Decentralized Low-Gain Integrators for LTI Systems with Application to Decentralized Feedback-Based Optimization ⋆

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Abstract: Decentralized control architectures are widely used for the control of large-scale and networked systems. We review classical formulations and results in decentralized integral control, and illustrate how the results can be applied to derive stability conditions for decentralized feedback-based optimization controllers.

Keywords: Large scale systems, feedback control systems, optimization : theory and algorithms, linear systems.

1. INTRODUCTION

Output regulation (i.e., tracking and disturbance rejection) is a key design goal to be achieved via feedback design. The essential technique used to achieve output regulation in the presence of constant references and disturbances is *integral control*, wherein the accumulation of a tracking error over time is fed back, allowing regulation to be achieved robustly in the presence of modelling error. A particular case of these controllers, the so-called *low-gain* integral controllers, have been developed for both linear and nonlinear stable systems (for example see Davison (1976a); Grosdidier et al. (1985); Simpson-Porco (2021)). Such controllers are relatively simple to design, as they require only the DC gain of the system to be known. Additionally, these controllers can be tuned online until a desired level of performance is achieved. These advantages come at the cost of a time scale separation from the plant.

Typically, integral controllers utilize a centralized architecture, where information is shared across the system. However large-scale and network systems often favour a simpler decentralized control architecture, where local information is used to control local inputs (see Davison (1976b); Campo and Morari (1994)). The need for output regulation of large-scale and network systems inspired the development of decentralized low-gain integral controllers (see Campo and Morari (1994); Abed (1986a)). In contrast to centralized low-gain integral controllers, decentralized low-gain integral controllers offer extra tuning flexibility and implementation simplicity at the cost of more stringent design criteria.

A (relatively) recent trend in control theory has been the development of feedback-based optimization, a technique where optimization algorithms are connected in feedback to a stable system with the goal of long-term cost optimization (see Hauswirth et al. (2024) for a comprehensive review). These controllers are structurally similar to low-gain integrators, and their development has leveraged and extended analysis techniques used for low-gain integrator systems. Despite this, there has been little development in decentralized feedback-based optimization (see Wang et al. (2024); Agarwal et al. (2023); Belgioioso et al. (2025)). Thus, the purpose of this extended abstract is to review decentralized low-gain integrator theory and relate decentralized low-gain integrators to decentralized feedback-based optimization. For presentation clarity, we consider only LTI systems and formally present only one of several problem formulations related to decentralized low-gain integrators. For a comprehensive analysis on decentralized low-gain integrators, we direct the reader to a soon to be available preprint by the same authors.

1.1 Notation

For $N \in \mathbb{N}$ we let $\mathcal{I}_N \triangleq \{1, \dots, N\}$ be an index set. Let $\alpha, \beta \subset \mathcal{I}_N$ denote *ordered index sets* of cardinality $n_\alpha = |\alpha|$ and $n_\beta = |\beta|$ respectively. Denote $\alpha(i)$ the i^{th} element of α. When no ordering is explicitly prescribed, the ordering is assumed strictly increasing. For a block partitioned matrix

$$A \triangleq \begin{bmatrix} A_{1,1} & \cdots & A_{1,N} \\ \vdots & & \vdots \\ A_{N,1} & \cdots & A_{N,N} \end{bmatrix},$$

where $A_{i,j}$ are of appropriate sizes, we denote

$$A_{\alpha,\beta} \triangleq \begin{bmatrix} A_{\alpha(1),\beta(1)} & \cdots & A_{\alpha(1),\beta(n_\beta)} \\ \vdots & & \vdots \\ A_{\alpha(n_\alpha),\beta(1)} & \cdots & A_{\alpha(n_\alpha),\beta(n_\beta)} \end{bmatrix}$$

the submatrix formed from the blocks prescribed in α and β. Similarly, for block row, block column, or block diagonal matrices, we denote A_α the submatrix formed from the blocks prescribed in α.

