \tag{25}$$

where $y^{\circ}_t := y_t - v_t \in \mathbb{R}^p$ is the output excluding measurement noise, and $\rho_t := \mathcal{O} w_t \in \mathbb{R}^{pL}$ stacks the system's response to process noise w_t on time interval $[t+1, t+L]$. The auxiliary signals $\mathbf{x}_t, \mathbf{w}_t$ together with u_t, y_t, v_t then satisfy the relations given by Lemma 3.

Lemma 3 (Auxiliary Model [18])**.** *For system* (1)*, signals* u_t, y_t, v_t *and the auxiliary signals* $\mathbf{x}_t, \mathbf{w}_t$ *in* (25) *satisfy*

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}u_t + \mathbf{w}_t \tag{26a}$$

$$y_t = \mathbf{C}\mathbf{x}_t + v_t \tag{26b}$$

for $\mathbf{A} \in \mathbb{R}^{n_{\mathsf{aux}}\times n_{\mathsf{aux}}}$, $\mathbf{B} \in \mathbb{R}^{n_{\mathsf{aux}}\times m}$ *and* $\mathbf{C} \in \mathbb{R}^{p\times n_{\mathsf{aux}}}$ *given by*

$$\mathbf{A} := \left[\begin{array}{c|c|c} \begin{matrix} & I_{m(L-1)} \\ 0_{m\times m} & \end{matrix} & 0 & 0 \\ \hline \begin{matrix} 0 \\ \mathbf{\Gamma}_{\mathsf{U}} \end{matrix} & \begin{matrix} 0 \;\; I_{p(L-1)} \\ \hline \mathbf{\Gamma}_{\mathsf{Y}} \end{matrix} & \begin{matrix} 0 \\ \mathbf{F} - \mathbf{\Gamma}_{\mathsf{Y}}\mathbf{E} \end{matrix} \\ \hline 0 & 0 & \begin{matrix} & I_{pL(L-1)} \\ 0_{pL\times pL} & \end{matrix} \end{array}\right],$$

$$\mathbf{B} := \left[\begin{array}{c} 0_{m(L-1)\times m} \\ I_m \\ \hline 0_{pL\times m} \\ \hline 0_{pL^2\times m} \end{array}\right], \quad \mathbf{C} := \left[\begin{array}{c|c|c} \mathbf{\Gamma}_{\mathsf{U}} & \mathbf{\Gamma}_{\mathsf{Y}} & \mathbf{F} - \mathbf{\Gamma}_{\mathsf{Y}}\mathbf{E} \end{array}\right],$$

with matrices $\mathbf{\Gamma}_{\mathsf{U}}$ *and* $\mathbf{\Gamma}_{\mathsf{Y}}$ *in* (24)*, and zero-one matrices* $\mathbf{E} \in \mathbb{R}^{pL\times pL^2}$ *and* $\mathbf{F} \in \mathbb{R}^{p\times pL^2}$ *composed by selection matrices* $S_j := [0_{p\times(j-1)p}, I_p, 0_{p\times(L-j)p}] \in \mathbb{R}^{p\times pL}$ *for* $j \in \mathbb{Z}_{[1,L]}$.

$$\left[\begin{array}{c} \mathbf{E} \\ \hline \mathbf{F} \end{array}\right] := \left[\begin{array}{cccc} 0_{p\times pL} & & & \\ S_1 & 0_{p\times pL} & & \\ \vdots & \ddots & \ddots & \\ S_{L-1} & \cdots & S_1 & 0_{p\times pL} \\ \hline S_L & \cdots & S_2 & S_1 \end{array}\right]$$

The output noise signal v_t in (26) is precisely the same as in (1); the signal $\mathbf{w}_t$ appears now as a new disturbance; $\mathbf{w}_t$ and v_t are uncorrelated; $\mathbf{w}_t$ has zero mean and the variance

$$\mathbf{\Sigma}^{\mathsf{w}} := \mathrm{Diag}(0_{(n_{\mathsf{aux}}-pL)\times(n_{\mathsf{aux}}-pL)}, \Sigma^{\rho}),$$

where $\Sigma^{\rho} := \mathcal{O}\Sigma^{\mathsf{w}}\mathcal{O}^{\mathsf{T}} \in \mathbb{S}^{pL}_{+}$ is the variance of ρ_t. The matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are known given offline data described in Section III-B-1, since they only depend on matrix $\mathbf{\Gamma} = [\mathbf{\Gamma}_{\mathsf{U}}, \mathbf{\Gamma}_{\mathsf{Y}}]$ which is data-representable via Lemma 2. Hence, the auxiliary model (26) can be interpreted as a non-minimal data-representable realization of system (1).

