ECE1659H: Robust and Optimal Control

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Acknowledgements

This course is based on material from many sources, including notes by C. Scherer, S. Weiland, M. Peet, B. Bamieh, S. Boyd, S. Lall, A. Megretski, A. D. Lewis, and the books by Dullerud/Paganini, Zhou/Doyle, Desoer/Vidayasagar, and Boyd/El Ghaoui/Feron/Balakrishnan, and Skogestad/Postlethwaite. All typos are my own.

Table of Contents

1. Introduction

Administrative information

- Course website is the authoritative administrative source

https://www.control.utoronto.ca/~jwsimpson/robust/

Course requirements:

- Completion of assignments
- Completion of individual course project
- Participation in course Piazza
- There is no midterm or final exam

What is in this course?

- Fundamental systems theory: $\mathcal{H}_2/\mathcal{H}_\infty$ norms, dissipativity, input-output stability theory, the Kalman-Yakubovich-Popov (KYP) Lemma, ...
- The generalized plant framework for analysis and control
- LMI-based system analysis and controller design methods
- Robustness analysis: stability and performance of uncertain systems, leading to integral quadratic constraints

I will need to occasionally tell half-truths, jump over technical details, and skip many robust control topics entirely...

Your job: prove things, do examples, and have fun with the material

Why study robust control now?

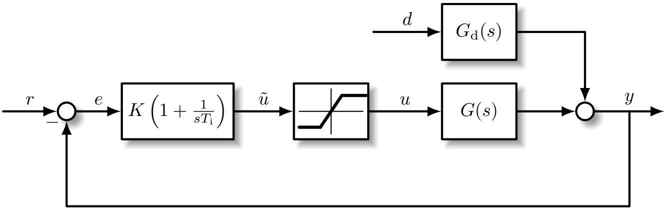
- Increasing complexity and decentralization
- Data-driven control and optimization
- Autonomy and stacked control layers
- Civilization-critical applications: energy, synthetic biology, robotics, aerospace, medicine, smart materials

Robust control ideas are ***already*** playing an essential role in . . .

- Data-driven feedback control
- Feedback-based optimizing controllers, game theory
- Analysis and design of new optimization algorithms
- Hybrid systems, risk-sensitive control

Many fresh topics to study for the class project

Motivating example: actuator saturation



- SISO LTI plant controlled with error-feedback PI controller
- Saturation block enforces control limits $\Longrightarrow$ ***nonlinear system***
- Saturation degrades performance and may induce ***instability***

How can we assess closed-loop stability and performance?

Motivating example: fragility of LQG control

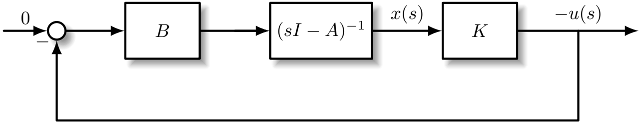
- **Recall:** optimal state-feedback controller design problem

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0, \qquad u = -Kx, \qquad (A, B) \text{ stabilizable}$$

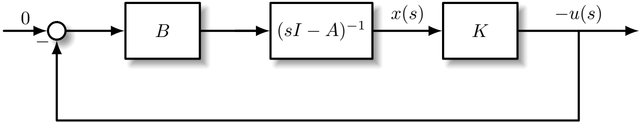
- Classical linear-quadratic design method ($Q \succeq 0$, $R \succ 0$)

$$J(x_0) = \underset{K}{\text{minimize}} \int_0^\infty x(t)^\mathsf{T} Q x(t) + u(t)^\mathsf{T} R u(t) \, \mathrm{d}t$$

produces optimal controller $K = R^{-1} B^\mathsf{T} P$ where $P \succeq 0$ solves ARE.



Motivating example: fragility of LQG control



- One can check ***gain/phase margins*** of ***open-loop*** TF

$$G_{\mathrm{LQR}}(s) = K(sI - A)^{-1}B$$

- By these metrics, LQR produces a ***very*** robust closed-loop system:
  - Upper gain margin is $+\infty$ (wow)
  - Lower gain margin is $\frac{1}{2}$ (pretty good)
  - Phase margin of $\pm 60^\circ$ in each channel (fantastic)

Motivating example: fragility of LQG control

What about ***output*** feedback? Do these robustness results hold when we include an ***optimal state observer*** in the loop? ***No!***

Guaranteed Margins for LQG Regulators

JOHN C. DOYLE

***Abstract*—There are none.**

INTRODUCTION

Considerable attention has been given lately to the issue of robustness of linear–quadratic (LQ) regulators. The recent work by Safonov and Athans [1] has extended to the multivariable case the now well-known guarantee of 60° phase and 6 dB gain margin for such controllers. However, for even the single-input, single-output case there has remained the question of whether there exist any guaranteed margins for the full LQG (Kalman filter in the loop) regulator. By counterexample, this note answers that question; there are none.

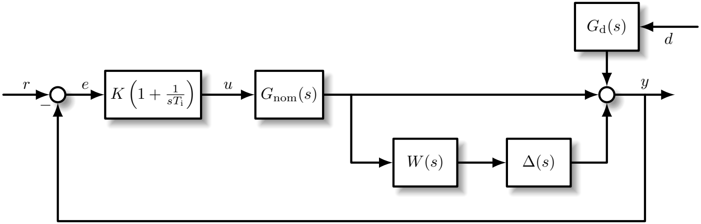
Need a ***joint theory*** of optimal and robust control.

Motivating example: robust performance

- As we will see, we can model uncertain plants as

$$G(s) = G_{\text{nom}}(s)(1 + \Delta(s)W(s))$$

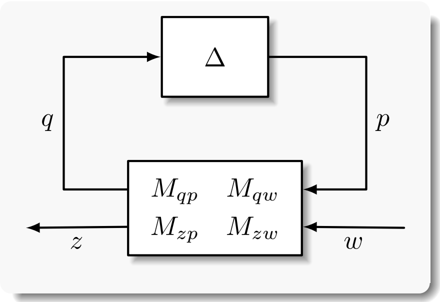
where $G_{\text{nom}}(s)$ is our nominal model, $\Delta(s)W(s)$ is uncertainty.



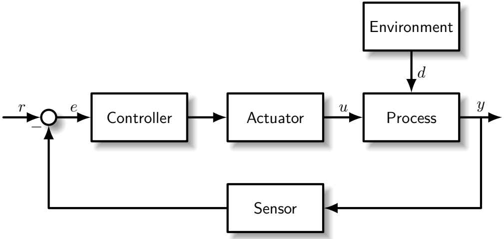
Does our design achieve ***robust performance***, i.e., good performance despite the uncertainty described by $\Delta(s)$?

Course objectives

- Formulate and solve standard ***optimal control*** formulations
- Formulate models of ***uncertain systems***
- Formulate tractable ***robust stability/performance*** tests
- Explore a new exciting topic in your project



Why use feedback?



Feedback allows us to:

1. stabilize (remove "asymptotic trajectory uncertainty")
2. reduce/remove environmental effects ("exogenous uncertainty")
3. reduce sensitivity to process uncertainty ("endogenous uncertainty")

The ***whole point*** is uncertainty management!

Exogenous vs. endogenous uncertainty

- ***Exogenous*** uncertainty = disturbances from the "environment"
  - Certain important disturbance signals (constant, ramp, sinusoids) can be ***asymptotically rejected*** using the ***internal model principle*** of linear control theory

- ***Endogenous*** uncertainty = imperfections in our process model
  - Much more subtle to model, analyze, and design for
  - Ability to tolerate endogenous uncertainty = ***robustness***

Robustness in classical control

- 1890's: Routh–Hurwitz allows simple parametric sensitivity studies

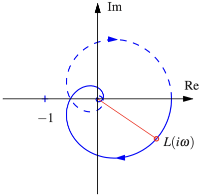
$$\Pi(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

- 1930's: Bode's plot (gain and phase margins)
- 1930's: Nyquist plot $\omega \mapsto (\mathrm{Re}(\hat{G}(\mathbf{j}\omega)), \mathrm{Im}(\hat{G}(\mathbf{j}\omega)))$

Key features:

(i) graphical freq. domain tests

(ii) MIMO extensions difficult



Incredibly effective and practical tools, which highlight the importance of ***frequency-domain*** analysis.

Robustness in modern control

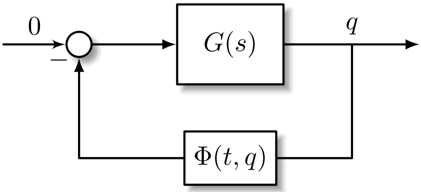
- 1960's: Kalman's state-space control revolution
  - LQ theory $\Longrightarrow$ systematic computation of MIMO controllers
  - LQR showed excellent gain/phase margins (Anderson, Safonov, etc.)
  - Endogenous uncertainty had disappeared from the picture, and was replaced with stochastic exogenous disturbances; this concerned some (Horowitz, Athans, Rosenbrock, McMorran, . . .)

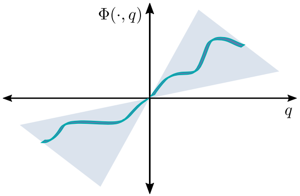
Indeed things were too good to be true . . .

- High-profile failures of LQG (see Trident, F-8C Crusader) when implemented on nonlinear systems
- Doyle's 1978 LQG counter-example showed ***zero*** robustness margins

Absolute stability

- 1950's/60's: Soviet scientists (Lur'e, Postnikov, Popov, Yakubovich ...) studied stability of SISO LTI loops with time-varying *sector-bounded* nonlinearities $\Phi(t, q)$





- Under certain conditions on ***frequency response*** $G(\mathbf{j}\omega)$, loop is stable ***for all*** Φ within class of interest ("absolute" stability)

Extension of linear analyis tools to nonlinear settings!

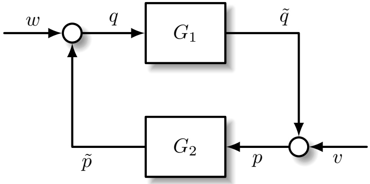
Input-output stability theory

- 1960's: parallel to Kalman's theory, others (Zames, Sandberg, Willems, Vidyasagar, Desoer, Safonov, ...) tried to formalize and extend classical Laplace-domain methods via functional analysis

- Model components as ***causal operators*** on ***signal spaces***

$$G : \{\text{Space of signals}\} \rightarrow \{\text{Space of signals}\}$$

- Stability: ***finite-norm inputs*** must produce ***finite-norm outputs***



Very general framework for feedback analysis, leading to many ***deep theoretical results*** (small-gain theorem, passivity theorem)

The robust control revolution

- 1978: debut of ***singular values*** of the transfer matrix as key robustness indicators (Stein, Laub, Doyle, Safonov . . .)
- 1981: Zames introduces the $\mathcal{H}_\infty$ space to the field and solves the first SISO $\mathcal{H}_\infty$ control problem
- 1980's: further developments of $\mathcal{H}_\infty$ control based on analytic transfer function methods, driven by Francis, Zames, Doyle, . . .
- 1989: DGKF Paper

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 34, NO. 8, AUGUST 1989 831

State-Space Solutions to Standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control Problems

JOHN C. DOYLE, KEITH GLOVER, MEMBER, IEEE, PRAMOD P. KHARGONEKAR, MEMBER, IEEE, AND BRUCE A. FRANCIS, FELLOW, IEEE

The computational classical-modern synthesis

- 1988: Nesterov and Nemirovskii develop efficient ***interior point methods*** for numerically solving LMI problems
- 1988–Present: ***Explosion*** of activity on LMI analysis and design methods for control (Boyd, Balakrishnan, Feron, El Ghaoui, Scherer, Khargonekar, Poolla, Zhou, Glover, Chilali, Gahinet, Iwasaki, Dullerud, Paganini, many more . . .). Many problems convexified, including
  (i) $\mathcal{H}_\infty$ and $\mathcal{H}_2$ output feedback problems
  (ii) Extensions: regional pole constraints, multiobjective designs
  (iii) Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design, robust $\mathcal{H}_2$ control
  (iv) LPV analysis, gain-scheduled controller design, . . .
- 1997–Present: ***IQCs*** (Megretski, Rantzer, Jönsson, Scherer, . . .): A unifying analysis perspective, which connects frequency-domain methods, absolute stability, nonlinear input-output theory, robust control, and the more recent LMI revolution . . .

2. Vector Spaces and Linear Operators

Definition of a vector space

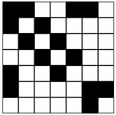
Definition 2.1 (Vector space). A ***vector space*** over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ is a set V of ***vectors*** equipped with the following two operations:

1. vector addition, which is a mapping $+$: V $\times$ V $\rightarrow$ V taking two vectors $v_1, v_2 \in$ V and producing a new vector $v_1 + v_2 \in$ V s.t.
   - *commutativity:* $v_1 + v_2 = v_2 + v_1$;
   - *associativity:* $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$;
   - *zero vector:* there exists an element $0 \in$ V such that $v + 0 = v$;
   - *additive inverse:* $\forall v \in$ V $\exists u \in$ V s.t. $v + u = 0$;
2. scalar multiplication, denoted by $\alpha v \in$ V for $\alpha \in \mathbb{F}$, s.t.
   - *associativity:* $\alpha_1(\alpha_2 v) = (\alpha_1 \alpha_2)v$;
   - *vector distributivity:* $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$;
   - *scalar distributivity:* $(\alpha_1 + \alpha_2)v = \alpha_1 v + \alpha_2 v$;
   - *multiplicative identity element:* $1v = v$.

Examples of finite-dimensional vector spaces

- Our favourite vector space $\mathbb{F}^n$ over $\mathbb{F}$
- The set $\mathcal{M}_{m,n}(\mathbb{F})$ of all $m \times n$ matrices $A \in \mathbb{F}^{m \times n}$ over $\mathbb{F}$
- The sets of all $n \times n$ ***Hermitian*** or ***symmetric*** matrices

$$\mathbb{H}^n = \{A \in \mathbb{C}^{n \times n} \mid A = A^*\}$$
$$\mathbb{S}^n = \{A \in \mathbb{R}^{n \times n} \mid A = A^{\mathsf{T}}\}$$



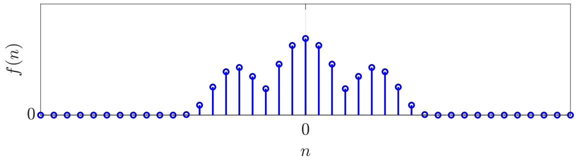
- The set of all discrete-time N_0-***periodic*** signals

$$\mathsf{c}_{\mathrm{per}}(\mathbb{Z}; \mathbb{F}^n) = \{f : \mathbb{Z} \to \mathbb{F}^n \mid f(n + N_0) = f(n) \text{ for all } n \in \mathbb{Z}\}.$$

Examples of infinite-dimensional vector spaces

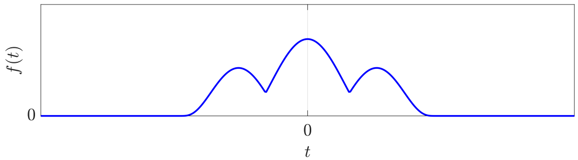
- The set $\mathsf{c}_{\mathrm{fin}}(\mathbb{Z};\mathbb{F})$ of all ***finite-duration*** DT signals

$$\mathsf{c}_{\mathrm{fin}}(\mathbb{Z};\mathbb{F}) = \{f : \mathbb{Z} \to \mathbb{F} \mid \exists\, N \in \mathbb{Z}_{\geq 0} \text{ s.t. } f(n) = 0\ \forall |n| \geq N\}.$$



- The set $\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{F})$ of all ***cont. compactly-supported*** CT signals

$$\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{F}) = \{f \in \mathsf{C}^0(\mathbb{R};\mathbb{F}) \mid \exists\, T \geq 0 \text{ s.t. } f(t) = 0\ \forall |t| \geq T\}.$$



Norms on vector spaces

Definition 2.2 (Seminorm). A ***seminorm*** on a $\mathbb{F}$-vector space V is a map $\|\cdot\|_{\mathsf{V}} : \mathsf{V} \to \mathbb{R}$ satisfying

(i) *homogeneity:* $\|\alpha v\|_{\mathsf{V}} = |\alpha|\,\|v\|_{\mathsf{V}}$ for all $\alpha \in \mathbb{F}, v \in \mathsf{V}$;

(ii) *nonnegativity:* $\|v\|_{\mathsf{V}} \geq 0$ for all $v \in \mathsf{V}$;

(iii) *triangle inequality:* $\|v_1 + v_2\|_{\mathsf{V}} \leq \|v_1\|_{\mathsf{V}} + \|v_2\|_{\mathsf{V}}$ for all $v_1, v_2 \in \mathsf{V}$.

If additionally $\|\cdot\|_{\mathsf{V}}$ satisfies

(iv) *non-degeneracy:* $\|v\|_{\mathsf{V}} = 0$ if and only if $v = 0_{\mathsf{V}}$

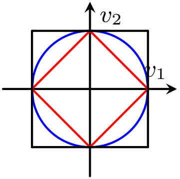
then V is a ***norm*** on V. We call $(\mathsf{V}, \|\cdot\|_{\mathsf{V}})$ a normed vector space.