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2. PROBLEM FORMULATION

We consider the MIMO open-loop LTI system with inputs and outputs partitioned into $N \in \mathbb{N}$ collections,

$$\begin{aligned} \dot{x} &= Ax + \sum_{j \in \mathcal{I}_N} B_j^u u_j + B^w w, \qquad && x(0) = x_0, \\ e_i &= C_i x + \sum_{j \in \mathcal{I}_N} D_{i,j}^u u_j + D_i^w w, \qquad && i \in \mathcal{I}_N, \end{aligned} \tag{1}$$

with state $x(t) \in \mathbb{R}^n$, local control inputs $u_i(t) \in \mathbb{R}^{m_i}$, and local errors $e_i(t) \in \mathbb{R}^{p_i}$. The system is subject to a constant disturbance $w \in \mathbb{R}^{n_w}$. More compactly, we write

$$\begin{aligned} \dot{x} &= Ax + B^u u + B^w w, \qquad x(0) = x_0, \\ e &= Cx + D^u u + D^w w, \end{aligned} \tag{2}$$

where $u \triangleq \operatorname{col}(u_1, \ldots, u_N) \in \mathbb{R}^m$ represents our total control input, and $e \triangleq \operatorname{col}(e_1, \ldots, e_N) \in \mathbb{R}^p$ represents our total error. Our control objective is to robustly regulate the errors $e(t)$ to zero using simple local integral controllers of the form

$$\dot{\eta}_i = -\varepsilon_i e_i, \qquad u_i = K_i \eta_i, \qquad i \in \mathcal{I}_N, \tag{3}$$

where the set of feedback gains $K_i \in \mathbb{R}^{m_i \times p_i}$ are to be designed *offline*, and the set of tuning parameters $\varepsilon_i > 0$ are to be adjusted *online*. We let $\eta \triangleq \operatorname{col}(\eta_1, \ldots, \eta_N)$, $K \triangleq \operatorname{blkdiag}(K_1, \ldots, K_N)$, $\varepsilon \triangleq \operatorname{col}(\varepsilon_1, \ldots, \varepsilon_N)$, and $\mathcal{E} = \operatorname{blkdiag}(\varepsilon_1 I_{p_1}, \ldots, \varepsilon_N I_{p_N})$ denote appropriate concatenations of our integrator states, control gains, and tuning gains. Some natural questions arise regarding the design and implementation of the control loops (3) that do not arise in a centralized setting. We ask:

(i) In what way will the control loops be implemented?
(ii) How much freedom should we allow adjusting the tuning parameters?

For the purpose of this extended abstract, we focus on the following setup:

(i) The control loops may be connected (or disconnected) in any arbitrary order.
(ii) The tuning parameters may be adjusted arbitrarily up to some maximum value.

We formally define the problem. Consider a set $\sigma \subset \mathcal{I}_N$ and its complement $\bar{\sigma} = \mathcal{I}_N \setminus \sigma$. We introduce the closed loop system $\Sigma_{\text{cl}}^{\sigma}$:

$$\begin{aligned} \dot{x} &= Ax + B^u u + B^w w \qquad && u_\sigma = K_\sigma \eta_\sigma, \\ \dot{\eta}_\sigma &= -\mathcal{E}_\sigma e_\sigma, && u_{\bar{\sigma}} \in \mathbb{R}^{m_{\bar{\sigma}}}, \\ e &= Cx + D^u u + D^w w && w \in \mathbb{R}^{n_w} \end{aligned} \tag{4}$$

where $m_{\bar{\sigma}} = \sum_{i \in \bar{\sigma}} m_i$ and any unused control inputs are treated as arbitrary constant inputs. We interpret $\Sigma_{\text{cl}}^{\sigma}$ as the system with loop closures indexed by σ. We wish to uphold stability given *any* configuration of closed loops.

Problem 1. *Find a set of integral feedback gains $\{K_i\}_{i=1}^N$ and an upper tuning gain value $\varepsilon^\star > 0$ such that for each disturbance $w \in \mathbb{R}^{n_w}$ and configuration of closed loops $\sigma \subset \mathcal{I}_N$, the closed-loop system* (4)

(i) possess a unique equilibrium point which is independent of ε;
(ii) the aforementioned equilibrium point is exponentially stable for all $\varepsilon \in (0, \varepsilon^\star)^N$.

Any controller solving Problem 1 has several properties advantageous for large-scale and network systems. Such a controller is local, may be implemented (or disconnected) asynchronously, and may be tuned independently for desired local performance (with the condition $\varepsilon_i < \varepsilon^*$).