3) Data-Driven State Estimation, Feedback Policy and DR-CVaR Constraint: The auxiliary model (26) will now be used for both state estimation and constrained feedback control purposes. Suppose we are at a control step $t = k$ in a receding-horizon process. Similar to (7), we know the prior mean and variance of the auxiliary state $\mathbf{x}_k$

$$\mathbb{E}[\mathbf{x}_k] = \boldsymbol{\mu}^{\mathsf{x}}_k, \qquad \mathrm{Var}[\mathbf{x}_k] = \mathbf{\Sigma}^{\mathsf{x}}_k \tag{27}$$

through a Kalman filter to be introduced next. At the initial time $k = 0$, the initial auxiliary-state mean $\boldsymbol{\mu}^{\mathsf{x}}_0$ and variance $\mathbf{\Sigma}^{\mathsf{x}}_0$ are given as parameters. The Kalman filter for the auxiliary model (26) is analogous to (8) and (9),

$$\hat{\mathbf{x}}_t := \hat{\mathbf{x}}^{-}_t + \mathbf{L}_t(y_t - \mathbf{C}\hat{\mathbf{x}}^{-}_t), \qquad t \in \mathbb{Z}_{[k,k+N)} \tag{28a}$$

$$\hat{\mathbf{x}}^{-}_{t+1} := \mathbf{A}\hat{\mathbf{x}}_t + \mathbf{B}u_t, \qquad t \in \mathbb{Z}_{[k,k+N)} \tag{28b}$$

$$\hat{\mathbf{x}}^{-}_k := \boldsymbol{\mu}^{\mathsf{x}}_k \tag{28c}$$

where $\hat{\mathbf{x}}_t$ and $\hat{\mathbf{x}}^{-}_t$ are the posterior and prior estimates of $\mathbf{x}_t$, respectively, and the Kalman gain $\mathbf{L}_t \in \mathbb{R}^{n_{\mathsf{aux}}\times p}$ in (28a) is calculated as

$$\mathbf{L}_t := \mathbf{P}^{-}_t\mathbf{C}^{\mathsf{T}}(\mathbf{C}\mathbf{P}^{-}_t\mathbf{C}^{\mathsf{T}} + \Sigma^{\mathsf{v}})^{-1}, \quad t \in \mathbb{Z}_{[k,k+N)} \tag{29a}$$

$$\mathbf{P}_t := (I_{n_{\mathsf{aux}}} - \mathbf{L}_t\mathbf{C})\mathbf{P}^{-}_t, \qquad t \in \mathbb{Z}_{[k,k+N)} \tag{29b}$$

$$\mathbf{P}^{-}_{t+1} := \mathbf{A}\mathbf{P}_t\mathbf{A}^{\mathsf{T}} + \mathbf{\Sigma}^{\mathsf{w}}, \qquad t \in \mathbb{Z}_{[k,k+N)} \tag{29c}$$

$$\mathbf{P}^{-}_k := \mathbf{\Sigma}^{\mathsf{x}}_k. \tag{29d}$$

The affine output-error-feedback policy (12) from SMPC is now extended as $\boldsymbol{\pi}_t(\cdot)$,

$$u_t \leftarrow \boldsymbol{\pi}_t\big(\hat{\mathbf{x}}^{-}_{[k,t]}\big) := u^{\mathsf{nom}}_t + \textstyle\sum_{s=k}^{t} M^s_t(y_s - \mathbf{C}\hat{\mathbf{x}}^{-}_s) \tag{30}$$

where the nominal input $u^{\mathsf{nom}}_t \in \mathbb{R}^m$ and the gain matrices $M^s_t \in \mathbb{R}^{m\times p}$ are the decision variables. The SOC formulation of the DR-CVaR constraint (6) is similar to (17),

$$2\sqrt{\tfrac{1-\alpha}{\alpha}}\,\Big\|\mathbf{\Sigma}^{\eta\,\frac{1}{2}}_k\mathbf{\Lambda}^{\mathsf{T}}_t e_i\Big\|_2 \le -e^{\mathsf{T}}_i\begin{bmatrix} u^{\mathsf{nom}}_t \\ \mathbf{y}^{\mathsf{nom}}_t \end{bmatrix} + f_i, \; i \in \mathbb{Z}_{[1,q]} \tag{31}$$

for $t \in \mathbb{Z}_{[k,k+N)}$, with the auxiliary nominal output $\mathbf{y}^{\mathsf{nom}}_t$ produced by the noise-free auxiliary model,

$$\mathbf{x}^{\mathsf{nom}}_{t+1} := \mathbf{A}\mathbf{x}^{\mathsf{nom}}_t + \mathbf{B}u^{\mathsf{nom}}_t, \qquad t \in \mathbb{Z}_{[k,k+N)} \tag{32a}$$

$$\mathbf{y}^{\mathsf{nom}}_t := \mathbf{C}\mathbf{x}^{\mathsf{nom}}_t, \qquad t \in \mathbb{Z}_{[k,k+N)} \tag{32b}$$