- A norm allows us to measure the ***size*** of a vector, and helps us identify two vectors: $v = u$ if and only if $\|v - u\|_{\mathsf{V}} = 0$

Examples of norms

- $\mathbb{F}^n$ is a normed vector space with any of

$$\|v\|_1 = |v_1| + |v_2| + \cdots + |v_n|$$
$$\|v\|_2 = \sqrt{|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2}$$
$$\|v\|_\infty = \max_{i\in\{1,\dots,n\}} |v_i|$$



- For $p \in [1,\infty)$, $\mathsf{c}_{\mathrm{fin}}(\mathbb{Z}_{\geq 0};\mathbb{F})$ is a normed vector space with any of

$$\|f\|_p = \Big(\sum_{n=0}^{\infty} |f(n)|^p\Big)^{1/p}, \qquad \|f\|_\infty = \sup_{n\geq 0} |f(n)|$$

- For $p \in [1,\infty)$, $\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R}_{\geq 0};\mathbb{F})$ is a normed vector space with any of

$$\|f\|_p = \Big(\int_0^{\infty} |f(x)|^p \,\mathrm{d}x\Big)^{1/p}, \qquad \|f\|_\infty = \sup_{x\geq 0} |f(x)|$$

Inner products on vector spaces

Definition 2.3 (Inner product). An ***inner product*** on V is a map $\langle \cdot, \cdot \rangle_{\mathsf{V}} : \mathsf{V} \times \mathsf{V} \to \mathbb{F}$ satisfying

(i) *conjugate symmetry:* $\langle v_1, v_2 \rangle_{\mathsf{V}} = \langle v_2, v_1 \rangle_{\mathsf{V}}^*$

(ii) *partial linearity:* $\langle v_1, \alpha_2 v_2 + \alpha_3 v_3 \rangle_{\mathsf{V}} = \alpha_2 \langle v_1, v_2 \rangle_{\mathsf{V}} + \alpha_3 \langle v_1, v_3 \rangle_{\mathsf{V}}$,

(iii) *non-negativity:* $\langle v, v \rangle_{\mathsf{V}} \geq 0$ for all $v \in \mathsf{V}$, and

(iv) *non-degeneracy:* $\langle v, v \rangle_{\mathsf{V}} = 0$ if and only if $v = 0_{\mathsf{V}}$.

We call the pair $(\mathsf{V}, \langle \cdot, \cdot \rangle_{\mathsf{V}})$ an ***inner product space***.

- Inner products let us discuss ***orthogonality***: $u \perp v$ means $\langle u, v \rangle_{\mathsf{V}} = 0$
- Every inner product $\langle \cdot, \cdot \rangle_{\mathsf{V}}$ ***induces a norm*** $\|x\|_{\mathsf{V}} \triangleq \sqrt{\langle x, x \rangle_{\mathsf{V}}}$
- ***Cauchy-Schwarz Inequality:*** $|\langle u, v \rangle_{\mathsf{V}}| \leq \|u\|_{\mathsf{V}} \|v\|_{\mathsf{V}}$.

Examples of inner products (and associated norms)

- $\mathbb{F}^n$ is an inner product space with ***Euclidean*** i.p.

$$\langle x, y\rangle_2 \triangleq x^* y = x_1^* y_1 + \cdots + x_n^* y_n, \qquad \|x\|_2 = \sqrt{\langle x, x\rangle_2}$$

- $\mathcal{M}_{m,n}(\mathbb{F})$ is an inner product space with the ***Frobenius*** i.p.

$$\langle X, Y\rangle_{\mathsf{F}} \triangleq \mathrm{trace}(X^* Y), \qquad \|X\|_{\mathsf{F}} = \sqrt{\mathrm{trace}(X^* X)}$$

- $\mathsf{c}_{\mathrm{fin}}(\mathbb{Z}_{\geq 0}; \mathbb{F})$ is an inner product space with

$$\langle f, g\rangle_2 \triangleq \sum\nolimits_{n=0}^{\infty} f(n)^* g(n), \qquad \|f\|_2 = \sqrt{\sum\nolimits_{n=0}^{\infty} |f(n)|_2^2}$$

- $\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R}_{\geq 0}; \mathbb{F})$ is an inner product space with

$$\langle f, g\rangle_2 \triangleq \int_0^{\infty} f(x)^* g(x)\,\mathrm{d}x, \qquad \|f\|_2 = \sqrt{\int_0^{\infty} |f(x)|_2^2\,\mathrm{d}x}$$

Convergence and completeness in vector spaces

Definition 2.4 (Convergence, Cauchy, Completeness). Let $(\mathsf{V}, \|\cdot\|_{\mathsf{V}})$ be a normed vector space. A sequence $(v_k)_{k\in\mathbb{Z}_{\geq 0}}$ in V

(i) ***converges*** to $v \in \mathsf{V}$ if $\lim_{k\to\infty} \|v_k - v\|_{\mathsf{V}} = 0$;

(ii) is ***Cauchy*** if $\lim_{k,j\to\infty} \|v_k - v_j\|_{\mathsf{V}} = 0$.

If all Cauchy sequences in $(\mathsf{V}, \|\cdot\|_{\mathsf{V}})$ converge, then $(\mathsf{V}, \|\cdot\|_{\mathsf{V}})$ is a *complete* normed vector space or ***Banach space***.

- Why care about completeness?
    1. We can check convergence by checking Cauchy-ness
    2. Sensible limits will always exist "within" the space

- All ***finite-dimensional*** normed vector spaces are complete in *all possible* norms; infinite-dimensional spaces are often not complete

$(\mathsf{c}_{\text{fin}}(\mathbb{Z}_{\geq 0};\mathbb{F}), \|\cdot\|_2)$ **is not complete**

- Consider the sequence $(f_j)_{j\in\mathbb{Z}_{\geq 0}}$ in $\mathsf{c}_{\text{fin}}(\mathbb{Z}_{\geq 0};\mathbb{F})$ given by

$$f_j(n) = \begin{cases} \frac{1}{n+1}, & n \leq j \\ 0, & n > j. \end{cases}$$

- For $k, \ell \in \mathbb{Z}_{\geq 0}$ with $k > \ell$ we have that

$$\|f_\ell - f_k\|_2^2 = \sum_{n=\ell+1}^{k} \frac{1}{(n+1)^2} \to 0 \text{ as } k, \ell \to \infty$$

  so the sequence is Cauchy.

- The sequence does not converge though, since the "obvious" limiting signal $f(n) = \frac{1}{n+1}$ for $n \geq 0$ does not belong to $\mathsf{c}_{\text{fin}}(\mathbb{Z}_{\geq 0};\mathbb{F})$.

Completions (and not) of $\mathsf{c}_{\mathrm{fin}}(\mathbb{Z}_{\geq 0}; \mathbb{F})$

- If a normed vector space is incomplete, one can complete it. The resulting complete space depends on the norm you use.

Theorem 2.1 (Complete sequence spaces).

(i) The completion of $\mathsf{c}_{\mathrm{fin}}(\mathbb{Z}_{\geq 0}; \mathbb{F})$ in the norm $\|\cdot\|_p$ for $p \in [1, \infty)$ is

$$\ell^p(\mathbb{Z}_{\geq 0}; \mathbb{F}) = \left\{ f : \mathbb{Z}_{\geq 0} \to \mathbb{F} \;\middle|\; \|f\|_p < \infty \right\}.$$

(ii) The completion of $\mathsf{c}_{\mathrm{fin}}(\mathbb{Z}_{\geq 0}; \mathbb{F})$ in the norm $\|\cdot\|_\infty$ is

$$\mathsf{c}_0(\mathbb{Z}_{\geq 0}; \mathbb{F}) = \left\{ f : \mathbb{Z}_{\geq 0} \to \mathbb{F} \;\middle|\; \lim_{n \to \infty} f(n) = 0 \right\}.$$

(iii) The space $\ell^\infty(\mathbb{Z}_{\geq 0}; \mathbb{F}) = \left\{ f : \mathbb{Z}_{\geq 0} \to \mathbb{F} \;\middle|\; \sup_{n \geq 0} |f(n)| < \infty \right\}$ with norm $\|\cdot\|_\infty$ is a Banach space.

Operators on vector spaces

- An ***operator*** is a fancy name for a mapping $A : \mathsf{U} \to \mathsf{Y}$ between vector spaces U and Y. We will assume that $A(0) = 0$; if this doesn't hold, just subtract off $A(0)$ and redefine A.
- If U and Y are Banach spaces, we can measure the "size" of A by comparing the relative size of inputs u and outputs $A(u)$

Definition 2.5 (Boundedness of operators). An operator $A : \mathsf{U} \to \mathsf{Y}$ is ***bounded*** if there exists $L \geq 0$ such that $\|A(u)\|_{\mathsf{Y}} \leq L\|u\|_{\mathsf{U}}$ for all $u \in \mathsf{U}$. In this case, the least upper bound on this ratio, given by

$$\|A\|_{\mathsf{U}\to\mathsf{Y}} \triangleq \sup_{u\in\mathsf{U}\setminus\{0\}} \frac{\|A(u)\|_{\mathsf{Y}}}{\|u\|_{\mathsf{U}}}$$

is called the ***induced norm*** or ***gain*** of A.

Bounded operators

- With the induced norm $\|\cdot\|_{\mathsf{U}\to\mathsf{Y}}$, the set of all bounded operators between two Banach spaces is ***itself*** a Banach space!
- Even more, it is an ***algebra***, because we can compose two operators $A, B : \mathsf{V} \to \mathsf{V}$ via the formula $(A \circ B)(v) = A(B(v))$
- ***Crucial in robust control:*** A, B bounded $\implies A \circ B$ bounded!

Lemma 2.1 (Induced norms are submultiplicative). If A, B are bounded operators on V, then $\|A \circ B\|_{\mathsf{V}\to\mathsf{V}} \leq \|A\|_{\mathsf{V}\to\mathsf{V}} \cdot \|B\|_{\mathsf{V}\to\mathsf{V}}$.

"Norm of the product is less than the product of the norms"

Linear operators on vector spaces

As is always the case, linearity is of special importance.

Definition 2.6 (Linear operators). Let U and Y be Banach spaces over $\mathbb{F}$. A mapping $A : \mathsf{U} \to \mathsf{Y}$ is a ***linear operator*** if it is

1. *distributive:* $A(u_1 + u_2) = A(u_1) + A(u_2)$ for all $u_1, u_2 \in \mathsf{U}$, and
2. *homogeneous:* $A(\alpha u) = \alpha A(u)$ for all $u \in \mathsf{U}$ and $\alpha \in \mathbb{F}$.

Properties of linear operators:

- subspaces are mapped to subspaces
- boundedness is *equivalent* to Lipschitz continuity
- linear operators are ***always bounded*** when U, Y are *finite-dimensional*

Examples of linear operators

- A matrix $A \in \mathbb{C}^{m \times n}$ defines a (bounded) linear operator $f_A : \mathbb{C}^n \to \mathbb{C}^m$ via matrix-vector mult. $f_A(x) = Ax$
- For a fixed $A \in \mathbb{R}^{n \times n}$ the ***(continuous-time) Lyapunov operator*** $\mathsf{Lyap} : \mathbb{S}^n \to \mathbb{S}^n$ by defined by $\mathsf{Lyap}(X) = A^\mathsf{T} X + XA$ is a (bounded) linear operator
- With the norm $\|\cdot\|_\infty$ on the domain/codomain

$$\mathcal{I} : \mathsf{C}^0([0,T];\mathbb{R}) \to \mathsf{C}^1([0,T];\mathbb{R}), \quad \mathcal{I}(f)(x) \triangleq \int_0^x f(\xi)\,\mathrm{d}\xi$$

  defines a bounded linear operator. The derivative mapping

$$\mathcal{D} : \mathsf{C}^1([0,T];\mathbb{R}) \to \mathsf{C}^0([0,T];\mathbb{R}), \qquad \mathcal{D}(f)(x) \triangleq \frac{\mathrm{d}f}{\mathrm{d}x}(x)$$

  is also a linear operator, but is not bounded.

The singular value decomposition

Every matrix $A \in \mathbb{C}^{m \times n}$ admits a ***singular value decomposition***

$$A = U \Sigma V^*, \quad U^* U = I_m, \quad V^* V = I_n$$

where $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ are unitary matrices and $\Sigma \in \mathbb{R}^{m \times n}$ is the matrix of ***singular values***, which depending on the relative sizes of m and n has the form

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_n \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & \vdots & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \sigma_m & 0 & \cdots & 0 \end{bmatrix}$$

Observations:

- $AA^* = U \Sigma \Sigma^\mathsf{T} U^* \implies U$ are the eigenvectors of AA^*
- $A^* A = V \Sigma^\mathsf{T} \Sigma V^* \implies V$ are the eigenvectors of $A^* A$
- σ_i^2 are the eigenvalues of AA^* (or $A^* A$).

The singular value decomposition

Proposition 2.1 (Properties of singular values). Let $A \in \mathbb{F}^{m \times n}$ and let $p = \min\{m, n\}$.

(i) The singular values of A are real, nonnegative, and ordered as $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_p(A) \geq 0 = \cdots = 0$

(ii) $\sigma_i(A) = \sigma_i(A^*)$ for $i \in \{1, \ldots, p\}$.

(iii) the number of non-zero singular values is equal to $\operatorname{rank}(A)$.

Warning: Singular values ***are not*** eigenvalues.

$$A = \begin{bmatrix} 1 & 10^6 \\ 0 & 1 \end{bmatrix}, \quad \operatorname{eig}(A) = \{1, 1\}, \quad \sigma_1(A) \approx 10^6 \ldots$$

The singular value decomposition

The SVD yields “input directions” and “output directions”

$$A = U\Sigma V^* = \underbrace{\begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}}_{\text{output directions}} \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & \vdots & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \sigma_m & 0 & \cdots & 0 \end{bmatrix} \underbrace{\begin{bmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_n^* \end{bmatrix}}_{\text{input directions}}$$

- If $x = v_k$, then $Ax = \sigma_k u_k$, so $\|Ax\|_2 = \sigma_k$!
- Singular values σ_k measure “gain” on the v_k–u_k axis

$$\textstyle\sum_k \sigma_k u_k (V^* x)_k \longleftarrow \boxed{A} \longleftarrow x$$

Induced norms and singular values

For $A \in \mathbb{C}^{m\times n}$, consider the following two ways of measuring its size

$$\|A\|_{\mathsf{F}} \triangleq \sqrt{\operatorname{trace}(A^*A)}, \qquad \|A\|_2 \triangleq \|f_A\|_{2\to 2} = \sup_{x\in\mathbb{C}^n\setminus\{0\}} \frac{\|Ax\|_2}{\|x\|_2}.$$

Are $\|A\|_{\mathsf{F}}$ and $\|A\|_2$ related? Yes, using ***singular values***.

Theorem 2.2 (2-norm and Frobenius norm). For $A \in \mathbb{F}^{m\times n}$

$$\|A\|_{\mathsf{F}} = \sqrt{\sum_{k=1}^{\min\{m,n\}} \sigma_k^2(A)}, \qquad \|A\|_2 = \sigma_1(A) = \sigma_{\max}(A).$$

As an immediate consequence, it always holds that $\|A\|_2 \leq \|A\|_{\mathsf{F}}$.

Proof of Theorem 2.2

The Frobenius norm formula is immediate. For the 2-norm, we compute that

$$\|A\|_2^2 = \|f_A\|_{2\to 2}^2 = \sup_{v \in \mathbb{F}^m \setminus \{0\}} \frac{\|Av\|_2^2}{\|v\|_2^2} = \sup_{\|v\|_2 = 1} \|Av\|_2^2 = \sup_{v^* v = 1} v^* A^* A v$$

Since A^*A is symmetric, there exists a unitary matrix $U \in \mathbb{C}^{n \times n}$ such that $A^*A = U\Gamma U^*$ where $\Gamma = \mathrm{diag}(\sigma_1(A)^2, \ldots, \sigma_n(A)^2)$. Therefore

$$\begin{aligned} \|A\|_2^2 = \sup_{v^*v=1} v^* U \Gamma U^* v = \sup_{u^*u=1} u^* \Gamma u = \sup_{u^*u=1} \sum_{k=1}^{n} \Gamma_{kk} |u_k|^2 &\leq \max_k \Gamma_{kk} \\ &= \sigma_{\max}(A)^2 \end{aligned}$$

where we have used the fact that since U is unitary, $\|U^* v\|_2 = \|u\|_2 = 1$. Therefore, $\|A\|_2 \leq \sigma_{\max}(A)$. To show that this is the least upper bound, note that by selecting $u = \mathbb{e}_1 = (1, 0, \ldots, 0)$ we obtain

$$u^* \Gamma u = \sigma_{\max}(A)^2$$

and therefore $v \triangleq U\mathbb{e}_1$ is the (unique) maximizer of the original problem. •

3. Mathematical Optimization and Linear Matrix Inequalities (LMIs)

Mathematical optimization

- A mathematical optimization problem is generally notated as

$$\underset{x \in \mathsf{X}}{\text{minimize}} \;\; f(x) \qquad \text{subject to} \qquad x \in \mathsf{C}.$$

The ingredients are:

1. an ambient vector space X of ***all*** candidate decisions for x
2. a set $\mathsf{C} \subseteq \mathsf{X}$ of ***feasible*** decisions
3. a ***cost function*** $f : \mathsf{C} \to \mathbb{R}$ quantifying the cost $f(x)$ of each feasible decision $x \in \mathsf{C}$

- Sometimes we do not care about cost, and simply want to find *any* feasible decision. Then one often writes

$$\text{find } \; x \qquad \text{subject to} \qquad x \in \mathsf{C}.$$

Examples of mathematical optimization problems

- General nonlinear program

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \;\; f(x) \qquad \text{subject to} \qquad g(x) \leq 0, \quad h(x) = 0.$$

- Quadratic program (e.g., least squares, regression, MPC)

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \;\; \tfrac{1}{2}x^{\mathsf{T}}Qx + c^{\mathsf{T}}x \qquad \text{subject to} \qquad A_1x \leq b_1, \quad A_2x = b_2.$$

- ***Semidefinite program*** (SDP)

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \;\; c^{\mathsf{T}}x \qquad \text{subject to} \qquad \underbrace{A_0 + \sum\nolimits_{i=1}^{n} A_i x_i \succeq 0}_{\text{Matrix is positive semidefinite}}$$

We will study SDPs in detail shortly.