3. PROBLEM SOLUTION

3.1 Preliminaries

Consider the open loop system (2), which we assume is exponentially stable. If this is not true, the open loop system can always be stabilized before implementation of a low-gain integral controller. Let

$$\begin{aligned} G^{\text{ol},u} &\triangleq D^u - CA^{-1}B^u \\ G^{\text{ol},w} &\triangleq D^w - CA^{-1}B^w \end{aligned} \tag{5}$$

denote the (appropriately partitioned) DC gains of (2) from our inputs u and disturbance w to our error e.

3.2 Solution to Problem 1

Theorem 1. *Consider the set of closed-loop systems* (4) *under the assumption that* (2) *is exponentially stable. Then Problem 1 is solvable if $-G^{\text{ol},u}K$ is robustly block D-stable.*

We defer to Appendix A for an introduction to D-stability and related concepts. Notably, the condition presented in Theorem 1 is dependent only on the DC gains of (2), not its dynamics. Typically, the DC gain matrix $G^{\text{ol},u} \in \mathbb{R}^{p \times m}$ is of lower dimension than that of the full closed loop system (4). Additionally, the condition presented in Theorem 1 is independent of the loop closure configuration σ. This simplicity makes designing decentralized low-gain integrators computationally favourable for large-scale and network systems.

Proof. The proofs (Abed, 1986a, Section IV) and (Campo and Morari, 1994, Theorem 7) may be adapted to show Theorem 1. Additionally, we have recently developed a Lyapunov based proof of this result that relies on recent advancements in D-stability theory. This result will soon be available in the aforementioned preprint by the authors. □

3.3 Numerical Example

The following numerical example is inspired by Wang et al. (2024). Consider a model for a 3-node DC power system, with nodes connected in a ring,

$$\begin{aligned} \begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{i} \end{bmatrix} &= \begin{bmatrix} -G & -B \\ B^\top & -R \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} i^c + \begin{bmatrix} i^r + i^d \\ 0 \end{bmatrix} \\ v^e &= v^r - v^m \\ &= v^r - (v + v^d), \end{aligned} \tag{6}$$

where $v \in \mathbb{R}^3$ represents the voltage, $v^e \in \mathbb{R}^3$ represents the voltage error, $v^m \in \mathbb{R}^3$ represents the measured voltage, $v^r \in \mathbb{R}^3$ represents the reference voltage, $v^d \in \mathbb{R}^3$ represents the voltage measurement disturbance, $i \in \mathbb{R}^3$ represents the line current, $i^c \in \mathbb{R}^3$ represents the node control current, $i^r \in \mathbb{R}^3$ represents the node reference injected current, $i^d \in \mathbb{R}^3$ represents the node disturbance current, $C = I \in \mathbb{R}^{3\times 3}$ represents the node capacitance, $G = I \in \mathbb{R}^{3\times 3}$ represents the node conductance, $L = I \in \mathbb{R}^{3\times 3}$ represents the line inductance, $R = 2I \in \mathbb{R}^{3\times 3}$ represents the line resistance, and $B \in \mathbb{R}^{3\times 3}$ represents the incidence matrix of the network. We assume that our reference and disturbance signals v_r, i_r, v_d, i_d are constant. We wish to design a decentralized low-gain integrator for (6), consisting of the controllers

$$\dot{\eta}_i = -\varepsilon_i v_i^e, \qquad i_i^c = K_i \eta_i. \tag{7}$$

Setting $K_i = -1$, we calculate

$$-G^{\text{ol},u}K = \begin{bmatrix} -0.6 & -0.2 & -0.2 \\ -0.2 & -0.6 & -0.2 \\ -0.2 & -0.2 & -0.6 \end{bmatrix},$$

which is block diagonally dominant, and thus robustly block D-stable. We have therefore satisfied the conditions of Theorem 1. We implement our controllers, and adjust our tuning parameters online to the values $\varepsilon_i = 1, i \in \mathcal{I}_3$. During simulation, we close our three control loops at 0, 25, and 50 seconds respectively. Simulation results are shown in Figure 1. Clearly, the closed loop system is stable, and the external disturbance is rejected at system steady state.