$$\mathbf{x}^{\mathsf{nom}}_k := \boldsymbol{\mu}^{\mathsf{x}}_k \tag{32c}$$

and matrices $\mathbf{\Sigma}^{\eta}_k := \mathrm{Diag}(\mathbf{\Sigma}^{\mathsf{x}}_k, I_N \otimes \mathbf{\Sigma}^{\mathsf{w}}, I_N \otimes \Sigma^{\mathsf{v}}) \in \mathbb{S}^{n_{\eta\text{-aux}}}_{+}$ and $\mathbf{\Lambda}_t \in \mathbb{R}^{(m+p)\times n_{\eta\text{-aux}}}$ with $n_{\eta\text{-aux}} := n_{\mathsf{aux}} + n_{\mathsf{aux}}N + pN$,

$$\mathbf{\Lambda}_t := \begin{bmatrix} \mathbf{\Delta}^{\mathsf{U}}_{t-k} \\ \mathbf{\Delta}^{\mathsf{Y}}_{t-k} \end{bmatrix} \mathcal{M}\mathbf{\Delta}^{\mathsf{M}}_k + \begin{bmatrix} 0_{m\times n_{\eta\text{-aux}}} \\ \mathbf{\Delta}^{\mathsf{A}}_{t-k} \end{bmatrix} \tag{33}$$

where $\mathbf{\Delta}^{\mathsf{U}}_i \in \mathbb{R}^{m\times mN}$, $\mathbf{\Delta}^{\mathsf{Y}}_i \in \mathbb{R}^{p\times mN}$, $\mathbf{\Delta}^{\mathsf{A}}_i \in \mathbb{R}^{p\times n_{\eta\text{-aux}}}$ and $\mathbf{\Delta}^{\mathsf{M}}_k \in \mathbb{R}^{pN\times n_{\eta\text{-aux}}}$ can be found in Appendix A, and where $\mathcal{M} \in \mathbb{R}^{mN\times pN}$ is a concatenation of M^s_t as in (16).

4) SDDPC Optimization Problem and Control Algorithm: With the results above, we are now ready to mirror the steps of getting (20) and formulate a distributionally robust optimized-gain Stochastic Data-Driven Predictive Control (SDDPC) optimization problem,

$$\begin{aligned} &\underset{u^{\mathsf{nom}}, M^s_t}{\text{minimize}} && (35) \\ &\text{subject to} && (33), (31) \text{ for } t \in \mathbb{Z}_{[k,k+N)}, \text{ and } (32) \end{aligned} \tag{34}$$

where the cost function is analogous to (19).

$$\sum_{t=k}^{k+N-1}\Big[J(u_t^{\mathsf{nom}},\mathbf{y}_t^{\mathsf{nom}}) + \big\|\left[\begin{smallmatrix} R & \\ & Q\end{smallmatrix}\right]^{\frac{1}{2}}\mathbf{\Lambda}_t(\mathbf{\Sigma}_k^{\eta})^{\frac{1}{2}}\big\|_{\mathsf{F}}^2\Big] \quad (35)$$

Problem (34) has a unique optimal solution if feasible, similar as problem (20); the solution $(u^{\mathsf{nom}}, M_t^s)$ to (34) finishes parameterization of the control policies $\boldsymbol{\pi}_{[k,k+N)}$ via (30). We apply the first N_{c} control policies to the system, and then $t = k + N_{\mathrm{c}}$ is set as the next control step. The initial condition (27) at the new control step is iterated as the prior mean and prior variance of the auxiliary state $\mathbf{x}_{k+N_{\mathrm{c}}}$

$$\boldsymbol{\mu}_{k+N_{\mathrm{c}}}^{\mathsf{x}} = \hat{\mathbf{x}}_{k+N_{\mathrm{c}}}^{-}, \quad \mathbf{\Sigma}_{k+N_{\mathrm{c}}}^{\mathsf{x}} = \mathbf{P}_{k+N_{\mathrm{c}}}^{-}, \qquad (36)$$

obtained from the Kalman filter (28) and (29). The method is formally summarized in Algorithm 2.

Algorithm 2 Distributionally Robust Optimized-Gain Stochastic Data-Driven Predictive Control (DR/O-SDDPC)

Input: horizon lengths L, N, N_{c}, offline data $u_{[1,T_{\mathrm{d}}]}^{\mathrm{d}}$, $y_{[1,T_{\mathrm{d}}]}^{\mathrm{d}}$, noise variances $\Sigma^{\rho}, \Sigma^{\mathsf{v}}$, initial-state mean $\boldsymbol{\mu}_0^{\mathsf{x}}$ and variance $\mathbf{\Sigma}_0^{\mathsf{x}}$, cost matrices Q, R, constraint coefficients E and f, and CVaR level α.