Basic questions for mathematical optimization

$$\underset{x \in \mathsf{X}}{\text{minimize}} \;\; f(x) \qquad \text{subject to} \qquad x \in \mathsf{C}.$$

(i) Does there exist *any* $x \in \mathsf{C}$? (***feasibility***)

(ii) What is the least possible cost $f_{\mathrm{opt}} = \inf_{x \in \mathsf{C}} f(x)$?

- if $\mathsf{C} = \emptyset$, then we define $f_{\mathrm{opt}} = +\infty$
- if f is not bounded below on C, then $f_{\mathrm{opt}} = -\infty$

(iii) Does there exist $x \in \mathsf{C}$ such that $f(x) = f_{\mathrm{opt}}$? If so, x is a ***minimizer*** and the *minimum is attained*, so $f_{\mathrm{opt}} = \min_{x \in \mathsf{C}} f(x)$.

(iv) If the minimum is attained, can we characterize the ***optimal set***

$$\mathsf{X}_{\mathrm{opt}} \triangleq \{x \in \mathsf{X} \mid x \text{ is a minimizer}\} = \underset{x \in \mathsf{C}}{\operatorname{argmin}} \, f(x).$$

When can we answer these questions?

- In general, answering any of the questions (i)–(iv) is computationally intractable – optimization problems are not typically solvable!
- In order to obtain tractable classes of problems, additional assumptions must be placed on the cost f and the feasible set C

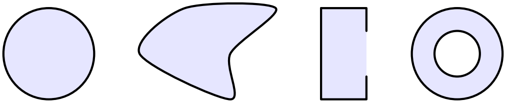
> A broad and practical property to impose on both f and C is ***convexity***.

- Convexity will provide us with theoretical guarantees, and powerful algorithms have been developed for solving convex optimization problems (we will exploit these, but not study them).

Convex sets

Definition 3.1 (Convex set). A subset $\mathsf{C} \subseteq \mathsf{V}$ of a vector space V is ***convex*** if $\alpha v_1 + (1-\alpha)v_2 \in \mathsf{C}$ for all $v_1, v_2 \in \mathsf{C}$ and $\alpha \in [0,1]$.

"The line segment between any two points is contained in the set"



Examples of convex sets:

- Linear equalities $\{x \mid Ax = b\}$ and inequalities $\{x \mid Ax \leq b\}$
- ϵ-norm ball centered at x_0: $\mathscr{B}_\epsilon(x_0) \triangleq \{x \in \mathsf{V} \mid \|x - x_0\| \leq \epsilon\}$

Obvious, but very important! The intersection of convex sets is a convex set, and the interior of a convex set is a convex set.

Affine mappings

- We will often want to express sets as mappings of other sets
- What kind of mappings play nice with convexity?

Definition 3.2 (Affine map). A map $f : \mathsf{V} \to \mathsf{W}$ between vector spaces V, W over $\mathbb{F}$ is ***affine*** if for all $v_1, v_2 \in \mathsf{V}$ and all $\alpha \in \mathbb{F}$

$$f(\alpha v_1 + (1-\alpha)v_2) = \alpha f(v_1) + (1-\alpha) f(v_2).$$

Properties of affine maps:

- Affine maps are *almost* linear; every affine mapping is of the form $f(v) = A(v) + b$ for some linear operator $A : \mathsf{V} \to \mathsf{W}$ and $b \in \mathsf{W}$.
- if $\mathsf{C} \subset \mathsf{V}$ is cvx, then the image $f(\mathsf{C}) = \{f(v) \mid v \in \mathsf{C}\}$ is cvx
- if $\mathsf{C} \subset \mathsf{W}$ is cvx, then the preimage $f^{-1}(\mathsf{C}) = \{v \mid f(v) \in \mathsf{C}\}$ is cvx

Convex optimization problems

- A ***convex*** optimization problem with affine cost is

$$\underset{x \in \mathsf{X}}{\text{minimize}} \;\; f(x) \qquad \text{subject to} \qquad x \in \mathsf{C}.$$

where C is a convex set and $f : \mathsf{C} \to \mathbb{R}$ is an affine mapping.

Key fact: Any ***locally*** optimal solution $x^\star \in \mathsf{C}$ is ***globally*** optimal.

- **Warning:** convexity alone does not guarantee feasibility, a finite optimal cost, the existence or uniqueness of an optimal solution, or the existence of an efficient algorithm for solving the problem!
- ***Luckily***, our problems of interest will generally be "nice enough", and will not unduly suffer from these issues.

The vector space of symmetric matrices

- **Recall**: The vector spaces $\mathbb{H}^n$ and $\mathbb{S}^n$ of Hermitian and symmetric matrices are both Hilbert spaces with inner product

$$\langle X, Y\rangle_{\mathsf{F}} = \operatorname{trace}(X^*Y) = \operatorname{trace}(XY)$$

- You probably already know the following result.

Lemma 3.1 (Properties of Hermitian Matrices). If $A \in \mathbb{H}^n$ then

(i) the eigenvalues of A are real, i.e., $\operatorname{eig}(A) \subset \mathbb{R}$;

(ii) the eigenvectors of A are orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_2$ on $\mathbb{C}^n$;

(iii) there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $A = U\Lambda U^*$ where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$.

Definite matrices

Definition 3.3 (Definite matrices). A matrix $A \in \mathbb{H}^n$ is

(i) **positive semidefinite ($A \succeq 0$)** if $x^* A x \geq 0$ for all $x \in \mathbb{C}^n$;

(ii) **positive definite ($A \succ 0$)** if $x^* A x > 0$ for all $x \in \mathbb{C}^n \setminus \{0\}$;

(iii) **negative semidefinite ($A \preceq 0$)** if $-A$ is positive semidefinite;

(iv) **negative definite ($A \prec 0$)** if $-A$ is positive definite;

(v) **indefinite** othwerwise.

In the real symmetric case, we let $\mathbb{S}^n_{\geq 0}, \mathbb{S}^n_{>0}, \mathbb{S}^n_{\leq 0}, \mathbb{S}^n_{<0} \subset \mathbb{S}^n$ denote the sets of positive semidefinite, positive definite, negative semidefinite, and negative definite matrices.

- Note that we are ***only*** considering symmetric matrices. You could define the same properties for non-symmetric matrices, but there is apparently little use in doing so.

Positive definiteness and eigenvalues

Definiteness is intimately related to *eigenvalues*.

Proposition 3.1 (Definite matrices). All eigenvalues of $A \in \mathbb{H}^n$ are nonnegative (resp. positive) if and only if $A \succeq 0$ (resp. $A \succ 0$).

- Proof is very simple; use eigenvalue decomposition of A
- Equivalently, we can talk in terms of minimum/maximum eigenvalues:

$$
\begin{aligned}
A \succeq 0 \qquad &\iff \qquad \lambda_{\min}(A) \geq 0 \\
A \preceq 0 \qquad &\iff \qquad \lambda_{\max}(A) \leq 0 \\
A \succ 0 \qquad &\iff \qquad \lambda_{\min}(A) > 0 \\
A \prec 0 \qquad &\iff \qquad \lambda_{\max}(A) < 0.
\end{aligned}
$$

Positive definiteness and matrix decomposition

Proposition 3.2 (PSD Decomposition). Let $A \in \mathbb{H}^n$. Then $A \succeq 0$ if and only if there exists B such that $A = B^*B$

- When $A \in \mathbb{S}^n$, one can of course take B to be real
- While there is no unique choice of B, there is a special choice called the ***square root*** of A

Proposition 3.3 (Square Root of a PSD Matrix). Let $A \in \mathbb{H}^n$. Then $A \succeq 0$ if and only if there exists a unique matrix $A^{1/2} \succeq 0$ such that $A = A^{1/2}A^{1/2}$.

Operations which preserve definiteness

You can perform certain transformations on definite matrices which preserve definiteness; very useful

- ***Conic combination:*** If $A_1, A_2 \succeq 0$ and $\alpha_1, \alpha_2 \geq 0$, then $\alpha_1 A_1 + \alpha_2 A_2 \succeq 0$.
- ***Inversion:*** $A \succ 0$ if and only if $A^{-1} \succ 0$
- ***Similarity Transform:*** Given a nonsingular $T \in \mathbb{R}^{n \times n}$, A is positive (semi)definite if and only if $T^{-1}AT$ is positive (semi)definite
- ***Congruence Transform:*** Given nonsingular $T \in \mathbb{R}^{n \times n}$, A is positive (semi)definite if and only if $T^{\mathsf{T}}AT$ is positive (semi)definite.
- ***Projection Result:*** Given full column rank $T \in \mathbb{R}^{n \times \bullet}$, if A is positive definite then $T^{\mathsf{T}}AT$ is positive definite.

Definiteness and the trace

- Recall that for $X \in \mathbb{R}^{n\times n}$ we have that $\mathrm{trace}(X) = \sum_{i=1}^{n} \lambda_i(X)$ where $\{\lambda_i(X)\}_{i=1}^{n}$ are the eigenvalues of X. Obviously:

$$\boxed{X \succeq 0 \quad \Longrightarrow \quad \mathrm{trace}(X) \geq 0}$$

- Moreover, the trace has the following ***cyclic property***

$$\mathrm{trace}(XYZ) = \mathrm{trace}(ZXY) = \mathrm{trace}(YZX)$$

- If $Y \succeq 0$ and $F \preceq 0$, then

$$\mathrm{trace}(YF) = \mathrm{trace}(Y^{\frac{1}{2}}Y^{\frac{1}{2}}F) = \mathrm{trace}(\underbrace{Y^{\frac{1}{2}}FY^{\frac{1}{2}}}_{\preceq 0}) \leq 0$$

$$\boxed{Y \succeq 0,\ F \preceq 0 \quad \Longrightarrow \quad \langle Y, F\rangle_{\mathbb{S}^n} = \mathrm{trace}(YF) \leq 0}$$

The Schur complement

Lemma 3.2 (Schur Complement Lemma). Let $Q \in \mathbb{S}^p$, $S \in \mathbb{R}^{p\times m}$, and $R \in \mathbb{S}^m$. The following statements are equivalent:

(i) $\begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \prec 0$

(ii) $Q \prec 0$ and $R - S^\mathsf{T} Q^{-1} S \prec 0$

(iii) $R \prec 0$ and $Q - S R^{-1} S^\mathsf{T} \prec 0$.

- An ***endlessly useful*** result for block matrices
- Various semidefinite versions hold as well, e.g., if $Q \prec 0$ then $\begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \preceq 0$ if and only if $R - S^\mathsf{T} Q^{-1} S \preceq 0$

(i) $\Rightarrow$ (ii) used for *reducing dimension* of a block matrix, while (ii) $\Rightarrow$ (i) is useful for ***linearizing*** the nonlinear inequality $R - S^\mathsf{T} Q^{-1} S \prec 0$.

Convexity of $\mathbb{S}^n_{\geq 0}$

Proposition 3.4. The sets $\mathbb{S}^n_{\geq 0}, \mathbb{S}^n_{>0}, \mathbb{S}^n_{<0}, \mathbb{S}^n_{\leq 0} \subset \mathbb{S}^n$ are all convex.

Let $X_1, X_2 \in \mathbb{S}^n_{\geq 0}$, $\alpha \in [0,1]$, and $x \in \mathbb{R}^n$. We compute

$$x^{\mathsf{T}}(\alpha X_1 + (1-\alpha)X_2)x = \alpha x^{\mathsf{T}} X_1 x + (1-\alpha) x^{\mathsf{T}} X_2 x \geq 0$$

since each term is nonnegative, so $\alpha X_1 + (1-\alpha)X_2 \in \mathbb{S}^n_{\geq 0}$. •

In short, this means we can efficiently optimize over these sets; this leads to a class of optimization problems called ***semidefinite programs***.

Linear matrix inequalities (LMIs)

Definition 3.4 (LMI). Let X be a finite-dimensional Hilbert space over $\mathbb{R}$ and let $F : \mathsf{X} \to \mathbb{S}^n$ be an affine mapping. We call the inequality $F(x) \preceq 0$ a ***linear matrix inequality*** or ***LMI***, and the inequality $F(x) \prec 0$ a ***strict LMI***.

(n.s.d. inequality is just a convention; doesn't matter).

Proposition 3.5 (LMIs define convex sets). The set of points satisfying an LMI or strict LMI is convex.

Proof: $\mathbb{S}^n_{\preceq 0}$ is a convex set, and the preimage of a convex set under an affine map is convex, so $\{x \in \mathsf{X} \mid F(x) \preceq 0\}$ is convex. •

- ***Note:*** multiple simultaneous LMIs $F_1(x) \preceq 0, \ldots, F_N(x) \preceq 0$ all together also define an LMI (why?)

LMI feasibility and linear SDP problems

Definition 3.5 (LMI Feasibility and Linear SDP). A ***LMI feasibility problem*** is the convex feasibility problem

$$\text{find } x \in \mathsf{X} \qquad \text{subject to} \qquad F(x) \preceq 0$$

where $F(x) \preceq 0$ is an LMI. A ***linear semidefinite program (SDP)*** is the convex optimization problem

$$\underset{x \in \mathsf{X}}{\text{minimize}} \;\; \varphi(x) \qquad \text{subject to} \qquad F(x) \preceq 0$$

where $F(x) \preceq 0$ is a LMI and $\varphi : \mathsf{X} \to \mathbb{R}$ is a linear map.

- ***Short story:*** Analytical solutions very rare, but we can (usually) numerically compute accurate solutions to these problems.
- You can also add affine ***equality*** constraints without issue.

Remarks on writing LMIs and SDPs

- LMIs often naturally appear with ***matrix variables***. For example,

$$F(X) = \sum_{k=1}^{r} A_k X B_k^{\mathsf{T}} + B_k X A_k^{\mathsf{T}} + Q_k + Q_k^{\mathsf{T}} \preceq 0$$

  where $X \in \mathsf{X} \triangleq \mathbb{S}^m$ and A_k, B_k, Q_k are matrices of appropriate sizes.

- This is a *perfectly acceptable representation*: there is no need to play around with bases for $\mathbb{S}^m$ to rewrite the problem, nor is there a need to translate the problem to standard forms that you may find in other references. The map F is affine, and that's all that matters.

- In this case, you will typically see linear costs expressed as $\varphi(x) = \langle C, X \rangle_{\mathbb{S}^n} = \mathrm{trace}(CX)$ for some $C \in \mathbb{S}^n$ (Riesz Theorem).

Example: minimum induced norm

- Let $A_0, \ldots, A_N \in \mathbb{S}^n$ and consider the problem

$$\underset{x \in \mathbb{R}^N}{\text{minimize}} \quad \|A(x)\|_2, \qquad \text{where } A(x) = A_0 + \sum_{i=1}^{N} x_i A_i \in \mathbb{S}^n.$$

- ***Key observation***: From Theorem 2.2, $\|A\|_2^2 = \sigma_{\max}(A)^2$, which (by definition) is the maximum eigenvalue of $A^{\mathsf{T}}A \succeq 0$. We have that

$$\begin{aligned} \|A\|_2 < \gamma \;\Leftrightarrow\; \lambda_{\max}(A^{\mathsf{T}}A) < \gamma^2 \quad &\Longleftrightarrow \quad \lambda_{\max}(A^{\mathsf{T}}A - \gamma^2 I_n) < 0 \\ &\Longleftrightarrow \quad A^{\mathsf{T}}A - \gamma^2 I_n \prec 0 \\ &\Longleftrightarrow \quad \begin{bmatrix} \gamma I_n & A \\ A^{\mathsf{T}} & \gamma I_n \end{bmatrix} \succ 0 \end{aligned}$$

- So the problem can be equivalently written as the SDP

$$\underset{\gamma \geq 0, x \in \mathbb{R}^N}{\text{minimize}} \quad \gamma \qquad \text{subject to} \quad \begin{bmatrix} \gamma I_n & A(x) \\ A(x)^{\mathsf{T}} & \gamma I_n \end{bmatrix} \succ 0$$

Solving SDPs

- Some excellent options available for `MATLAB`
- Two ingredients: a ***parser*** (front-end to make your life easy) and a ***solver*** (algorithm which does the computation)
- *My two cents*: use `YALMIP` as your parser, and have `SDPT3` and `SeDuMi` installed as two different solvers to try out.
- You can also try `cvx`, which is a popular and flexible platform for optimization. I have personally found the parser to be less reliable than `YALMIP`, particularly for larger problems.