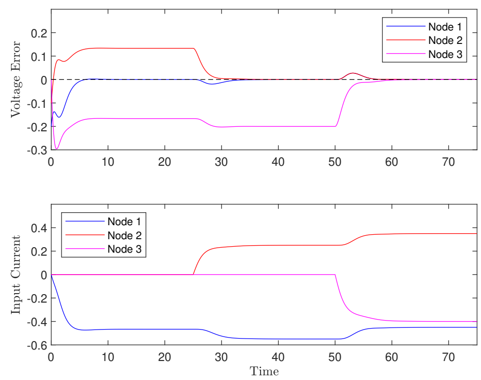

Fig. 1. Decentralized low-gain integrator implementation.

4. RELATION TO FEEDBACK-BASED OPTIMIZATION AND QUADRATIC GAMES

4.1 Low-Gain Regulation of Quadratic Games

We now relate decentralized low-gain integrators to decentralized feedback-based optimization, inspired by Wang et al. (2024). Consider an open loop system of the form (2) which is assumed exponentially stable. We can consider the system as having N agents, which we assume are non-cooperative. As a result, each agent only values its own measured error e_i and control effort u_i which it wishes to minimize. If we assume our control inputs u and disturbance w are constant, our measured error will converge to a steady state value $\bar{e}$, described by the relation

$$\bar{e} = G^{\text{ol},u}u + G^{\text{ol},w}w. \tag{8}$$

We define the set of steady state cost functions $J_i\colon \mathbb{R}^{p_i} \times \mathbb{R}^{m_i} \to \mathbb{R}$,

$$J_i(\bar{e}_i, u_i) \triangleq \frac{1}{2}\left(\bar{e}_i^\mathsf{T} Q_i \bar{e}_i + u_i^\mathsf{T} R_i u_i\right), \qquad i \in \mathcal{I}_N, \tag{9}$$

where $Q_i \succeq 0$ and $R_i \succ 0$ are weighting parameters. Consider the quadratic game

$$\forall i \in \mathcal{I}_N, \qquad \min_{u_i} J_i(\bar{e}_i, u_i), \qquad \text{s.t.} \quad (8), \tag{10}$$

describing a situation in which each agent attempts to minimize its own cost independently but is subject to the global equality constraint (8). We denote any solutions to this game (assuming they exist) the Nash equilibria of the game, for which no agent can improve its cost. Any solution to such a game will render every local gradient $\nabla_{u_i} J_i = 0$. For our quadratic game, we can express each agents local gradient in terms of our system parameters

$$\begin{aligned}\nabla_{u_i} J_i &= (G_{i,i}^{\text{ol},u})^\mathsf{T} Q_i \bar{e}_i + R_i u_i \\ &= (G_{i,i}^{\text{ol},u})^\mathsf{T} Q_i (C_i \bar{x} + D_{i,[1:N]}^u u + D_i^w w) + R_i u_i.\end{aligned} \tag{11}$$

We define the related *pseudo-gradient* of all agents costs

$$\begin{aligned}\nabla_p J &\triangleq \text{col}(\nabla_{u_1} J_1, \dots, \nabla_{u_N} J_N) \\ &= (G_{\text{diag}}^{\text{ol},u})^\mathsf{T} Q (C\bar{x} + D^u u + D^w w) + Ru,\end{aligned} \tag{12}$$

where

$$Q \triangleq \text{blkdiag}(Q_1, \dots, Q_N), \qquad R \triangleq \text{blkdiag}(R_1, \dots, R_N),$$
$$G_{\text{diag}}^{\text{ol},u} \triangleq \text{blkdiag}(G_{1,1}^{\text{ol},u}, \dots, G_{N,N}^{\text{ol},u}).$$