Compute matrix $\mathbf{\Gamma}$ as in Section III-B-1 using data $u^{\mathrm{d}}, y^{\mathrm{d}}$, and formulate matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ as in Section III-B-2.
Compute $\mathbf{\Delta}_{[0,N)}^{\mathsf{U}}$, $\mathbf{\Delta}_{[0,N)}^{\mathsf{Y}}$, $\mathbf{\Delta}_{[0,N)}^{\mathsf{A}}$ through Appendix A.
Initialize the control step $k \leftarrow 0$ and the initial condition $\boldsymbol{\mu}_k^{\mathsf{x}} \leftarrow \boldsymbol{\mu}_0^{\mathsf{x}}$, $\mathbf{\Sigma}_k^{\mathsf{x}} \leftarrow \mathbf{\Sigma}_0^{\mathsf{x}}$.
while `true` **do**
    Compute Kalman gains $\mathbf{L}_{[k,k+N)}$ via (29).
    Compute matrix $\mathbf{\Delta}_k^{\mathsf{M}}$ through Appendix A.
    Solve nominal inputs $u_{[k,k+N)}^{\mathsf{nom}}$ and feedback-gain matrices M_t^s from problem (34), and thus formulate control policies $\boldsymbol{\pi}_{[k,k+N)}$ as in (30).
    for t **from** k **to** $k + N_{\mathrm{c}} - 1$ **do**
        Compute $\hat{\mathbf{x}}_t^{-}$ via (28).
        Measure y_t from the system (1).
        Input $u_t \leftarrow \boldsymbol{\pi}_t\big(\hat{\mathbf{x}}_{[k,t]}^{-}\big)$ to the system (1).
    $\boldsymbol{\mu}_{k+N_{\mathrm{c}}}^{\mathsf{x}} \leftarrow \hat{\mathbf{x}}_{k+N_{\mathrm{c}}}^{-}$ and $\mathbf{\Sigma}_{k+N_{\mathrm{c}}}^{\mathsf{x}} \leftarrow \mathbf{P}_{k+N_{\mathrm{c}}}^{-}$ as in (36).
    Set $k \leftarrow k + N_{\mathrm{c}}$.

C. Theoretical Equivalence of SMPC and SDDPC

We show in this section that under idealized tuning conditions, our proposed SDDPC method produces the same control actions as produced by model-based SMPC. In other words, our proposal is a genuine data-driven version of SMPC.

1) Relation of State Means and Variances: We start by noting an underlying relation between the means and between the variances of x_k and $\mathbf{x}_k$, at each control step $t = k$. This result will be leveraged in establishing equivalence between SMPC and our proposed SDDPC.

Lemma 4 (Mean-Variance Relation of x_k and $\mathbf{x}_k$ [18])**.** *Let*

$$\Phi_{\mathsf{orig}} := \big[\mathcal{C},\, A^L,\, \mathcal{C}_{\mathsf{W}}\big], \quad \Phi_{\mathsf{aux}} := \begin{bmatrix} I_{mL} & & \\ \mathbf{G} & \mathcal{O} & \mathbf{G}_{\mathsf{W}} \\ & & I_L \otimes \mathcal{O} \end{bmatrix},$$

with the matrices $\mathcal{C}$, $\mathcal{O}$, $\mathbf{G}$ defined in Section III-B-1 and

$$\mathcal{C}_{\mathsf{W}} := [A^{L-1}, \dots, A, I_n], \; \mathbf{G}_{\mathsf{W}} := \begin{bmatrix} 0_{p\times n} & & & \\ C & 0_{p\times n} & & \\ \vdots & \ddots & \ddots & \\ CA^{L-2} & \cdots & C & 0_{p\times n} \end{bmatrix}.$$

If the prior means and variances of x_k and $\mathbf{x}_k$ at time $t = k$ are (7) *and* (27) *respectively, then they satisfy*

$$\mu_k^{\mathsf{x}} = \Phi_{\mathsf{orig}}\, \mu_k^{\xi}, \qquad \Sigma_k^{\mathsf{x}} = \Phi_{\mathsf{orig}}\, \Sigma_k^{\xi}\, \Phi_{\mathsf{orig}}^{\mathsf{T}}, \qquad (37a)$$

$$\boldsymbol{\mu}_k^{\mathsf{x}} = \Phi_{\mathsf{aux}}\, \mu_k^{\xi}, \qquad \mathbf{\Sigma}_k^{\mathsf{x}} = \Phi_{\mathsf{aux}}\, \Sigma_k^{\xi}\, \Phi_{\mathsf{aux}}^{\mathsf{T}}, \qquad (37b)$$

for some $\mu_k^{\xi} \in \mathbb{R}^{mL+n(L+1)}$ and some $\Sigma_k^{\xi} \in \mathbb{S}_{+}^{mL+n(L+1)}$.