Example: Minimum induced norm via YALMIP

```
%% Define Problem Data
n = 5; N = 7; A = randn(n,n,N+1);
for k=1:N; A(:,:,k)=A(:,:,k) + A(:,:,k)'; end

%% Define SDP Problem
gamma = sdpvar(1,1); x = sdpvar(N,1);
Ax = A(:,:,1);
for k=1:N; Ax = Ax + x(k)*A(:,:,k+1); end
M = [gamma*eye(n),Ax;Ax',gamma*eye(n)];
Constraints = [gamma ≥ 0, M ≥ eye(2*n)];
Cost = gamma;

%% Solve
options = sdpsettings('solver','sdpt3','verbose',1);
sol = optimize(Constraints,Cost,options);

value(x) %print value
```

Other comments on LMIs

- ***Strict vs. non-strict LMIs:*** For both numerical and theoretical reasons, strict LMIs are typically preferred to non-strict LMIs. Most parsers however accept only non-strict LMIs. In code, one therefore replaces $F(x) \prec 0$ with $F(x) \preceq -\epsilon I_n$ for some small $\epsilon > 0$.

- ***Linear LMIs:*** If the function F is a linear function (as opposed to affine), then $F(x) \prec 0$ is feasible if and only if $F(x) \preceq -I$ is feasible. Additionally, note that if x is feasible, then $F(\alpha x) = \alpha F(x) \prec 0$ for all $\alpha > 0$, so αx is a solution. Numerically, things can now go crazy, because solvers can generate solutions with arbitrarily large norms. To fix this, one should additionally constrain (or minimize) the norm of x.

The adjoint of a linear operator

Definition 3.6 (Adjoint). Let X, Y be Hilbert* spaces over $\mathbb{F}$ and let $F : \mathsf{X} \to \mathsf{Y}$ be a bounded linear operator. The ***adjoint*** of F is the mapping $F^{\mathrm{adj}} : \mathsf{Y} \to \mathsf{X}$ satisfying

$$\langle y, F(x) \rangle_{\mathsf{Y}} = \langle F^{\mathrm{adj}}(y), x \rangle_{\mathsf{X}}, \qquad x \in \mathsf{X},\ y \in \mathsf{Y}.$$

- One can show that F^{adj} always exists, is unique, and *is itself* a bounded linear operator with induced norm $\|F^{\mathrm{adj}}\|_{\mathsf{Y}\to\mathsf{X}} = \|F\|_{\mathsf{X}\to\mathsf{Y}}$.
- **Example:** If $A \in \mathbb{C}^{m\times n}$ and $F(x) = Ax$, then

$$\begin{aligned}\langle y, F(x) \rangle_2 &= \langle y, Ax \rangle_2 = y^*(Ax) \\ &= (A^*y)^*x = \langle A^*y, x \rangle_2 = \langle F^{\mathrm{adj}}(y), x \rangle_2\end{aligned}$$

so $F^{\mathrm{adj}}(y) = A^*y$; the adjoint is defined by the Hermitian transpose A^*

*A Hilbert space is a complete inner product space.

Example: adjoint of the Lyapunov operator

- For a fixed $A \in \mathbb{R}^{n\times n}$, recall that the mapping $\mathsf{Lyap} : \mathbb{S}^n \to \mathbb{S}^n$ by defined by $\mathsf{Lyap}(X) = A^\mathsf{T} X + XA$ is a (bounded) linear operator.

- For any $X, Y \in \mathbb{S}^n$ we have that

$$\begin{aligned}
\langle Y, \mathsf{Lyap}(X)\rangle_{\mathbb{S}^n} &= \mathrm{trace}(Y^\mathsf{T}(A^\mathsf{T} X + XA)) \\
&= \mathrm{trace}(Y(A^\mathsf{T} X + XA)) \\
&= \mathrm{trace}(YA^\mathsf{T} X) + \mathrm{trace}(YXA) \\
&= \mathrm{trace}(YA^\mathsf{T} X) + \mathrm{trace}(AYX) \\
&= \mathrm{trace}((AY + YA^\mathsf{T})X) \\
&= \langle AY + YA^\mathsf{T}, X\rangle_{\mathbb{S}^n}
\end{aligned}$$

from which we conclude that $\mathsf{Lyap}^{\mathrm{adj}}(Y) = AY + YA^\mathsf{T}$.

Example: adjoint of the convolution operator

- For a causal FD-LTI system $(A, B, C, \mathbb{0})$ with impulse response $m(t) = Ce^{At}B1(t) \in \mathbb{R}^{p\times m}$, the linear ***convolution operator*** is

$$\mathsf{Conv}_m(u)(t) = \int_{-\infty}^{\infty} m(t-\tau)u(\tau)\,\mathrm{d}\tau$$

- Bounded iff the system is BIBO stable $\Leftrightarrow \lim_{t\to\infty} m(t) = 0$

- For any signals $u(t) \in \mathbb{R}^m$ and $z(t) \in \mathbb{R}^p$, we have

$$\begin{aligned}\langle z, \mathsf{Conv}_m(u)\rangle_{\mathcal{L}_2} &= \int_{-\infty}^{\infty} z(t)^\mathsf{T} \int_{-\infty}^{\infty} m(t-\tau)u(\tau)\,\mathrm{d}\tau\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} u(\tau)^\mathsf{T} \int_{-\infty}^{\infty} m(-(\tau-t))^\mathsf{T} z(t)\,\mathrm{d}t\,\mathrm{d}\tau = \langle u, \mathsf{Conv}_n(z)\rangle_{\mathcal{L}_2}\end{aligned}$$

- Adjoint is a conv. operator of the ***anti-causal*** FD-LTI system $(A^\mathsf{T}, C^\mathsf{T}, B^\mathsf{T}, \mathbb{0})$ with impulse response $n(t) = B^\mathsf{T} e^{-A^\mathsf{T}t} C^\mathsf{T} 1(-t)$.

SDP duality

- We now consider the formulation of ***dual problems*** for semidefinite programs; while the exposition is self-contained, previous background in duality theory for linear programming would be beneficial

- Consider the "***primal***" linear SDP

$$\underset{x \in \mathsf{X}}{\text{minimize}}\ \langle c, x\rangle_{\mathsf{X}} \qquad \text{subject to} \qquad F_0 + F_1(x) \preceq 0$$

where $c \in \mathsf{X}$, $F_0 \in \mathbb{S}^n$ and $F_1 : \mathsf{X} \to \mathbb{S}^n$ is a (bounded) linear operator.

- The optimal value p_{opt} of this problem is of course

$$p_{\text{opt}} = \inf_{x \in \mathsf{C}} \langle c, x\rangle_{\mathsf{X}}$$

where $\mathsf{C} \triangleq \{\xi \in \mathsf{X} \mid F_0 + F_1(\xi) \preceq 0\}$ denotes the feasible set

SDP duality

- The ***(conic) Lagrangian*** $\mathcal{L}$ of this primal SDP is the function $\mathcal{L} : \mathsf{X} \times \mathbb{S}^n \to \mathbb{R}$ defined by

$$\mathcal{L}(x, Y) = \langle c, x \rangle_{\mathsf{X}} + \langle Y, F_0 + F_1(x) \rangle_{\mathbb{S}^n}$$

- $Y \in \mathbb{S}^n$ is the ***dual variable*** associated with the LMI constraint
- **Recall:** If $Y \succeq 0$ and $\mathcal{F} \preceq 0$, then

$$\langle Y, \mathcal{F} \rangle_{\mathbb{S}^n} = \mathrm{trace}(Y\mathcal{F}) \leq 0$$

- **Fact:** For any $\mathcal{F} \in \mathbb{S}^n$ we have

$$\sup_{Y \succeq 0} \langle Y, \mathcal{F} \rangle_{\mathbb{S}^n} = \begin{cases} 0 & \text{if } \mathcal{F} \preceq 0 \\ +\infty & \text{if otherwise} \end{cases}$$

SDP duality

- It follows immediately that

$$\sup_{Y \succeq 0} \mathcal{L}(x, Y) = \langle c, x \rangle_{\mathsf{X}} + \sup_{Y \succeq 0} \langle Y, F_0 + F_1(x) \rangle_{\mathbb{S}^n}$$

$$= \begin{cases} \langle c, x \rangle_{\mathsf{X}} & \text{if } x \in \mathsf{C} \\ +\infty & \text{if } x \notin \mathsf{C} \end{cases}$$

- We conclude that

$$\inf_{x \in \mathsf{X}} \sup_{Y \succeq 0} \mathcal{L}(x, Y) = \inf_{x \in \mathsf{C}} \langle c, x \rangle_{\mathsf{X}} = p_{\mathrm{opt}}$$

The *maximin* problem $\inf_{x \in \mathsf{X}} \sup_{Y \succeq 0} \mathcal{L}(x, Y)$ is ***equivalent*** to the primal problem!

SDP duality

Lemma 3.3 (Max-Min Inequality). Let $\mathscr{X}, \mathscr{Y}$ be any sets and let $f : \mathscr{X} \times \mathscr{Y} \to \mathbb{R}$. Then

$$\sup_{y \in \mathscr{Y}} \inf_{x \in \mathscr{X}} f(x, y) \leq \inf_{x \in \mathscr{X}} \sup_{y \in \mathscr{Y}} f(x, y)$$

- The ***Lagrange dual*** SDP associated with the primal SDP is obtained by interchanging $\sup$ and $\inf$ in our maximin problem

$$d_{\mathrm{opt}} = \sup_{Y \succeq 0} \underbrace{\inf_{x \in \mathsf{X}} \mathcal{L}(x, Y)}_{\triangleq g(Y)} \leq \inf_{x \in \mathsf{X}} \sup_{Y \succeq 0} \mathcal{L}(x, Y) = p_{\mathrm{opt}}$$

or simply $d_{\mathrm{opt}} = \sup_{Y \succeq 0} g(Y)$

- We therefore always have so-called ***weak duality:*** $d_{\mathrm{opt}} \leq p_{\mathrm{opt}}$

SDP duality

- To compute the dual function, note that

$$\begin{aligned}\mathcal{L}(x,Y) &= \langle c, x\rangle_{\mathsf{X}} + \langle Y, F_0 + F_1(x)\rangle_{\mathbb{S}^n} \\ &= \langle c, x\rangle_{\mathsf{X}} + \langle Y, F_0\rangle_{\mathbb{S}^n} + \langle Y, F_1(x)\rangle_{\mathbb{S}^n} \\ &= \langle c + F_1^{\mathrm{adj}}(Y), x\rangle_{\mathsf{X}} + \langle F_0, Y\rangle_{\mathbb{S}^n}\end{aligned}$$

- We can now compute that

$$g(Y) = \inf_{x\in\mathsf{X}} \mathcal{L}(x,Y) = \begin{cases} -\infty & \text{if } c + F_1^{\mathrm{adj}}(Y) \neq 0 \\ \langle F_0, Y\rangle_{\mathbb{S}^n} & \text{if } c + F_1^{\mathrm{adj}}(Y) = 0. \end{cases}$$

- The dual problem is therefore

$$d_{\mathrm{opt}} = \sup_{Y\succeq 0,\ c+F_1^{\mathrm{adj}}(Y)=0} \langle F_0, Y\rangle_{\mathbb{S}^n}.$$

Example: The minimum may not be achieved

- Consider the example

$$\inf_{(x_1,x_2)\in\mathbb{R}^2} x_1 \quad \text{subject to} \quad \begin{pmatrix} x_1 & 1 \\ 1 & x_2 \end{pmatrix} \succeq 0.$$

- The PSD conditions are that $x_1 \geq 0$, $x_2 \geq 0$, and $x_1 x_2 \geq 1$; note also the problem is ***strictly*** feasible. For $\epsilon > 0$, set $x_1 = \epsilon$ and $x_2 = 1/\epsilon$. The LMI is satisfied, and $\lim_{\epsilon\to 0} p_{\text{opt}}(\epsilon) = 0$, but the optimal value ***is never achieved.***

It turns out the issue here is with the dual problem.

$$\sup_{Y\succeq 0} -2y_{12} \quad \text{subject to} \quad y_{11} = 1,\, y_{22} = 0$$

- Here, $d_{\text{opt}} = 0$ is achieved by $Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \succeq 0$, but the dual problem is not ***strictly*** feasible.

SDP and dual SDP

Primal SDP:

$$p_{\text{opt}} = \inf_{F_0+F_1(x)\preceq 0} \langle c, x\rangle_{\mathsf{X}}$$

Dual SDP:

$$d_{\text{opt}} = \sup_{Y\succeq 0,\ c+F_1^{\text{adj}}(Y)=0} \langle F_0, Y\rangle_{\mathbb{S}^n}.$$

Theorem 3.1 (Strong Duality). Suppose that the primal and dual SDP problems are both feasible.

(i) If the primal is strictly feasible, i.e., there exists x s.t. $F_0 + F_1(x) \prec 0$, then $p_{\text{opt}} = d_{\text{opt}}$ and the dual optimum is achieved by some $Y \succeq 0$.

(ii) If the dual is strictly feasible, i.e., there exists $Y \succ 0$ s.t. $c + F_1^{\text{adj}}(Y) = 0$, then $p_{\text{opt}} = d_{\text{opt}}$ and the primal optimum is achieved by some $x \in \mathsf{C}$.

(iii) If $p_{\text{opt}} = d_{\text{opt}}$, then any primal-feasible x and dual-feasible Y are optimal if and only if $(F_0 + F_1(x))Y = 0$.

Strong alternatives for LMIs

- Consider the case of a ***primal feasibility*** problem

Primal SDP:

$$p_{\rm opt} = \inf_{F_0 + F_1(x) \preceq 0} \quad 0$$

Dual SDP:

$$d_{\rm opt} = \sup_{Y \succeq 0,\ F_1^{\rm adj}(Y) = 0} \quad \langle F_0, Y \rangle_{\mathbb{S}^n}.$$

Theorem 3.2 (Strong Alternatives). Exactly one of the following statements is true:

(i) There exists $x \in \mathsf{X}$ such that $F_0 + F_1(x) \prec 0$.

(ii) There exists a non-zero $Y \succeq 0$ such that $F_1^{\rm adj}(Y) = 0$ and $\langle F_0, Y \rangle_{\mathbb{S}^n} \geq 0$.

Proof of Theorem 3.2

Suppose that ***both*** statements are true. Then by our previous arguments we have

$$\langle Y, F_0 + F_1(x)\rangle_{\mathbb{S}^n} \leq 0.$$

Moreover though, since Y is non-zero and $F_0 + F_1(x) \prec 0$, one can strengthen our previous argument to show that in fact

$$\langle Y, F_0 + F_1(x)\rangle_{\mathbb{S}^n} < 0.$$

This now implies that

$$\langle Y, F_0\rangle_{\mathbb{S}^n} + \underbrace{\langle F_1^{\mathrm{adj}}(Y), x\rangle_{\mathsf{X}}}_{=0} < 0$$

and therefore $\langle Y, F_0\rangle_{\mathbb{S}^n} < 0$, which is a contradiction with the second statement. Therefore, ***at most*** one of these statements is true. The remainder of the proof is omitted; see, e.g., Balakrishnan & Vandenberghe. •

Supplement: The geometry of $\mathbb{S}^n_{\geq 0}$

- $\mathbb{S}^n_{\geq 0}$ has a special structure: it is a ***proper convex cone*** in the vector space $\mathbb{S}^n$, and $\mathbb{S}^n_{>0} = \mathrm{interior}(\mathbb{S}^n_{\geq 0})$.
- All matrices $X \in \mathrm{interior}(\mathbb{S}^n_{\geq 0})$ have all positive eigenvalues, while all matrices $X \in \mathrm{bd}(\mathbb{S}^n_{\geq 0})$ have at least one eigenvalue equal to zero.
- The proper convex cone structure implies that $\mathbb{S}^n_{\geq 0}$ and $\mathbb{S}^n_{>0}$ can be used to define a ***partial order*** and a ***strict partial order on*** $\mathbb{S}^n$, which allows us to order (some) elements of the space
- Indeed, this is why we use the notation $A \succeq B$ to mean that $A - B \in \mathbb{S}^n_{\geq 0}$, and $A \succ B$ to mean that $A - B \in \mathbb{S}^n_{>0}$.
- You can go much deeper on the geometry of this space and look at faces, etc. ...we have everything we need though.