We propose implementing a decentralized low-gain integrator which drives the *pseudo-gradient* to zero, instead of the error. Consider the system

$$\begin{aligned}\dot{x} &= Ax + B^u u + B^w w, \qquad x(0) = x_0, \\ y &= \mathcal{C}x + \mathcal{D}^u u + \mathcal{D}^w w,\end{aligned} \tag{13}$$

where we have set $y = \nabla_p J(x, u, w) = \nabla_p J(e, u)$ as the output we wish to regulate, and have defined

$$\mathcal{C} \triangleq (G_{\text{diag}}^{\text{ol},u})^\mathsf{T} Q C, \qquad \mathcal{D}^w \triangleq (G_{\text{diag}}^{\text{ol},u})^\mathsf{T} Q D^w,$$
$$\mathcal{D}^u \triangleq (G_{\text{diag}}^{\text{ol},u})^\mathsf{T} Q D^u + R.$$

We suggest the set of decentralized low-gain integral controllers

$$\dot{\eta}_i = -\varepsilon_i \nabla_{u_i} J_i(e_i, u_i), \qquad u_i = K_i \eta_i, \tag{14}$$

with the goal of solving Problem 1. We calculate the control DC gain of (13) to be

$$\mathcal{G}^u \triangleq \mathcal{D}^u - \mathcal{C}A^{-1}B^u = R + (G_{\text{diag}}^{\text{ol},u})^\mathsf{T} Q G^{\text{ol},u}. \tag{15}$$

If we can find a controller K such that $-\mathcal{G}^u K$ is robustly block D-stable, Theorem 1 will be satisfied. This would imply there exists a decentralized low-gain integral controller which drives our pseudo-gradient to zero, or equivalently, drives our system to the desired Nash equilibria. We now consider the special case where $K = I$. In this case, our controller becomes mathematically equivalent to the controller

$$\dot{u}_i = -\varepsilon_i \nabla_{u_i} J_i(e_i, u_i), \qquad u_i(0) = 0. \tag{16}$$

This controller is structurally similar to the gradient descent based approaches for centralized feedback-based optimization (presented in Hauswirth et al. (2024)) and decentralized feedback-based optimization (presented in Wang et al. (2024)). The difference between our proposed controller and the work presented in Wang et al. (2024) is the inclusion of local tuning parameters ε_i instead of a global parameter $\varepsilon_1 = \ldots = \varepsilon_N$. This extra flexibility comes at the cost of more stringent design criteria.

Remark 1. *Consider the case where $K = I$ and each agent can adjust its weighting parameters Q_i, R_i centrally and cooperatively before controller implementation. Then by making R "large" relative to Q, one can ensure block diagonal dominance of $-\mathcal{G}^u K$, and consequently block robust D-stability of $-\mathcal{G}^u K$. However, the larger R is* *designed relative to Q, the closer the Nash equilibria move towards the equilibrium point of the open loop system* (2) *under input* $u = 0$.

In a future paper, we will analyze the non-linear version of this problem with non-quadratic costs using similar decentralized low-gain integral control concepts.

4.2 Numerical Example

We once again consider the example system (6). Instead of implementing a decentralized low-gain integral controller, we wish to implement a decentralized low-gain inspired feedback-based optimization controller. Selecting $K = Q = 10R = I$, we calculate

$$-\mathcal{G}^u K = \begin{bmatrix} -0.46 & -0.12 & -0.12 \\ -0.12 & -0.46 & -0.12 \\ -0.12 & -0.12 & -0.46 \end{bmatrix} \quad (17)$$

which is once again block diagonally dominant, and thus robustly block D-stable. We have thus satisfied Theorem 1. We implement our set of controllers and tune our parameters online to the values $\varepsilon_i = 1, i \in \mathcal{I}_3$. Simulation results are shown in Figure 2, where once again the control loops are closed at 0, 25, and 50 seconds. Unlike the results in Section 3.3, the system errors do not converge to zero. However, this is by design, as we are minimizing our cost, not our error. We do, however, reach our desired Nash equilibria at steady state, indicated by the dashed lines.

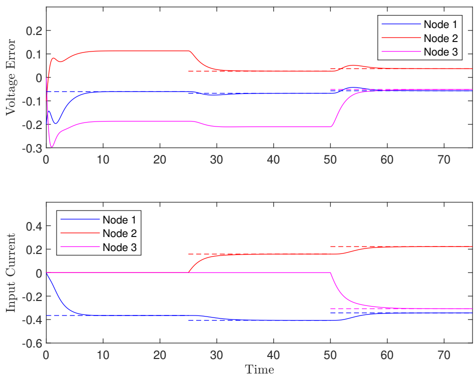

Fig. 2. Decentralized feedback-based optimization implementation.