2) Equivalence of Optimization Problems: We now establish that the SDDPC problem (34) and the SMPC problem (20) have equal feasible and optimal sets, when the respective state means and variances are related as in (37) of Lemma 4.

Proposition 5 (Equivalence of Optimization Problems)**.** *If the parameters $\mu_k^{\mathsf{x}}, \Sigma_k^{\mathsf{x}}, \boldsymbol{\mu}_k^{\mathsf{x}}, \mathbf{\Sigma}_k^{\mathsf{x}}$ satisfy* (37)*, then the optimal (resp. feasible) solution set of the SDDPC problem* (34) *is equal to the optimal (resp. feasible) solution set of the SMPC problem* (20)*.*

Proof. We first claim that, for all u^{nom} and M_t^s, we have

$$y_t^{\mathsf{nom}} = \mathbf{y}_t^{\mathsf{nom}}, \quad \Lambda_t \Sigma_k^{\eta} \Lambda_t^{\mathsf{T}} = \mathbf{\Lambda}_t \mathbf{\Sigma}_k^{\eta} \mathbf{\Lambda}_t^{\mathsf{T}} \qquad (38)$$

for $t \in \mathbb{Z}_{[k,k+N)}$, which is explained in Appendix B. Given (38), the objective function (19) of problem (20) and objective function (35) of problem (34) are equal, and the constraint (17) in problem (20) and constraint (31) in problem (34) are equivalent. Thus the problems (20) and (34) have the same objective function and constraints, and the result follows. ∎

3) Equivalence of SMPC and SDDPC Control Methods: We present in Theorem 7 our main theoretical result, saying that under idealized conditions our proposed SDDPC control method and the benchmark SMPC method will result in identical control actions.

Assumption 6 (SDDPC Parameter Choice w.r.t. SMPC)**.** Given the parameters in Algorithm 1, we assume the parameters in Algorithm 2 satisfy the following.

(a) L is sufficiently large so that $\mathcal{O}$ has full column rank and $\mathcal{C}$ has full row rank.
(b) Data $u^{\mathrm{d}}, y^{\mathrm{d}}$ comes from the deterministic system (22); the input data u^{d} is persistently exciting of order $L + 1 + n$.
(c) Given Σ^{w} in Algorithm 1, parameter Σ^{ρ} in Algorithm 2 is set equal to $\mathcal{O}\Sigma^{\mathsf{w}}\mathcal{O}^{\mathsf{T}}$.
(d) Given $\mu_0^{\mathsf{x}}, \Sigma_0^{\mathsf{x}}$ in Algorithm 1, for some μ_0^{ξ}, Σ_0^{ξ} satisfying (37a) at $k = 0$, the parameters $\boldsymbol{\mu}_0^{\mathsf{x}}, \mathbf{\Sigma}_0^{\mathsf{x}}$ in Algorithm 2 are selected as in (37b) at $k = 0$. (Such μ_0^{ξ}, Σ_0^{ξ} always exist because Φ_{orig} has full row rank.)

Theorem 7 (**Equivalence of SMPC and SDDPC**)**.** *Consider the stochastic system* (1) *with initial state x_0, and consider the following two control processes:*

a) decide control actions $\{u_t\}_{t=0}^{\infty}$ by executing Algorithm 1;
b) decide control actions $\{u_t\}_{t=0}^{\infty}$ by executing Algorithm 2, where the parameters satisfy Assumption 6.

Let the noise realizations $\{w_t, v_t\}_{t=0}^{\infty}$ be the same in process a) and in process b). Then the state-input-output trajectories

$\{x_t, u_t, y_t\}_{t=0}^{\infty}$ *resulting from process a) and from process b) are the same.*

Proof. The proof is similar to the proof of [18, Thm. 9] and is omitted here. The proof requires Proposition 5 and the fact that both problems (20) and (34) have unique optimal solutions if feasible. ■

The equivalence between SMPC and SDDPC is built in an idealized setting as stated in Assumption 6. While in practice these assumptions may not hold, noisy offline data can be accommodated as discussed in Section III-B-1, and Σ^{ρ} becomes a tuning parameter of our SDDPC method.