4. Lyapunov Stability and Inequalities

State-space LTI systems

- In ECE557 you learned all about the causal CT FD LTI model

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \in \mathbb{R}^n \tag{1}$$

  with state $x(t) \in \mathbb{R}^n$ and input $u(t) \in \mathbb{R}^m$.
- For now, we will side-step precisely what types of inputs and what kinds of solutions are being considered.

Two questions you answered in 557:

(i) What is exponential stability, and how do you check it for (1)?

(ii) How to design state feedback / LQR controllers $u(t) = -Kx(t)$?

We will begin by approaching these same questions via LMIs

Stability of autonomous systems

- Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and consider the ***nonlinear*** differential equation

$$\dot{x} = f(x), \qquad \text{where } x(0) \in \mathbb{R}^n \text{ and } f(0) = 0. \tag{2}$$

Definition 4.1. The equilibrium point $x = 0$ of (2) is

(i) ***stable*** if, for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x(0)\|_2 \leq \delta \quad \Longrightarrow \quad \|x(t)\|_2 \leq \varepsilon \qquad \text{for all } t \geq 0;$$

(ii) ***globally asymptotically stable*** if it is stable and if for all $x(0) \in \mathbb{R}^n$ we have $\lim_{t\to\infty} x(t) = 0$;

(iii) ***globally exponentially stable*** if there exist constants $c, M > 0$ such that $\|x(t)\|_2 \leq Me^{-ct}\|x(0)\|_2$ for all $t \geq 0$ and all $x(0) \in \mathbb{R}^n$.

Stability of autonomous LTI systems

Theorem 4.1 (Stability). The equilibrium point $x = 0$ of $\dot{x} = Ax$ is

(i) stable if and only if all eigenvalues of A have nonpositive real part *and* any eigenvalue $\lambda \in \mathrm{eig}(A)$ with $\mathrm{Re}(\lambda) = 0$ has equal geometric and algebraic multiplicity;

(ii) globally asymptotically stable if and only if all eigenvalues of A have negative real part (A is ***Hurwitz***);

(iii) globally exp. stable if and only if it is globally asymptotically stable.

This characterization is problematic, in that

(i) it does not extend to nonlinear systems, and

(ii) the set of Hurwitz matrices is ***not*** a convex set (we can't optimize)

We need to develop a more flexible characterization of stability.

Lyapunov theorems for stability

- We now return to $\dot{x} = f(x)$ with $f(0) = 0$

Theorem 4.2 (Lyapunov). Suppose there exists a continuously differentiable map $V : \mathbb{R}^n \to \mathbb{R}$ satisfying $V(0) = 0$ and such that

$$V(x) \geq c_1 \|x\|_2^2 \qquad \text{and} \qquad \nabla V(x)^{\mathsf{T}} f(x) \leq -c_3 \|x\|_2^2$$

for some $c_1 > 0$, $c_3 \geq 0$, and all $x \in \mathbb{R}^n$. Then $x = 0$ is stable. Moreover,

(i) if $c_3 > 0$, then $x = 0$ is globally asymptotically stable;

(ii) if $c_3 > 0$ and $V(x) \leq c_2 \|x\|_2^2$ for some $c_2 > 0$, then $x = 0$ is globally exponentially stable.

Asymptotic stability guaranteed by finding a scalar-valued positive-definite function that ***decreases*** along trajectories.

Proof of Theorem 4.2 for Exp. Stability Case

Differentiating V along trajectories of $\dot{x} = f(x)$, we have that

$$\dot{V}(x(t)) = \nabla V(x(t))^{\mathsf{T}}\dot{x} = \nabla V(x(t))^{\mathsf{T}} f(x(t)) \leq -c_3\|x(t)\|_2^2 \leq -\frac{c_3}{c_2}V(x(t))$$

which implies (e.g., via the so-called comparison lemma) that

$$V(x(t)) \leq \exp\left(-\frac{c_3}{c_2}t\right)V(x(0)).$$

Lower bounding the LHS and upper bounding the RHS, we obtain

$$c_1\|x(t)\|_2^2 \leq c_2\exp\left(-\frac{c_3}{c_2}t\right)\|x(0)\|_2^2$$

from which it follows that

$$\|x(t)\|_2 \leq \sqrt{\frac{c_2}{c_1}}\exp\left(-\frac{c_3}{2c_2}t\right)\|x(0)\|_2$$

showing global exponential stability. •

Quadratic Lyapunov functions for LTI systems

For our LTI system $\dot{x} = Ax$ what happens if we look for a ***quadratic Lyapunov function*** $V(x) = x^{\mathsf{T}}Px$ for some matrix $P \in \mathbb{R}^{n\times n}$?

- We can assume $P \in \mathbb{S}^n$, and $V(x) \leq c_2\|x\|_2^2$ is satisfied (why?)
- To satisfy $x^{\mathsf{T}}Px \geq c_1\|x\|_2^2$, we need that $\lambda_{\min}(P) > 0 \iff P \succ 0$.
- The condition $\nabla V(x)^{\mathsf{T}}f(x) \leq -c_3\|x\|_2^2$ becomes

$$\underbrace{2x^{\mathsf{T}}P}_{\nabla V(x)^{\mathsf{T}}}\underbrace{Ax}_{f(x)} = x^{\mathsf{T}}(PA + A^{\mathsf{T}}P)x \leq -c_3\|x\|_2^2, \quad \forall x \in \mathbb{R}^n,$$

or equivalently $A^{\mathsf{T}}P + PA \preceq -c_3 I_n$. There exists $c_3 > 0$ satisfying this ***if and only if***

$$\boxed{A^{\mathsf{T}}P + PA \prec 0.}$$

Exponential stability of LTI systems (revised)

Theorem 4.3 (Exponential stability of LTI systems). Consider the LTI state-space system (1). The following statements are equivalent:

(i) the origin $x = \mathbb{0}_n$ of (1) is globally exponentially stable;

(ii) all eigenvalues of A have negative real part (A is ***Hurwitz***);

(iii) there exists $P \succ \mathbb{0}$ satisfying the ***Lyapunov LMI***

$$A^\mathsf{T} P + PA \prec \mathbb{0}$$

(iv) $x = \mathbb{0}_n$ of (1) admits a ***Lyapunov function*** $V(x) = x^\mathsf{T} P x$.

LMI Problem! $\quad \text{find } \; P \in \mathbb{S}^n \quad$ subject to $\quad \begin{aligned} P &\succ 0 \\ A^\mathsf{T} P + PA &\prec 0. \end{aligned}$

Proof of Theorem 4.5

(i) $\iff$ (ii): This is in ECE 557. (iv) $\implies$ (i): This is the result of Theorem 4.2.

(iii) $\iff$ (iv): This is basically our argument preceding the Theorem.

(ii) $\implies$ (iii): For any $Q \succ 0$ define $P = \int_0^\infty e^{A^\mathsf{T} t} Q e^{At}\,\mathrm{d}t$. Obviously P is symmetric. Since A is Hurwitz, $\|e^{At}\|_2 \to 0$ as $t \to \infty$, and it is easy to show as a result that P is well-defined. To check positive-definiteness, let $v \in \mathbb{R}^n$ be non-zero and compute that

$$v^\mathsf{T} P v = \int_0^\infty v^\mathsf{T} e^{A^\mathsf{T} t} Q e^{At} v\,\mathrm{d}t = \int_0^\infty (e^{At}v)^\mathsf{T} Q (e^{At} v)\,\mathrm{d}t = \int_0^\infty \xi(t)^\mathsf{T} Q \xi(t)\,\mathrm{d}t,$$

where $\xi(t) = e^{At}v$. Since $Q \succ \mathbb{0}$, the integrand is nonnegative for all $t \geq 0$, so we conclude that at least $P \succeq \mathbb{0}$. Further, we can have $v^\mathsf{T} P v = 0$ only if $\xi(t) = e^{At} v = \mathbb{0}_n$ for all $t \geq 0$. Since e^{At} is always nonsingular, this implies that $v = \mathbb{0}_n$, and therefore $P \succ \mathbb{0}$. Finally, we compute that

$$\begin{aligned} A^\mathsf{T} P + PA &= \int_0^\infty \left(A^\mathsf{T} e^{A^\mathsf{T} t} Q e^{At} + e^{A^\mathsf{T} t} Q e^{At} A \right)\,\mathrm{d}t = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} \left(e^{A^\mathsf{T} t} Q e^{At} \right)\,\mathrm{d}t \\ &= \lim_{t\to\infty} \left[e^{A^\mathsf{T} t} Q e^{At} \right] - \lim_{t \to 0} \left[e^{A^\mathsf{T} t} Q e^{At} \right] = -Q \prec 0 \end{aligned}$$

where we have used that $Ae^{At} = e^{At}A$ and that $e^{A0} = I_n$. •

Application: diagonal Lyapunov functions

- **Important case:** the solution $P \succ 0$ to the Lyapunov equation is ***diagonal*** or ***block diagonal***

$$\text{find } P \succ 0 \quad \text{subject to} \quad A^{\mathsf{T}}P + PA \prec 0, \quad P_{ij} = 0 \ \forall i \neq j.$$

- Applications in economics, biology, ecology, numerical analysis, and stability of systems over networks.
- You can obviously include other (affine) constraints on P to enforce any structure you would like.
- Diagonal stability is ***restrictive***, and a solution might not exist even if A is Hurwitz. If the LMI is feasible, you can compute a solution. If the LMI is infeasible, then no such solution exists.

Application to large-scale system analysis

Suppose we have N ***interconnected nonlinear systems***

$$\dot{x}_i = f_i(x_i) + \sum_{j=1}^{N} g_{ij}(x_j), \qquad i \in \{1, \ldots, N\},$$

where each f_i admits a Lyapunov function V_i (Theorem 4.2) and the coupling functions g_{ij} satisfy the boundedness condition

$$|\nabla V_i(x_i)^\mathsf{T} g_{ij}(x_j)| \leq \gamma_{ij} \|x_i\|_2 \|x_j\|_2, \qquad \text{for some } \gamma_{ij} > 0.$$

Let $D = \mathrm{diag}(d_1, \ldots, d_N) \succ 0$ and define $V(x) = \sum_{i=1}^{N} d_i V_i(x_i)$. Then

$$\begin{aligned}
\dot{V}(x(t)) &= \sum_{i=1}^{N} d_i \nabla V_i(x_i)^\mathsf{T} f_i(x_i) + \sum_{i=1}^{N} d_i \sum_{j=1}^{N} \nabla V_i(x_i)^\mathsf{T} g_{ij}(x_j) \\
&\leq -\sum_{i=1}^{N} d_i c_i \|x_i\|_2^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} d_i \gamma_{ij} \|x_i\|_2 \|x_j\|_2 \\
&= \tfrac{1}{2} \phi(x)^\mathsf{T} \left[-2DC + \Gamma^\mathsf{T} D + D\Gamma \right] \phi(x)
\end{aligned}$$

where $\phi(x) = (\|x_1\|_2, \ldots, \|x_N\|_2)$.

Find diagonal $D \succ 0$ s.t. $-2DC + \Gamma^\mathsf{T} D + D\Gamma \prec 0$ to guarantee asymp. stability!

Stabilizing state feedback design

With our handy new stability LMI, we can start having some fun!

- **Problem:** ***stabilizing*** state feedback $u = Kx$ for $\dot{x} = Ax + Bu$.
- The closed-loop system is given by $\dot{x} = (A + BK)x$
- ***Closed-loop stability***: there exists $P \succ 0$ such that

$$
\begin{aligned}
& (A + BK)^{\mathsf{T}} P + P(A + BK) \prec 0 \\
\iff \qquad & A^{\mathsf{T}} P + PA + (PBK) + (PBK)^{\mathsf{T}} \prec 0
\end{aligned}
$$

- Perform a *congruence transformation* with $X = P^{-1} \succ 0$

$$
\iff \qquad XA^{\mathsf{T}} + AX + (BKX) + (BKX)^{\mathsf{T}} \prec 0
$$

- Now define $Z = KX$ as a new variable, and we get the LMI

$$
\text{find} \quad X \succ 0,\ Z \in \mathbb{R}^{m \times n} \qquad \text{subject to} \quad XA^{\mathsf{T}} + AX + BZ + (BZ)^{\mathsf{T}} \prec 0.
$$

Stabilizing state feedback design

Theorem 4.4 (LMI for Stabilizing State Feedback). There exists $K \in \mathbb{R}^{m \times n}$ such that $A + BK$ is Hurwitz if and only if there exist $X \succ 0$ and $Z \in \mathbb{R}^{m \times n}$ such that

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} X & Z^\mathsf{T} \end{bmatrix} \begin{bmatrix} A^\mathsf{T} \\ B^\mathsf{T} \end{bmatrix} \prec \mathbb{0}\,. \tag{3}$$

In particular, a stabilizing feedback gain is given by $K = ZX^{-1}$, with $P = X^{-1}$ satisfying the closed-loop Lyapunov LMI.

- **Neat Trick:** we removed a *product* of decision variables KX by introducing a new variable Z; this ***linearized*** the inequality
- **Observation:** the final ***synthesis inequality*** (3) involves the ***inverse*** $X = P^{-1}$ of the original Lyapunov variable P.

Example: Stabilizing state feedback

```
%% Define Two-Mass Positioning System
k = 5; b = 5.82e-3;
J1 = 1e-3; J2 = 2e-4;
Jtot = J1+J2; Jred = J1*J2/(J1+J2);
s = tf('s'); P = (b*s+k)/((Jtot*s^2)*(Jred*s^2 + b*s + ...
    k)); P = ss(P);
A = P.A; B = P.B; n = size(A,1); m = size(B,2);

%% Solve LMI Problem
X = sdpvar(n,n); Z = sdpvar(m,n,'full');
small = 1e-6;
Constraints = [X ≥ small*eye(n), [A,B]*[X;Z] + ...
    ([A,B]*[X;Z])'  ≤ -small*eye(n)];
Cost = 0;
options = sdpsettings('solver','sdpt3','verbose',1);
sol = optimize(Constraints,Cost,options);
K = value(Z)*inv(value(X));
```

Connection to stabilizability of (A, B)

Shouldn't it hold that (A, B) stabilizable $\iff$ synthesis LMI feasible?

- With $\mathsf{X} = \mathbb{S}^n \times \mathbb{R}^{m \times n}$ and $x = (X, Z)$ the state-feedback LMI is

$$F(x) = \begin{bmatrix} XA^\mathsf{T} + AX + BX + (BX)^\mathsf{T} & 0 \\ 0 & -X \end{bmatrix} \prec 0$$

- **Contraposition via strong alternatives:** if $F(x) \prec 0$ is ***infeasible***, then there exists a non-zero $Y = \left[\begin{smallmatrix} Y_1 & Y_2^\mathsf{T} \\ Y_2 & Y_3 \end{smallmatrix}\right] \succeq 0$ such that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = F^{\mathrm{adj}}(Y) = \begin{bmatrix} A^\mathsf{T} Y_1 + Y_1 A - Y_3 \\ B^\mathsf{T} Y_1 \end{bmatrix}$$

- **Note:** $Y_1 \neq 0$; otherwise, contradiction with $Y \neq 0$

Connection to stabilizability of (A, B)

- Since $\mathrm{rank}(Y_1) = r \geq 1$, let $Y_1 = UU^*$ with $U \in \mathbb{C}^{n \times r}$ full rank. Then

$$A^{\mathsf{T}} Y_1 + Y_1 A = A^{\mathsf{T}} UU^* + UU^* A = Y_3 \succeq 0$$

Lemma 4.1. $UU^* A + (UU^* A)^* \succeq 0$ if and only if $\exists D \succeq 0$ and $\exists S$ satisfying $S + S^* = 0$ such that $UU^* A = U(D + S)U^*$.

Proof of "if": Compute directly that

$$UU^* A + (UU^* A)^* = U(D + D^* + S + S*)U^* = 2UDU^* \succeq 0.$$

- Necessarily, we have $\mathrm{eig}(D + S) \subseteq \mathbb{C}_{\geq 0}$
- We let $J \Lambda J^{-1}$ denote the Jordan decomposition of $D + S^*$

Connection to stabilizability of (A, B)

- As U has full column rank, we conclude that $U^*A = (D + S)U^*$, or

$$A^{\mathsf{T}}U = U(D + S^*) \quad \Longrightarrow \quad A^{\mathsf{T}} \underbrace{UJ}_{\triangleq V} = \underbrace{UJ}_{\triangleq V} \Lambda \quad \Longrightarrow \quad A^{\mathsf{T}}V = V\Lambda$$

Therefore, $\mathcal{V} \triangleq \mathrm{range}(V)$ is a non-empty A^{T}-invariant subspace corresponding to some ***unstable*** eigenvalues Λ of A^{T}!