5. CONCLUSIONS

We have reviewed some fundamental ideas of decentralized low-gain integral control architectures, and have demonstrated how these classical concepts can be applied to derive stability conditions for a topic of current interest in decentralized feedback-based optimization control. Future work will expand significantly on this insight, with extensions of decentralized stability certificates to nonlinear systems and more complex objective functions.

REFERENCES

Abed, E. (1986a). Decomposition and stability for multiparameter singular perturbation problems. *IEEE Transactions on Automatic Control*, 31(10), 925–934.

Abed, E.H. (1986b). Strong d-stability. *Systems & Control Letters*, 7(3), 207–212.

Agarwal, A., Simpson-Porco, J.W., and Pavel, L. (2023). Model-free game-theoretic feedback optimization. In *2023 European Control Conference (ECC)*, 1–8.

Belgioioso, G., Liao-McPherson, D., Hudoba de Badyn, M., Bolognani, S., Smith, R.S., Lygeros, J., and Dörfler, F. (2025). Online feedback equilibrium seeking. *IEEE Transactions on Automatic Control*, 70(1), 203–218.

Campo, P. and Morari, M. (1994). Achievable closed-loop properties of systems under decentralized control: conditions involving the steady-state gain. *IEEE Transactions on Automatic Control*, 39(5), 932–943.

Davison, E. (1976a). Multivariable tuning regulators: The feedforward and robust control of a general servomechanism problem. *IEEE Transactions on Automatic Control*, 21(1), 35–47.

Davison, E. (1976b). The robust decentralized control of a general servomechanism problem. *IEEE Transactions on Automatic Control*, 21(1), 14–24.

Grosdidier, P., Morari, M., and Holt, B.R. (1985). Closed-loop properties from steady-state gain information. *Industrial & Engineering Chemistry Fundamentals*, 24(2), 221–235.

Hauswirth, A., He, Z., Bolognani, S., Hug, G., and Dörfler, F. (2024). Optimization algorithms as robust feedback controllers. *Annual Reviews in Control*, 57, 100941.

Kushel, O.Y. (2019). Unifying matrix stability concepts with a view to applications. *SIAM Review*, 61(4), 643–729.

Simpson-Porco, J.W. (2021). Analysis and synthesis of low-gain integral controllers for nonlinear systems. *IEEE Transactions on Automatic Control*, 66(9), 4148–4159.

Wang, W., He, Z., Belgioioso, G., Bolognani, S., and Dorfler, F. (2024). Decentralized feedback optimization via sensitivity decoupling: Stability and sub-optimality. In *2024 European Control Conference (ECC)*, 3201–3206.

Appendix A. BLOCK MATRIX STABILITY CONCEPTS

Consider a set of matrices $A_{i,j} \in \mathbb{R}^{n_i \times n_j}$, where $n_i, n_j, N \in \mathbb{N}$ and $i, j \in \mathcal{I}_N$. We denote A the related and appropriately concatenated square matrix. Consider the set of positive-definite block-diagonal matrices

$$\mathcal{D}_{\text{blk}} \triangleq \{D \mid D = \text{blkdiag}(d_1 I_{n_1}, \ldots, d_N I_{n_N}), \quad D \succ 0\}.$$

Definition 1 (Block D-Stability)**.** *The matrix A is* block D-stable *if DA is Hurwitz for all $D \in \mathcal{D}_{\text{blk}}$.*

Block D-stable matrices are not necessarily robust to perturbations. Thus, we define a second, more robust block D-stability concept in line with the definition given by Abed (1986b).

Definition 2 (Robust Block D-Stability)**.** *The matrix A is* robustly block D-stable *if there exists $\mu > 0$ such that $A + B$ is block D-stable for all B satisfying $\|B\| \leq \mu$.*

Robust block D-stability implies block D-stability, which in turn implies (Hurwitz) stability. While (robust) block D-stability is a difficult property to establish, there are several tractable sufficient conditions, the most useful of which is block diagonal Lyapunov stability. For a comprehensive history of block D-stability concepts, see Kushel (2019).