IV. Numerical Case Study

In this section, we numerically test our proposed method on a batch reactor system introduced in [27] and applied in [16], [28]. The system has $n = 4$ states, $m = 2$ inputs and $p = 2$ outputs, and the discrete-time system matrices with sampling period 0.1s are as follows:

$$\left[\begin{array}{c|c} A & B \\ \hline C & \end{array}\right] = \left[\begin{array}{cccc|cc} 1.178 & .001 & .511 & -.403 & .004 & -.087 \\ -.051 & .661 & -.011 & .061 & .467 & .001 \\ .076 & .335 & .560 & .382 & .213 & -.235 \\ 0 & .335 & .089 & .849 & .213 & -.016 \\ \hline 1 & 0 & 1 & -1 & & \\ 0 & 1 & 0 & 0 & & \end{array}\right].$$

The process/sensor noise on each state/output follows the t-distribution of 2 DOFs scaled by 10^{-4}, which is a heavy-tailed distribution. Control parameters are reported in TABLE I. We collected offline data of length $T_{\mathrm{d}} = 600$ from the noisy system, where the input data was the outcome of a PI controller $U(s) = \left[\begin{smallmatrix} 0 & -1/s \\ 2+1/s & 0 \end{smallmatrix}\right] Y(s)$ plus a white-noise signal of noise power 10^{-2}. In the online control process, the reference signal is $r_t = [0, 0]^{\mathsf{T}}$ from time 0s to time 30s, alternates between $[0, 0]^{\mathsf{T}}$ and $[0.3, 0]^{\mathsf{T}}$ from 30s to 60s, and is $r_t = [0.5, 0]^{\mathsf{T}}$ from 60s to 90s. With our proposed SDDPC method, the first output signal is in Fig. 1; the signal remains around 0.4 from 60s to 90s because of the safety constraint specified in TABLE I.

For comparison purposes, we implemented the simulation with different controllers. In addition to distributionally robust optimized-gain (DR/O) SMPC and SDDPC in this paper, we applied the SMPC and SDDPC frameworks from [18], which use chance constraints and a fixed feedback gain (CC/F). To observe separate impacts of using the DR constraint and optimized gains, we also implement SMPC and SDDPC with DR constraints and a fixed feedback gain (DR/F). We also compare to DeePC, SPC and deterministic MPC as benchmarks. The model used in MPC methods is identified from the same offline data in the data-driven controllers.

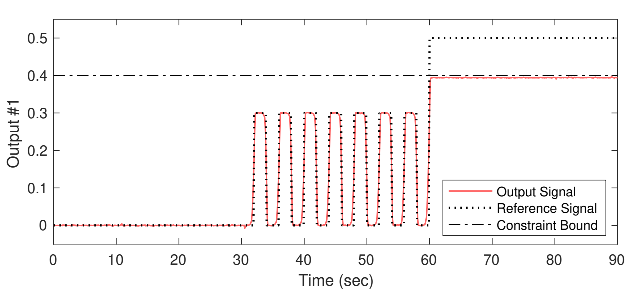

Fig. 1. The system's first output signal with DR/O-SDDPC.

TABLE I

CONTROL PARAMETERS

Time horizon lengths	$L = 5$, $N = 15$, $N_{\mathrm{c}} = 5$
Cost matrices	$Q = 10^3 I_p$, $R = I_m$
Safety constraint coefficients	$E = I_{m+p} \otimes \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$
	$f = [.1\ .1\ .5\ .1\ .4\ .4\ .4\ .4]^{\mathsf{T}}$
CVaR level[a]	$\alpha = 0.3$
Variance of v_t for SMPC/SDDPC	$\Sigma^{\mathrm{v}} = 5 \times 10^{-7} I_p$
Variance of ρ_t for SDDPC	$\Sigma^{\rho} = 10^{-7} I_{pL}$
Variance of w_t for SMPC[b]	$\Sigma^{\mathrm{w}} = \mathcal{O}^{\dagger} \Sigma^{\rho} \mathcal{O}^{\dagger\mathsf{T}}$

[a] α is used as the risk bound for chance constrained controllers.
[b] $\mathcal{O}$ is obtained given the identified model (A, B, C) in SMPC.

TABLE II

SIMULATION RESULT STATISTICS

Controller	Total Tracking Cost 0s to 30s	Total Tracking Cost 30s to 60s	Cumulative Violation from 60s to 90s
DR/O-SDDPC[a]	0.02	64.2	0
DR/F-SDDPC	0.02	68.9	0
CC/F-SDDPC	0.02	64.9	0.03
DR/O-SMPC	0.02	64.2	0
DR/F-SMPC	0.02	68.0	0
CC/F-SMPC	0.02	64.9	0.01
deterministic MPC	0.09	64.6	0.20
SPC	0.18	65.5	2.23
DeePC	0.18	64.7	0.19

[a] DR – distributionally robust constrained, CC – chance constrained, O – with optimized feedback gain, F – with fixed feedback gain.

The simulation results are summarized in TABLE II. We (i) evaluate the controllers' tracking performance through the tracking cost from 0s to 60s, and (ii) evaluate the controllers' ability to satisfy constraints according to the cumulative amount of constraint violation between 60s and 90s, when the first output signal hits the constraint margin. When the reference signal is constant (0s–30s), SMPC and SDDPC tracked better than other methods, aligning with the observation in [18]. Comparing DR/F and CC/F methods, the controllers with DR constraints achieved lower amounts of constraint violation (60s–90s), while the tracking performance is slightly worse during 30s–60s when the reference signal has frequent step changes. Comparing DR/O and DR/F methods, we observe that the methods with optimized feedback gain achieved better tracking performances when the reference signal changes frequently (30s–60s).