- **Fact*:** $\mathcal{V}$ must contain at least one eigenvector v of A^{T}, in this case with corresponding eigenvalue $\lambda \in \mathbb{C}_{\geq 0}$. Moreover,

$$B^{\mathsf{T}}Y_1 = 0 \quad \Leftrightarrow \quad B^{\mathsf{T}}U = 0 \quad \Leftrightarrow \quad B^{\mathsf{T}}V = 0 \quad \Rightarrow \quad B^{\mathsf{T}}v = 0$$

The system ***fails*** the eigenvector test for stabilizability!

*See, e.g., Hespanha, Property P12.2.

OK, but how do we achieve good "performance"?

- We want to design control systems that have good performance ... but what precisely does that even mean?
- In undergraduate control design, performance usually refers to the ***step response***: rise time, settling time, overshoot ...
  - this seems daunting to spec. for big MIMO systems
  - control *effort* is more of an afterthought
- In ECE 557, good performance meant "minimum LQR cost" which keeps a combination of the square-integrated states and control signals small ... but quite unclear how this relates to response under set-point changes or disturbances!

We need a more unified, systematic, and computationally-friendly framework for assessing performance

Appendix: Stability of discrete-time systems

- Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and consider the nonlinear difference equation

$$x(k+1) = f(x(k)), \qquad \text{where } f(0) = 0 \text{ and } k \in \{0, 1, 2, \ldots\} \tag{4}$$

Definition 4.2. The equilibrium point $x = 0$ of (4) is

(i) ***stable*** if, for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x(0)\|_2 \leq \delta \quad \Longrightarrow \quad \|x(k)\|_2 \leq \varepsilon \qquad \text{for all } k \geq 0;$$

(ii) ***globally asymptotically stable*** if it is stable and if for all $x(0) \in \mathbb{R}^n$ we have $\lim_{k \to \infty} x(k) = 0$;

(iii) ***globally exponentially stable*** if $\exists\, M > 0$ and $\rho \in [0, 1)$ such that $\|x(k)\|_2 \leq M\rho^k \|x(0)\|_2$ for all $k \geq 0$ and all $x(0) \in \mathbb{R}^n$.

Appendix: Stability of discrete-time LTI systems

Consider the discrete-time LTI system $x(k+1) = Ax(k)$? Here, a Lyapunov function needs to decrease at each step, so we ask for

$$c_1\|x\|_2^2 \leq V(x) \leq c_2\|x\|_2^2, \qquad V(Ax) - V(x) \leq -c_3\|x\|_2^2.$$

Theorem 4.5 (Exponential stability of LTI systems). Consider the (DT-LTI) system. The following statements are equivalent:

(i) the origin $x = \mathbb{0}_n$ of (DT-LTI) is globally exponentially stable;

(ii) all eigenvalues of A have magnitude less than 1 (A is ***Schur***);

(iii) there exists $P \succ \mathbb{0}$ satisfying the ***Lyapunov LMI***

$$A^{\mathsf{T}}PA - P \prec 0$$

(iv) $x = \mathbb{0}_n$ admits a Lyapunov function $V(x) = x^{\mathsf{T}}Px$.

5. The KYP Lemma and Dissipative Dynamical Systems

Introduction to dissipativity theory

- Lyapunov theory provides a tool (the Lyapunov function) for analyzing the *autonomous* behaviour of a dynamical system. We measure the "energy" of the state x using a Lyapunov function $V(x)$, and study how this energy evolves over time.

$$\dot{V}(x(t)) \leq 0 \quad \Longleftrightarrow \quad V(x(t_2)) \leq V(x(t_1)) \quad \forall\, t_1, t_2 \text{ s.t. } t_2 \geq t_1.$$

- Dissipativity theory generalizes Lyapunov theory to dynamical systems with ***inputs and outputs***. Two ingredients:
  - (i) a ***storage function*** $V(x)$ which measures the "energy" of the state
  - (ii) a ***supply rate*** $s(w,z)$ which captures the rate of change of energy entering the system through the input w and output z

$$\dot{V}(x(t)) \leq s(w(t), z(t)) \quad \Longleftrightarrow \quad V(x(t_2)) \leq V(x(t_1)) + \int_{t_1}^{t_2} s(w(\tau), z(\tau))\, \mathrm{d}\tau$$

Input-output causal CT-LTI systems

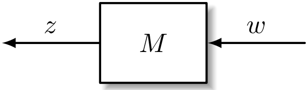
- We will focus on the FD CT state-space model

$$M: \left[\begin{array}{c}\dot{x}\\ \hline z\end{array}\right] = \left[\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right]\left[\begin{array}{c}x\\ \hline w\end{array}\right], \qquad x(0) = 0,$$

  with state $x \in \mathbb{R}^n$, input $w \in \mathbb{R}^m$, and output $z \in \mathbb{R}^p$.

- We will ***interpret*** the above ODE as defining a ***causal linear time-invariant system*** by restricting all signals to be ***right-sided***

- **Fact**: If $w(t)$ is "sufficiently nice" and right-sided, then the system will respond with a unique right-sided solution $x(t)$ and right-sided output $z(t)$, which both depend causally on $w(t)$:



Input-output causal CT-LTI systems

- With this, the state and output are given by

$$x(t) = \left[\int_0^t Ce^{A(t-\tau)} Bw(\tau)\,\mathrm{d}\tau\right] 1(t)$$
$$z(t) = Cx(t) + Dw(t)$$

  and $x(t)$ satisfies the ODE for almost every $t \in \mathbb{R}$.

- As you know, the system has a ***transfer function***

$$\hat{M}(s) = C(sI_n - A)^{-1}B + D, \quad s \in \text{ROC}.$$

- Assuming ***BIBO stability***, the system also has a ***frequency response***

$$\hat{M}(\mathbf{j}\omega) = C(\mathbf{j}\omega I_n - A)^{-1}B + D$$

Definition of dissipativity

Definition 5.1 (Dissipativity). Let $s : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ be a supply rate. The state-space system (CT-LTI) is ***dissipative*** if there exists a differentiable storage function $V : \mathbb{R}^n \to \mathbb{R}$ and $\varepsilon \geq 0$ such that

$$\nabla V(x)^{\mathsf{T}}(Ax + Bw) \leq s(w, z) - \varepsilon^2 \|w\|_2^2$$

for all $(x, w) \in \mathbb{R}^{n+m}$. If $\varepsilon > 0$, the system is ***input-strictly dissipative***.

- If $(w(t), x(t), z(t))$ is a system trajectory, then we have that

$$\underbrace{\dot{V}(x(t))}_{\text{Rate of Change of Stored Energy}} \leq \underbrace{s(w(t), z(t)) - \varepsilon^2 \|w(t)\|_2^2}_{\text{Externally Provided Power}}$$

- Often (not always) $V(x) \geq 0$

Quadratic supply rates

We now restrict our attention to fairly simple types of supply rates: homogeneous quadratic forms of (z, w)

Definition 5.2 (Quadratic supply rate). Let $\Pi = \left[\begin{smallmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{smallmatrix}\right] \in \mathbb{S}^{p+m}$. The mapping $s : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ defined by

$$s(w, z) = \begin{bmatrix} z \\ w \end{bmatrix}^\mathsf{T} \Pi \begin{bmatrix} z \\ w \end{bmatrix}$$

is called a ***quadratic supply rate***.

- ***Finite-gain:*** $s(w, z) = -z^\mathsf{T} z + \gamma^2 w^\mathsf{T} w$, $\Pi = \begin{bmatrix} -I_p & 0 \\ 0 & \gamma^2 I_m \end{bmatrix}$
- ***Passive:*** $s(w, z) = w^\mathsf{T} z$, $\Pi = \frac{1}{2} \left[\begin{smallmatrix} 0 & I_m \\ I_m & 0 \end{smallmatrix}\right]$

Why? Quadratic supply rates will play nice with LMIs

Integral characterization of dissipativity

- Suppose that $V(x) \geq 0$, that $V(\mathbb{0}_n) = 0$, and that $x(0) = \mathbb{0}_n$
- Integrating $\dot{V}(x(t)) \leq \left[\begin{smallmatrix} z(t) \\ w(t) \end{smallmatrix}\right]^{\mathsf{T}} \Pi \left[\begin{smallmatrix} z(t) \\ w(t) \end{smallmatrix}\right]$ over $[0, T]$ we obtain

$$\underbrace{V(x(T))}_{\geq 0} - \underbrace{V(x(0))}_{=0} \leq \int_0^T \begin{bmatrix} z(t) \\ w(t) \end{bmatrix}^{\mathsf{T}} \Pi \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \mathrm{d}t.$$

and therefore

$$\boxed{\int_0^T \begin{bmatrix} z(t) \\ w(t) \end{bmatrix}^{\mathsf{T}} \Pi \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \geq 0, \qquad \forall T \geq 0.}$$

- Dissipativity therefore specifies an ***integral-quadratic inequality*** involving the input and output signals.

Example: linear mechanical system

- Dynamics of a linearized mechanical system

$$\begin{bmatrix} \dot{q} \\ M\dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u, \qquad \begin{aligned} q &= \text{coordinates} \\ v &= \text{velocities} \\ u &= \text{torques} \end{aligned}$$

- inertia/damping/stiffness matrices $M, D, K \succ 0$
- Take ***energy*** $V(q,v) = \frac{1}{2}q^{\mathsf{T}}Kq + \frac{1}{2}v^{\mathsf{T}}Mv$ as storage and compute

$$\begin{aligned} \dot{V}(q,v) &= -v^{\mathsf{T}}Dv - v^{\mathsf{T}}Kq + v^{\mathsf{T}}u + q^{\mathsf{T}}Kv \\ &= -v^{\mathsf{T}}Dv + v^{\mathsf{T}}u \\ &= \begin{bmatrix} v \\ u \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} -D & \frac{1}{2}I \\ \frac{1}{2}I & 0 \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} \end{aligned}$$

- This is an instance of so-called *output-strict passivity*

Example: Linear mechanical system

- Sometimes you can manipulate one supply rate into another
- Our previous calculations show that for some $d > 0$ we have

$$\dot{V}(q,v) \leq -dv^{\mathsf{T}}v + v^{\mathsf{T}}u$$

- We can complete the square to obtain

$$\begin{aligned}\dot{V}(q,v) &\leq -\frac{1}{2d}(u-dv)^{\mathsf{T}}(u-dv) - \frac{d}{2}v^{\mathsf{T}}v + \frac{1}{2d}u^{\mathsf{T}}u \\ &\leq -\frac{d}{2}v^{\mathsf{T}}v + \frac{1}{2d}u^{\mathsf{T}}u \\ &= \frac{d}{2}\left(-v^{\mathsf{T}}v + \frac{1}{d^2}u^{\mathsf{T}}u\right)\end{aligned}$$

- Defining $V'(q,v) = V(q,v) \cdot \frac{2}{d}$, we finally have that

$$\dot{V}'(q,v) \leq -v^{\mathsf{T}}v + \frac{1}{d^2}u^{\mathsf{T}}u, \qquad \text{(finite-gain supply rate)}$$

Example: Linear electrical circuit

- Dynamics of a linear RLC circuit with shunt conductances

$$\begin{bmatrix} C\dot{v} \\ L\dot{i} \end{bmatrix} = \begin{bmatrix} -G & -\mathscr{B} \\ \mathscr{B}^{\mathsf{T}} & -R \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} I_{\mathrm{ext}}, \qquad \begin{aligned} V &= n \text{ cap. voltages} \\ i &= m \text{ inductor currents} \\ I_{\mathrm{ext}} &= \text{external currents} \end{aligned}$$

- $G, R, C, L \succ 0$ are diagonal matrices
- $\mathscr{B} \in \mathbb{R}^{n \times m}$ is the ***node-edge incidence matrix*** of the circuit graph
- Take energy $V(v, i) = \frac{1}{2} v^{\mathsf{T}} C v + \frac{1}{2} i^{\mathsf{T}} L i$ and compute

$$\begin{aligned} \dot{V} &= -v^{\mathsf{T}} G v - v^{\mathsf{T}} \mathscr{B} i + i^{\mathsf{T}} \mathscr{B}^{\mathsf{T}} v - i^{\mathsf{T}} R i + v^{\mathsf{T}} I_{\mathrm{ext}} \\ &\leq -v^{\mathsf{T}} G v + v^{\mathsf{T}} I_{\mathrm{ext}} \end{aligned}$$

Stability of dissipative LTI systems

One can sometimes go from dissipativity — an input-output property — to a statement about ***internal*** stability of the system. Here is one variation.

- Suppose that $V(x) \geq 0$, and in $\Pi = \left[\begin{smallmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{smallmatrix}\right]$ we have $\Pi_{11} \prec 0$
- With zero input $w = 0$, along trajectories of (CT-LTI) we have

$$\dot{V}(x(t)) \leq \begin{bmatrix} z(t) \\ w(t) \end{bmatrix}^{\mathsf{T}} \Pi \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} = z(t)^{\mathsf{T}} \Pi_{11} z(t) \leq -c_3 \|z(t)\|_2^2$$

for some $c_3 > 0$. Then, with $x(0) = x_0$, for any $T \geq 0$ we have

$$\underbrace{V(x(T))}_{\geq 0} - V(x_0) \leq -c_3 \int_0^T \|z(\tau)\|_2^2 \, \mathrm{d}\tau$$

and therefore $V(x_0) \geq c_3 \int_0^T \|z(\tau)\|_2^2 \, \mathrm{d}\tau$.

Stability of dissipative LTI systems

- Keep in mind though that $z(\tau) = Ce^{A\tau}x_0$. Therefore,

$$V(x_0) \geq c_3 \int_0^T \|Ce^{A\tau}x_0\|_2^2 \, \mathrm{d}\tau, \qquad \text{for all } T \geq 0.$$

- ***Claim:*** If (C, A) is observable, then $\exists c_1 > 0$ s.t. $V(x_0) \geq c_1\|x_0\|_2^2$

Proof: If $\exists x_0 \neq 0$ such that $V(x_0) = 0$, then the above implies that $\int_0^T \|Ce^{A\tau}x_0\|_2^2 \, \mathrm{d}\tau = 0$ for all $T \geq 0$, which implies $Ce^{At}x_0 = 0$. Thus, x_0 belongs to the ***unobservable subspace***. If (C, A) is observable, this subspace is just the origin, implying that $x_0 = 0$, a contradiction, and hence $V(x_0) > 0$ for all $x_0 \neq 0$. In fact, picking any $T^\star > 0$, we have

$$V(x_0) \geq c_3 x_0^\mathsf{T} \underbrace{\int_0^{T^\star} e^{A^\mathsf{T}t}C^\mathsf{T}Ce^{At} \, \mathrm{d}t}_{\triangleq W_\mathrm{o}(T^\star)} x_0 \geq c_1\|x_0\|_2^2$$

with $c_1 \triangleq c_3\lambda_{\min}(W_\mathrm{o}(T^\star))$. •

Stability of dissipative LTI systems

- Since $V(x) \geq c_1\|x\|_2^2$ and $\dot{V}(x(t)) \leq 0$, $x = 0$ is certainly ***stable***.
- However, $\dot{V}(x(t)) \leq -c_3\|z(t)\|_2^2$, so V is forced to decrease until $z(t) = 0$, i.e., $x(t) \to \mathcal{Z} \triangleq \{\xi \mid C\xi = 0\}$
- Once $x(t)$ reaches $\mathcal{Z}$, we would have $Cx(t) = Ce^{At}x_0 = 0$, but by observability this means $x_0 = 0$ and hence $x(t) = 0$
- So convergence to $\mathcal{Z}$ *implies* convergence to $x = 0$. We therefore have global ***asymptotic*** (hence, exponential) stability!