V. Conclusions

We proposed a Stochastic Data-Driven Predictive Control (SDDPC) framework that accommodates distributionally robust (DR) probability constraints and produces closed-loop control policies with feedback gains determined from optimization. In theory, our SDDPC method can produce equivalent control inputs with associated Stochastic MPC, under specific conditions. Simulation results indicated separate benefits of using DR constraints and optimized feedback gains. Looking forward, our future work will explore recursive feasibility and closed-loop stability of the control scheme.

Appendix A. Definition of $\Delta_i^{\mathsf{U}}, \Delta_i^{\mathsf{Y}}, \Delta_i^{\mathsf{A}}, \Delta_k^{\mathsf{M}}$

The matrices $\Delta_i^{\mathsf{U}} \in \mathbb{R}^{m\times mN}$, $\Delta_i^{\mathsf{Y}} \in \mathbb{R}^{p\times mN}$, $\Delta_i^{\mathsf{A}} \in \mathbb{R}^{p\times n_\eta}$ and $\Delta_k^{\mathsf{M}} \in \mathbb{R}^{pN\times n_\eta}$ for $i \in \mathbb{Z}_{[0,N)}$ in (15) are defined as follows, with $n_\eta := n + nN + pN$,

$$\begin{aligned}
\mathrm{col}\big(\Delta_0^{\mathsf{U}},\ldots,\Delta_{N-1}^{\mathsf{U}}\big) &:= I_{mN}\\
\mathrm{col}\big(\Delta_0^{\mathsf{Y}},\ldots,\Delta_{N-1}^{\mathsf{Y}}\big) &:= \bar{\Delta}^{\mathsf{P}}(I_N\otimes B)\\
\mathrm{col}\big(\Delta_0^{\mathsf{A}},\ldots,\Delta_{N-1}^{\mathsf{A}}\big) &:= [\bar{\Delta}^{\mathsf{O}},\bar{\Delta}^{\mathsf{P}},I_{pN}]\\
\Delta_k^{\mathsf{M}} &:= [\bar{\Delta}^{\mathsf{Q}},\bar{\Delta}^{\mathsf{R}},I_{pN}-\bar{\Delta}^{\mathsf{R}}\bar{\Delta}^{\mathsf{L}}]
\end{aligned}$$

where we define $\bar{\Delta}^{\mathsf{L}} := \mathrm{Diag}(AL_k, AL_{k+1},\ldots,AL_{k+N-1}) \in \mathbb{R}^{nN\times pN}$ and $[\bar{\Delta}^{\mathsf{O}},\bar{\Delta}^{\mathsf{P}}],[\bar{\Delta}^{\mathsf{Q}},\bar{\Delta}^{\mathsf{R}}] \in \mathbb{R}^{pN\times(n+nN)}$,

$$\big[\bar{\Delta}^{\mathsf{O}}\big|\bar{\Delta}^{\mathsf{P}}\big] := \left[\begin{array}{c|ccccc} C & 0_{p\times n} & & & & \\ CA & C & 0_{p\times n} & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ CA^{N-1} & CA^{N-2} & \cdots & C & 0_{p\times n} & \end{array}\right]$$

$$\big[\bar{\Delta}^{\mathsf{Q}}\big|\bar{\Delta}^{\mathsf{R}}\big] := \left[\begin{array}{c|cccc} \Psi_k^k & 0_{p\times n} & & & \\ \Psi_{k+1}^k & \Psi_{k+1}^{k+1} & 0_{p\times n} & & \\ \vdots & \vdots & \ddots & \ddots & \\ \Psi_{k+N-1}^k & \Psi_{k+N-1}^{k+1} & \cdots & \Psi_{k+N-1}^{k+N-1} & 0_{p\times n} \end{array}\right]$$

with $\Psi_t^s := C(A-AL_{t-1}C)(A-AL_{t-2}C)\cdots(A-AL_sC) \in \mathbb{R}^{p\times n}$ for $s<t$ and $\Psi_t^s := C$ for $s=t$.

Similarly, the matrices $\mathbf{\Delta}_i^{\mathsf{U}}$, $\mathbf{\Delta}_i^{\mathsf{Y}}$, $\mathbf{\Delta}_i^{\mathsf{A}}$ and $\mathbf{\Delta}_k^{\mathsf{M}}$ in (33) are computed (with underlying $\bar{\mathbf{\Delta}}^{\mathsf{L}},\bar{\mathbf{\Delta}}^{\mathsf{O}},\bar{\mathbf{\Delta}}^{\mathsf{P}},\bar{\mathbf{\Delta}}^{\mathsf{Q}},\bar{\mathbf{\Delta}}^{\mathsf{R}},\mathbf{\Psi}_t^s$) in the same way as above, with A,B,C,L_t,n replaced by $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{L}_t,n_{\mathsf{aux}}$, respectively.