A standard variation on this result:

$V(x)$ pos. def., $\Pi_{11} \prec 0$, (C, A) detectable $\implies$ exp. stability

Example: Linear electrical circuit

- Dynamics of a linear RLC circuit with shunt conductances

$$\begin{bmatrix} C\dot{v} \\ L\dot{i} \end{bmatrix} = \begin{bmatrix} -G & -\mathscr{B} \\ \mathscr{B}^{\mathsf{T}} & -R \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} I_{\mathrm{ext}}, \qquad \begin{aligned} V &= n \text{ cap. voltages} \\ i &= m \text{ inductor currents} \\ I_{\mathrm{ext}} &= \text{external currents} \end{aligned}$$

$$z = v$$

- $V(v,i) = \frac{1}{2}v^{\mathsf{T}}Cv + \frac{1}{2}i^{\mathsf{T}}Li$ is positive definite
- We had $\dot{V} \leq -v^{\mathsf{T}}Gv + v^{\mathsf{T}}I_{\mathrm{ext}}$, so we have $\Pi_{11} = -G \prec 0$.
- If the circuit contains no loops, then $\mathrm{null}(\mathscr{B}) = 0$. Eigenvector test:

$$\xi \in \mathrm{null}(C) = \mathrm{range}\begin{bmatrix} 0 \\ I \end{bmatrix}, \quad A\xi = \begin{bmatrix} -C^{-1}\mathscr{B}\xi \\ -L^{-1}R\xi \end{bmatrix} = \text{not a multiple of } \xi$$

so the system is observable and thus is globally exp. stable

Strictly quadratically dissipative systems

Theorem 5.1 (Strict Dissipativity). Assume that A is Hurwitz. Then the following statements are equivalent:

(i) system (CT-LTI) is input-strictly dissipative with quadratic supply rate s and storage function $V(x) = x^\mathsf{T} P x$, where $P \in \mathbb{S}^n$;

(ii) there exists $P \in \mathbb{S}^n$ satisfying the strict LMI

$$\begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix} - \begin{bmatrix} C & D \\ \mathbb{0} & I_m \end{bmatrix}^\mathsf{T} \Pi \begin{bmatrix} C & D \\ \mathbb{0} & I_m \end{bmatrix} \prec \mathbb{0}.$$

(iii) for all $\omega \in \mathbb{R} \cup \{\infty\}$ the frequency response $\hat{M}(\mathbf{j}\omega)$ satisfies

$$\begin{bmatrix} \hat{M}(\mathbf{j}\omega) \\ I_m \end{bmatrix}^* \Pi \begin{bmatrix} \hat{M}(\mathbf{j}\omega) \\ I_m \end{bmatrix} \succ \mathbb{0}.$$

Comments on strict dissipativity theorem

- Existence of a storage function certifying strict dissipativity is ***equivalent*** to a strict LMI feasibility problem
- The set of $P \in \mathbb{S}^n$ satisfying the LMI is convex, which means the ***set of all quadratic storage functions*** is convex.
- The result is even stronger than written here; one can show there is no loss of generality in the restriction to quadratic storage functions.
- The inequality in (iii) is called a ***frequency-domain inequality (FDI)***. It is essentially a statement about the ***Nyquist plot*** of the frequency response. We will look at this in a bit more detail soon.
- The hard implication to prove is that (iii) $\Longrightarrow$ (ii), which relies on the ***Kalman-Yakubovich-Popov Lemma***

Comments on strict dissipativity theorem

- The FDI is strict. Since it must also hold at $\omega = +\infty$, this ***may*** place additional requirements on the feedthrough term D for the model M. For example, with $\Pi = \frac{1}{2}\left[\begin{smallmatrix} 0 & I_m \\ I_m & 0 \end{smallmatrix}\right]$, the FDI becomes

$$\hat{M}(\mathbf{j}\omega) + \hat{M}(\mathbf{j}\omega)^* \succ 0 \qquad \forall \omega \in \mathbb{R} \cup \{\infty\} \quad \Longrightarrow \quad D + D^{\mathsf{T}} \succ 0.$$

- Why are we assuming that A is Hurwitz?
  (i) We will typically be applying the result to ***closed-loop*** systems
  (ii) You can relax the Hurwitz assumption; see Appendix

Equivalent ways to write the LMI

With $\Pi = \left[\begin{smallmatrix} Q & S \\ S^\mathsf{T} & R \end{smallmatrix}\right]$ one often sees the LMI written in equivalent forms:

$$\begin{bmatrix} A^\mathsf{T}P + PA & PB \\ B^\mathsf{T}P & \mathbb{0} \end{bmatrix} - \begin{bmatrix} C & D \\ \mathbb{0} & I_m \end{bmatrix}^\mathsf{T} \begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \begin{bmatrix} C & D \\ \mathbb{0} & I_m \end{bmatrix} \prec \mathbb{0}$$

$$\begin{bmatrix} A^\mathsf{T}P + PA - C^\mathsf{T}QC & PB - C^\mathsf{T}S^\mathsf{T} - C^\mathsf{T}QD \\ B^\mathsf{T}P - SC - D^\mathsf{T}QC & -R - SD - D^\mathsf{T}S^\mathsf{T} - D^\mathsf{T}QD \end{bmatrix} \prec \mathbb{0}$$

$$\left[\begin{array}{cc} I_n & \mathbb{0} \\ A & B \\ \hline C & D \\ \mathbb{0} & I_m \end{array}\right]^\mathsf{T} \left[\begin{array}{cc|cc} \mathbb{0} & P & \mathbb{0} & \mathbb{0} \\ P & \mathbb{0} & \mathbb{0} & \mathbb{0} \\ \hline \mathbb{0} & \mathbb{0} & -Q & -S \\ \mathbb{0} & \mathbb{0} & -S^\mathsf{T} & -R \end{array}\right] \left[\begin{array}{cc} I_n & \mathbb{0} \\ A & B \\ \hline C & D \\ \mathbb{0} & I_m \end{array}\right] \prec \mathbb{0}$$

Number #3 is the most intuitive, because you can easily left-right multiply by (x, w) and then substitute the dynamics.

Proof of Theorem 5.1

(i) $\Rightarrow$ (iii): Let $\omega_0 > 0$ and consider the input signal $w(t) = e^{\mathbf{j}\omega_0 t} w_0 \mathbf{1}(t)$ for some $w_0 \in \mathbb{R}^m$. The system M is causal and LTI, and since A is Hurwitz, the system M is BIBO stable. It follows by standard arguments that the state and output converge towards the steady-state signals

$$x_{\mathrm{ss}}(t) = (\mathbf{j}\omega_0 - A)^{-1} B w_0 e^{\mathbf{j}\omega_0 t}, \qquad z_{\mathrm{ss}}(t) = \hat{M}(\mathbf{j}\omega_0) w_0 e^{\mathbf{j}\omega_0 t}.$$

which are *periodic* with period $T_0 = 2\pi/\omega_0$. Note that

$$\begin{aligned} s(w(t), z_{\mathrm{ss}}(t)) - \varepsilon^2 \|w(t)\|_2^2 &= \begin{bmatrix} z_{\mathrm{ss}}(t) \\ w(t) \end{bmatrix}^* \Pi \begin{bmatrix} z_{\mathrm{ss}}(t) \\ w(t) \end{bmatrix} - \varepsilon^2 \|w(t)\|_2^2 \\ &= w(t)^* \begin{bmatrix} \hat{M}(\mathbf{j}\omega_0) \\ I_m \end{bmatrix}^* \Pi \begin{bmatrix} \hat{M}(\mathbf{j}\omega_0) \\ I_m \end{bmatrix} w(t) - \varepsilon^2 \|w(t)\|_2^2 \\ &= w_0^* \left(\begin{bmatrix} \hat{M}(\mathbf{j}\omega_0) \\ I_m \end{bmatrix}^* \Pi \begin{bmatrix} \hat{M}(\mathbf{j}\omega_0) \\ I_m \end{bmatrix} - \varepsilon^2 I_m \right) w_0 \end{aligned}$$

is actually *independent* of $t \geq 0$.

Proof of Theorem 5.1

By strict dissipativity, we have for any $t \geq 0$ that

$$V(x(t+T_0)) - V(x(t)) \leq \int_t^{t+T_0} s(w(\tau), z(\tau)) - \varepsilon^2 \|w(\tau)\|_2^2 \, \mathrm{d}\tau$$

for some $\varepsilon > 0$. Taking limits as $t \to \infty$, by periodicity we have that

$$\lim_{t\to\infty} V(x(t+T_0)) - V(x(t)) = 0$$

$$\lim_{t\to\infty} (\text{RHS}) = T_0 w_0^* \left(\begin{bmatrix} \hat{M}(\mathbf{j}\omega_0) \\ I_m \end{bmatrix}^* \Pi \begin{bmatrix} \hat{M}(\mathbf{j}\omega_0) \\ I_m \end{bmatrix} - \varepsilon^2 I_m \right) w_0.$$

Since $T_0 > 0$ and w_0 were arbitrary, we conclude that

$$\begin{bmatrix} \hat{M}(\mathbf{j}\omega_0) \\ I_m \end{bmatrix}^* \Pi \begin{bmatrix} \hat{M}(\mathbf{j}\omega_0) \\ I_m \end{bmatrix} \succ \varepsilon^2 I_m \succ 0.$$

Since A is Hurwitz, $\hat{M}$ has no poles on the $\mathbf{j}\omega$ axis, and hence the inequality (iii) must also hold by continuity at $\omega_0 = 0$ and as $\omega_0 \to \infty$. The case $\omega_0 < 0$ is handled similarly.

(iii) $\Rightarrow$ (ii): This is a consequence of the KYP Lemma, to be stated shortly.

Proof of Theorem 5.1

(ii) $\Rightarrow$ (i): Since the LMI is strict, there exists some $\varepsilon > 0$ such that

$$\begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix} - \begin{bmatrix} C & D \\ \mathbb{0} & I_m \end{bmatrix}^\mathsf{T} \Pi \begin{bmatrix} C & D \\ \mathbb{0} & I_m \end{bmatrix} \preceq \begin{bmatrix} \mathbb{0} & \mathbb{0} \\ \mathbb{0} & -\varepsilon^2 I_m \end{bmatrix}. \quad (5)$$

Let (x, w) be arbitrary, and left/right multiply this LMI by (x, w) to obtain

$$\begin{bmatrix} x \\ w \end{bmatrix}^\mathsf{T} \left(\begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix} - \begin{bmatrix} C & D \\ \mathbb{0} & I \end{bmatrix}^\mathsf{T} \Pi \begin{bmatrix} C & D \\ \mathbb{0} & I \end{bmatrix} \right) \begin{bmatrix} x \\ w \end{bmatrix} \leq -\varepsilon^2 \|w\|_2^2$$

or

$$\begin{bmatrix} x \\ Ax + Bw \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} x \\ Ax + Bw \end{bmatrix} - \begin{bmatrix} Cx + Dw \\ w \end{bmatrix}^\mathsf{T} \Pi \begin{bmatrix} Cx + Dw \\ w \end{bmatrix} \leq -\varepsilon^2 \|w\|_2^2.$$

With $V(x) = x^\mathsf{T} P x$ and $z = Cx + Dw$, this says precisely that

$$\nabla V(x)^\mathsf{T} (Ax + Bu) - \underbrace{\begin{bmatrix} z \\ w \end{bmatrix}^\mathsf{T} \Pi \begin{bmatrix} z \\ w \end{bmatrix}}_{= s(w,z)} \leq -\varepsilon^2 \|w\|_2^2.$$

Proof of Theorem 5.1

Just for fun, we can give a direct proof of (ii) $\Longrightarrow$ (iii): For the case $\omega = +\infty$, note that the (2,2) block of the LMI simply says that (multiply things out to convince yourself)

$$0 \prec \begin{bmatrix} D \\ I_m \end{bmatrix}^{\mathsf{T}} \Pi \begin{bmatrix} D \\ I_m \end{bmatrix}$$

which is (iii) at $\omega = +\infty$, since $\lim_{\omega \to +\infty} \hat{M}(\mathbf{j}\omega) = D$. Now let $\omega \in \mathbb{R}$ and $w \in \mathbb{C}^m$, and set $x = (\mathbf{j}\omega I_n - A)^{-1} B w$. Left/right multiplying (5) by (x, w), we obtain

$$\begin{aligned}
\begin{bmatrix} x \\ Ax + Bw \end{bmatrix}^* \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} x \\ Ax + Bw \end{bmatrix} - \begin{bmatrix} x \\ w \end{bmatrix}^* \begin{bmatrix} C & D \\ \mathbb{0} & I_m \end{bmatrix}^{\mathsf{T}} \Pi \begin{bmatrix} C & D \\ \mathbb{0} & I_m \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} &\leq -\varepsilon^2 \|w\|_2^2 \\
\begin{bmatrix} x \\ \mathbf{j}\omega x \end{bmatrix}^* \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} x \\ \mathbf{j}\omega x \end{bmatrix} - \begin{bmatrix} \hat{M}(\mathbf{j}\omega) w \\ w \end{bmatrix}^* \Pi \begin{bmatrix} \hat{M}(\mathbf{j}\omega) w \\ w \end{bmatrix} &\leq -\varepsilon^2 \|w\|_2^2 \\
0 - w^* \begin{bmatrix} \hat{M}(\mathbf{j}\omega) \\ I_m \end{bmatrix}^* \Pi \begin{bmatrix} \hat{M}(\mathbf{j}\omega) \\ I_m \end{bmatrix} w &\leq -\varepsilon^2 \|w\|_2^2.
\end{aligned}$$

which shows (iii) since w was arbitrary.

The KYP lemma

Theorem 5.2 (KYP Lemma I). Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and let $K = \left[\begin{smallmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{smallmatrix}\right] \in \mathbb{S}^{n+m}$. The following two statements are equivalent:

(i) there exists a symmetric matrix $P \in \mathbb{S}^n$ satisfying the strict LMI

$$\begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix} + K \prec \mathbb{0}$$

(ii) $K_{22} \prec \mathbb{0}$ and for all $\omega \in \mathbb{R}$ and $(x, w) \in \mathbb{C}^{n+m} \setminus \{0\}$

$$\begin{bmatrix} A - \mathbf{j}\omega I_n & B \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = \mathbb{0}_n \qquad \Longrightarrow \qquad \begin{bmatrix} x \\ w \end{bmatrix}^* K \begin{bmatrix} x \\ w \end{bmatrix} < 0.$$

Also: if (A, B) is controllable, then (i)$\Longleftrightarrow$(ii) with non-strict inequalities.

We will specialize to the case where A is Hurwitz.

The KYP lemma

Theorem 5.3 (KYP Lemma II). Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and let $K = \left[\begin{smallmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{smallmatrix}\right] \in \mathbb{S}^{n+m}$. If A is Hurwitz, then the following two statements are equivalent:

(i) there exists a symmetric matrix $P \in \mathbb{S}^n$ satisfying the strict LMI

$$\begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix} + K \prec \mathbb{0}$$

(ii) $K_{22} \prec \mathbb{0}$ and for all $\omega \in \mathbb{R} \cup \{\infty\}$

$$\begin{bmatrix} (\mathbf{j}\omega I_n - A)^{-1}B \\ I_m \end{bmatrix}^{*} K \begin{bmatrix} (\mathbf{j}\omega I_n - A)^{-1}B \\ I_m \end{bmatrix} \prec 0.$$

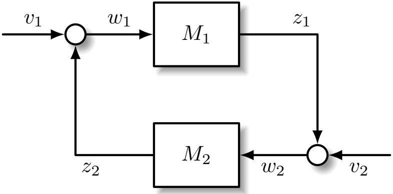
Also: if (A, B) is controllable, then (i)$\Longleftrightarrow$(ii) with non-strict inequalities.

A striking abstract relationship between time and frequency domain.

Comments on the KYP lemma

- The frequency-domain inequality is an *infinite-dimensional* analytic test; you need to check it for all ω. The KYP Lemma shows that this is equivalent to a *finite-dimensional* LMI. Truly amazing!
- The LMI ***does not*** require that $P \succeq 0$. However, note that if $K_{11} \succeq 0$, then from the (1,1) block of the LMI we conclude that $A^{\mathsf{T}}P + PA \prec 0$. Since A is Hurwitz, this implies that $P \succ 0$ (try to prove this). So sometimes definiteness of P comes for free.
- You will find many versions of this result in the literature, most of them looking quite different than this one!
- Many contributors other than Kalman, Yakubovich, and Popov: Anderson, Willems, Rantzer, Balakrishnan, Vandenberghe, . . .

Interconnections of dissipative systems



- Assume that each system *individually* is quadratically dissipative with positive-definite storage function:

$$\dot{V}_1(x_1) \leq \begin{bmatrix} z_1 \\ w_1 \end{bmatrix}^{\mathsf{T}} \Pi_1 \begin{bmatrix} z_1 \\ w_1 \end{bmatrix}, \qquad \dot{V}_2(x_2) \leq \begin{bmatrix} z_2 \\ w_2 \end{bmatrix}^{\mathsf{T}} \Pi_2 \begin{bmatrix} z_2 \\ w_2 \end{bmatrix}$$

- Interconnection conditions

$$w_1 = z_2 + v_1, \qquad w_2 = z_1 + v_2$$

Interconnections of dissipative systems

- Let $\alpha_1, \alpha_2 > 0$ and set $V(x) = \alpha_1 V_1(x_1) + \alpha_2 V_2(x_2)$.
- Trajectories of the unforced $(v_1 = v_2 = 0)$ system satisfy

$$\dot{V}(x(t)) \leq \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}^{\mathsf{T}} \underbrace{\left(\alpha_1 \Pi_1 + \alpha_2 \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] \Pi_2 \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]\right)}_{\triangleq \Pi(\alpha_1, \alpha_2)} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

- Suppose now that
  (i) $\exists \alpha_1, \alpha_2 > 0$ such that $\Pi(\alpha_1, \alpha_2) \prec 0$ (**LMI Problem!**) and
  (ii) (C_1, A_1) and (C_2, A_2) are observable.

$\Longrightarrow$ origin is globally exponentially stable!

A general and classic stability result (Hill/Moylan '77). Can you spot any results you already know as special cases?