Appendix B. Proof of (38)

Proof. The relation $y_t^{\mathsf{nom}} = \mathbf{y}_t^{\mathsf{nom}}$ in (38) was established in [18, Claim 7.5]. The other relation in (38) is equivalent to

$$\begin{aligned}
\Delta_{t-k}^{\mathsf{U}} &= \mathbf{\Delta}_{t-k}^{\mathsf{U}}, & \Delta_k^{\mathsf{M}}\Sigma_k^{\eta}(\Delta_k^{\mathsf{M}})^{\mathsf{T}} &= \mathbf{\Delta}_k^{\mathsf{M}}\mathbf{\Sigma}_k^{\eta}(\mathbf{\Delta}_k^{\mathsf{M}})^{\mathsf{T}},\\
\Delta_{t-k}^{\mathsf{Y}} &= \mathbf{\Delta}_{t-k}^{\mathsf{Y}}, & \Delta_{t-k}^{\mathsf{A}}\Sigma_k^{\eta}(\Delta_{t-k}^{\mathsf{A}})^{\mathsf{T}} &= \mathbf{\Delta}_{t-k}^{\mathsf{A}}\mathbf{\Sigma}_k^{\eta}(\mathbf{\Delta}_{t-k}^{\mathsf{A}})^{\mathsf{T}},
\end{aligned}$$

via the definitions of Λ_t and $\mathbf{\Lambda}_t$ in (15) and (33). Given the definitions in Appendix A, the above relations are implied by

1) $CA^qB = \mathbf{C}\mathbf{A}^q\mathbf{B}$ for $q \in \mathbb{Z}_{[0,N)}$,
2) $CA^q\Sigma_k^{\mathsf{x}}(CA^r)^{\mathsf{T}} = \mathbf{C}\mathbf{A}^q\mathbf{\Sigma}_k^{\mathsf{x}}(\mathbf{C}\mathbf{A}^r)^{\mathsf{T}}$ for $q,r \in \mathbb{Z}_{[0,N)}$,
3) $CA^q\Sigma^{\mathsf{w}}(CA^r)^{\mathsf{T}} = \mathbf{C}\mathbf{A}^q\mathbf{\Sigma}^{\mathsf{w}}(\mathbf{C}\mathbf{A}^r)^{\mathsf{T}}$ for $q,r \in \mathbb{Z}_{[0,N-2]}$,
4) $\Psi_q^k\Sigma_k^{\mathsf{x}}\Psi_r^{k\mathsf{T}} = \mathbf{\Psi}_q^k\mathbf{\Sigma}_k^{\mathsf{x}}\mathbf{\Psi}_r^{k\mathsf{T}}$ for $q,r \in \mathbb{Z}_{[k,k+N)}$,
5) $\Psi_r^q\Sigma^{\mathsf{w}}\Psi_s^{q\mathsf{T}} = \mathbf{\Psi}_r^q\mathbf{\Sigma}^{\mathsf{w}}\mathbf{\Psi}_s^{q\mathsf{T}}$ for integers q,r,s that satisfy $k<q\le z<k+N$ for $z\in\{r,s\}$, and
6) $\Psi_r^qAL_{q-1} = \mathbf{\Psi}_r^q\mathbf{A}\mathbf{L}_{q-1}$ for integers q,r that satisfy $k<q\le r<k+N$,

where the relations 1)–6) can be shown given the equalities

$$\begin{aligned}
A\Phi\Phi_{\mathsf{aux}} &= \Phi\mathbf{A}\Phi_{\mathsf{aux}}, & B &= \Phi\mathbf{B}, & L_t &= \Phi\mathbf{L}_t,\\
\mathbf{A}\Phi_{\mathsf{aux}} &= \Phi_{\mathsf{aux}}\widetilde{A}, & \mathbf{B} &= \Phi_{\mathsf{aux}}\widetilde{B}, & \mathbf{L}_t &= \Phi_{\mathsf{aux}}\widetilde{L}_t,\\
C\Phi\Phi_{\mathsf{aux}} &= \mathbf{C}\Phi_{\mathsf{aux}}, & \Sigma^{\mathsf{w}} &= \Phi\mathbf{\Sigma}^{\mathsf{w}}\Phi^{\mathsf{T}}, & \Sigma_k^{\mathsf{x}} &= \Phi\mathbf{\Sigma}_k^{\mathsf{x}}\Phi^{\mathsf{T}},
\end{aligned}$$

established with some matrices $\Phi,\widetilde{A},\widetilde{B},\widetilde{L}_t$ according to [18, Claim 7.1, Claim 7.2, Claim 7.6]. ∎

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