Appendix: Proof of (i)⇒(ii) for Theorem 5.2/5.3

(i) $\Longrightarrow$ (ii): Multiplying out the matrices, the LMI can be equivalently written as

$$\begin{bmatrix} A^{\mathsf{T}}P + PA + K_{11} & PB + K_{12} \\ B^{\mathsf{T}}P + K_{12}^{\mathsf{T}} & K_{22} \end{bmatrix} \prec \mathbb{0},$$

and so we conclude via Schur's Lemma that $K_{22} \prec 0$. Let $\omega \in \mathbb{R}$ and let $(x, w) \neq \mathbb{0}$ be such that $(A - \mathbf{j}\omega I_n)x + Bw = \mathbb{0}_n$, or equivalently $Ax + Bw = \mathbf{j}\omega x$. Right and left-multiplying the LMI by (x, w) we have

$$\begin{aligned}
& \begin{bmatrix} x \\ w \end{bmatrix}^* \begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} x \\ w \end{bmatrix}^* K \begin{bmatrix} x \\ w \end{bmatrix} < 0 \\
\Longrightarrow \quad & \begin{bmatrix} x \\ \mathbf{j}\omega x \end{bmatrix}^* \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} x \\ \mathbf{j}\omega x \end{bmatrix} + \begin{bmatrix} x \\ w \end{bmatrix}^* K \begin{bmatrix} x \\ w \end{bmatrix} < 0 \\
\Longrightarrow \quad & 0 + \begin{bmatrix} x \\ w \end{bmatrix}^* K \begin{bmatrix} x \\ w \end{bmatrix} < 0\,,
\end{aligned}$$

so we conclude that the inequality in (ii) holds in Theorem 5.2. For Theorem 5.3, since A has no imaginary axis eigenvalues the unique x is given by $x = (A - \mathbf{j}\omega I_n)^{-1}Bw$. Substituting this in immediately yields the FDI in Theorem 5.3 (ii).

Appendix: Proof of (ii)⇒(i) for Theorem 5.3

The proof is by contradiction. First note that if $K_{22} \prec 0$ is violated, then (i) is automatically false, so assume that $K_{22} \prec 0$. Assume now that the LMI is infeasible. This means that

$$p_{\mathrm{opt}} = \inf_{P \in \mathbb{S}^n, \gamma \geq 0 \text{ s.t. } \left[\begin{smallmatrix} I_n & \mathbb{0} \\ A & B \end{smallmatrix}\right]^{\mathsf{T}} \left[\begin{smallmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{smallmatrix}\right] \left[\begin{smallmatrix} I_n & \mathbb{0} \\ A & B \end{smallmatrix}\right] + K \preceq \gamma I} \gamma \qquad \geq 0.$$

Note that the constraints of this problem are *strictly feasible*, since we can always find γ sufficiently large such that the LMI holds as a strict LMI. It follows that the problem has zero duality gap, so the Lagrange dual problem has the same optimal value $d_{\mathrm{opt}} = p_{\mathrm{opt}}$. To compute the dual, we need the adjoint of the Lyapunov operator

$$F_1(P) \triangleq \begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix}.$$

Skipping the details, calculations show that $F_1^{\mathrm{adj}} : \mathbb{S}^{n+m} \to \mathbb{S}^n$ is given by

$$F_1^{\mathrm{adj}}(Y) = \begin{bmatrix} A & B \end{bmatrix} Y \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} I & 0 \end{bmatrix} Y \begin{bmatrix} A^{\mathsf{T}} \\ B^{\mathsf{T}} \end{bmatrix}.$$

Appendix: Proof of (ii)⇒(i) for Theorem 5.3

The dual problem is therefore

$$d_{\text{opt}} = \sup_{Y \succeq 0,\ [\,A\ \ B\,] Y \left[\begin{smallmatrix} I_n \\ 0 \end{smallmatrix}\right] + [\,I_n\ \ 0\,] Y \left[\begin{smallmatrix} A^\mathsf{T} \\ B^\mathsf{T} \end{smallmatrix}\right] = 0} \operatorname{trace}(KY) \geq 0.$$

By strong alternatives, we know that there is therefore a non-zero $Y \succeq 0$ such that

$$[\,A\ \ B\,]\, Y \begin{bmatrix} I_n \\ 0 \end{bmatrix} + [\,I_n\ \ 0\,]\, Y \begin{bmatrix} A^\mathsf{T} \\ B^\mathsf{T} \end{bmatrix} = 0, \qquad \operatorname{trace}(KY) \geq 0.$$

If we partition $Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^\mathsf{T} & Y_{22} \end{bmatrix}$, then a separate argument shows that $Y_{11} \neq 0$ and hence Y admits a factorization of the form

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^\mathsf{T} & Y_{22} \end{bmatrix} = \begin{bmatrix} V & 0 \\ W & U \end{bmatrix} \begin{bmatrix} V & 0 \\ W & U \end{bmatrix}^* = \begin{bmatrix} VV^* & VW* \\ WV^* & WW^* + UU^* \end{bmatrix}.$$

where V has full column rank. Substituting this in, we find that

$$AVV^* + BWV^* + (AVV^* + BWV^*)^* = 0$$

and therefore $AVV^* + BWV^*$ is skew-Hermitian.

Appendix: Proof of (ii)⇒(i) for Theorem 5.3

It follows that we may write $AVV^* + BWV^* = VJV^*$ for some $J + J^* = 0$. Since V has full column rank, this implies that $AV + BW = VJ$.

Our previous condition $\operatorname{trace}(KY) \geq 0$ can be written as

$$
\begin{aligned}
0 \leq \operatorname{trace}(YK) &= \operatorname{trace} \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^{\mathsf{T}} & K_{22} \end{bmatrix} \begin{bmatrix} V & 0 \\ W & U \end{bmatrix} \begin{bmatrix} V & 0 \\ W & U \end{bmatrix}^* \\
&= \operatorname{trace} \begin{bmatrix} V & 0 \\ W & U \end{bmatrix}^* \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^{\mathsf{T}} & K_{22} \end{bmatrix} \begin{bmatrix} V & 0 \\ W & U \end{bmatrix} \\
&= \operatorname{trace} \begin{bmatrix} V \\ W \end{bmatrix}^* \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^{\mathsf{T}} & K_{22} \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix} + \operatorname{trace} U^* K_{22} U \\
&\leq \operatorname{trace} \begin{bmatrix} V \\ W \end{bmatrix}^* \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^{\mathsf{T}} & K_{22} \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix}
\end{aligned}
$$

since $K_{22} \prec 0$. Let $J = QSQ^{-1}$ be a Schur decomposition of J; since J is skew-symmetric, S is diagonal with imaginary entries. The matrix Q is unitary satisfying $QQ^* = I$, and we may write

$$
Q = \begin{bmatrix} q_1 & \cdots & q_r \end{bmatrix}
$$

Appendix: Proof of (ii)⇒(i) for Theorem 5.3

$$\operatorname{trace}\begin{bmatrix} V \\ W \end{bmatrix}^* \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^{\mathsf{T}} & K_{22} \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix} = \operatorname{trace} Q^* \begin{bmatrix} V \\ W \end{bmatrix}^* \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^{\mathsf{T}} & K_{22} \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix} Q$$
$$= \sum_k q_k^* \begin{bmatrix} V \\ W \end{bmatrix}^* \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^{\mathsf{T}} & K_{22} \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix} q_k \geq 0$$

Obviously, at least one term in this sum must be nonnegative. Let k be the associated index, and define $x_k = Vq_k$, $w_k = Wq_k$, and let $\mathbf{j}\omega_k = \mathbb{e}_k^T S \mathbb{e}_k$ be the associated eigenvalue of J. Note that since V has full column rank, we have that $\operatorname{col}(x_k, w_k) \neq 0$. From $AV + BW = VJ = VQSQ^*$, we have that $AVQ + BWQ - VQS = 0$, the kth column of which reads as

$$0 = AVq_k + BWq_k - \mathbf{j}\omega_k Vq_k = Ax_k + Bw_k - \mathbf{j}\omega_k x_k = (A - \mathbf{j}\omega_k I_n)x_k + Bw_k.$$

We therefore have $\omega_k \in \mathbb{R}$ and a vector $\operatorname{col}(x_k, w_k) \neq 0$ such that $0 = (A - \mathbf{j}\omega_k I_n)x_k + Bw_k$ such that

$$\begin{bmatrix} x_k \\ w_k \end{bmatrix}^* K \begin{bmatrix} x_k \\ w_k \end{bmatrix} \geq 0$$

which contradicts statement (ii). •

Appendix: Quadratically dissipative systems

Theorem 5.4 (Dissipativity). The following are equivalent:

(i) system (CT-LTI) is dissipative with quadratic supply rate s and storage function $V(x) = x^\mathsf{T} P x$, where $P \in \mathbb{S}^n$;

(ii) there exists $P \in \mathbb{S}^n$ satisfying the LMI

$$\begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix} - \begin{bmatrix} C & D \\ \mathbb{0} & I_m \end{bmatrix}^\mathsf{T} \Pi \begin{bmatrix} C & D \\ \mathbb{0} & I_m \end{bmatrix} \preceq \mathbb{0}.$$

If additionally (A, B) is controllable, then a third equivalent statement is

(iii) for all $\omega \in \mathbb{R} \cup \{\infty\}$ such that $\mathbf{j}\omega \notin \mathrm{eig}(A)$ the frequency response $\hat{M}(\mathbf{j}\omega)$ satisfies

$$\begin{bmatrix} \hat{M}(\mathbf{j}\omega) \\ I_m \end{bmatrix}^* \Pi \begin{bmatrix} \hat{M}(\mathbf{j}\omega) \\ I_m \end{bmatrix} \succeq \mathbb{0}.$$

Appendix: Available Storage and Required Supply

- Dissipativity: $\dot{V}(x(t)) \leq s(w(t), z(t))$
- Controllable dissipative systems have two canonical storage functions

Definition 5.3. Consider the system (CT-LTI) with $x(0) = x_0 \in \mathbb{R}^n$, and let $s(w, z)$ be a supply rate. The ***available storage from $\boldsymbol{x_0}$*** is

$$V_{\mathrm{av}}(x_0) = \sup_{\substack{w(\cdot) \\ T \geq 0}} \left\{ -\int_0^T s(w(t), z(t))\, \mathrm{d}t \;:\; x(T) = 0 \right\}.$$

The ***required supply to $\boldsymbol{x_0}$*** is

$$V_{\mathrm{req}}(x_0) = \inf_{\substack{w(\cdot) \\ T \geq 0}} \left\{ \int_{-T}^0 s(w(t), z(t))\, \mathrm{d}t \;:\; x(-T) = 0 \right\}.$$

Appendix: Available Storage and Required Supply

Proposition 5.1. If (CT-LTI) is controllable and dissipative with storage function $V(x)$ satisfying $V(0) = 0$ and quadratic $s(w, z)$, then

(i) $V_{\mathrm{av}}(x)$ and $V_{\mathrm{req}}(x)$ are both storage functions,

(ii) $V_{\mathrm{av}}(x) \leq V(x) \leq V_{\mathrm{req}}(x)$,

(iii) there exists $P_{-} \in \mathbb{S}^n$ such that $V_{\mathrm{av}}(x) = x^{\mathsf{T}} P_{-} x$, and

(iv) there exists $P_{+} \in \mathbb{S}^n$ such that $V_{\mathrm{req}}(x) = x^{\mathsf{T}} P_{+} x$.

- $V_{\mathrm{av}}(x)$ finite for all $x \Longrightarrow$ you can only extract finite energy from a dissipative system from any state
- $V_{\mathrm{req}}(x)$ finite for all $x \Longrightarrow$ you need only provide finite energy to a dissipative system to transition to any state

Appendix: Available Storage and Required Supply

We prove the results for $V_{\rm av}(x)$; the results for $V_{\rm req}(x)$ are similar. Let $T \geq 0$ and let $(w(t), x(t), z(t))$ be a trajectory of (CT-LTI) such that $x(0) = x_0$ and $x(T) = 0$; by controllability, such a trajectory exists. By dissipativity, we know that

$$\underbrace{V(x(T))}_{=0} - V(x_0) \leq \int_0^T s(w(t), z(t))\,\mathrm{d}t$$

Taking the supremum over $T \geq 0$ and $w(\cdot)$, we find that $V_{\rm av}(x) \leq V(x_0)$ which shows (ii). To show (i), let $0 \leq \tau \leq T$ and note that, by definition

$$V_{\rm av}(x_0) \geq -\int_0^\tau s(w(t), z(t))\,\mathrm{d}t - \int_\tau^T s(w(t), z(t))\,\mathrm{d}t$$

The second term on the RHS is lower bounded by $V_{\rm av}(x(\tau))$, and thus

$$V_{\rm av}(x_0) - V_{\rm av}(x(\tau)) \geq -\int_0^\tau s(w(t), z(t))\,\mathrm{d}t$$

which shows (i). Item (iii) follows from the fact that the optimal value of a quadratic functional subject to linear dynamics is always quadratic function of the initial condition.

Appendix: Causal DT-LTI systems

- Consider the finite-dimensional discrete-time state-space model

$$M: \quad \begin{aligned} x(k+1) &= Ax(k) + Bw(k), \qquad x(0) = 0 \\ z(k) &= Cx(k) + Dw(k) \end{aligned}$$

- The state and output are of course given by

$$x(k) = \left[\sum_{\ell=0}^{k-1} CA^{k-\ell-1}Bw(\ell)\right]\mathsf{1}(k)$$

$$z(k) = Cx(k) + Dw(k)$$

- As you know, the system has a ***transfer function***

$$\hat{M}(z) = C(zI_n - A)^{-1}B + D, \qquad z \in \mathsf{ROC}$$

- Assuming ***BIBO stability***, the system also has a ***frequency response***

$$\hat{M}(e^{\mathbf{j}\omega}) = C(e^{\mathbf{j}\omega}I_n - A)^{-1}B + D$$

Appendix: Discrete-time dissipativity

Definition 5.4 (Dissipativity). Let $s : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ be a supply rate. The state-space system (DT-LTI) is ***dissipative*** if there exists a storage function $V : \mathbb{R}^n \to \mathbb{R}$ and $\varepsilon \geq 0$ such that

$$V(Ax + Bw) - V(x) \leq s(w, z) - \varepsilon^2 \|w\|_2^2$$

for all $(x, w) \in \mathbb{R}^{n+m}$. If $\varepsilon > 0$, the system is ***input-strictly dissipative***.

- If $(w(k), x(k), z(k))$ is a system trajectory, then we have that

$$\underbrace{V(x(k+1)) - V(x(k))}_{\text{Change in Stored Energy}} \leq \underbrace{s(w(k), z(k)) - \varepsilon^2 \|w(k)\|_2^2}_{\text{Externally Provided Power}}$$

- Often (not always) $V(x) \geq 0$

Appendix: dissipative DT systems

Theorem 5.5 (Strict Dissipativity). Assume that A is Schur. Then the following statements are equivalent:

(i) system (DT-LTI) is input-strictly dissipative with quadratic supply rate s and storage function $V(x) = x^\mathsf{T} P x$, where $P \in \mathbb{S}^n$;

(ii) there exists $P \in \mathbb{S}^n$ satisfying the strict LMI

$$\begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix}^\mathsf{T} \begin{bmatrix} -P & \mathbb{0} \\ \mathbb{0} & P \end{bmatrix} \begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix} - \begin{bmatrix} C & D \\ \mathbb{0} & I_m \end{bmatrix}^\mathsf{T} \Pi \begin{bmatrix} C & D \\ \mathbb{0} & I_m \end{bmatrix} \prec \mathbb{0}.$$

(iii) for all $\omega \in [0, \pi]$ the frequency response $\hat{M}(e^{\mathbf{j}\omega})$ satisfies

$$\begin{bmatrix} \hat{M}(e^{\mathbf{j}\omega}) \\ I_m \end{bmatrix}^* \Pi \begin{bmatrix} \hat{M}(e^{\mathbf{j}\omega}) \\ I_m \end{bmatrix} \succ \mathbb{0}.$$

Appendix: The discrete-time KYP lemma

Theorem 5.6 (Discrete KYP Lemma). Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and let $K = \left[\begin{smallmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{smallmatrix}\right] \in \mathbb{S}^{n+m}$. If A is Schur stable, then the following two statements are equivalent:

(i) there exists a symmetric matrix $P \in \mathbb{S}^n$ satisfying the strict LMI

$$\begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} -P & \mathbb{0} \\ \mathbb{0} & P \end{bmatrix} \begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix} + K \prec \mathbb{0}$$

(ii) $K_{22} \prec \mathbb{0}$ and for all $\omega \in [0, \pi]$

$$\begin{bmatrix} (e^{\mathbf{j}\omega} I_n - A)^{-1} B \\ I_m \end{bmatrix}^* K \begin{bmatrix} (e^{\mathbf{j}\omega} I_n - A)^{-1} B \\ I_m \end{bmatrix} \prec 0.$$

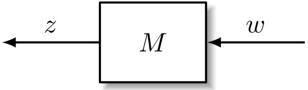
Also: if (A, B) is controllable, then (i)$\Longleftrightarrow$(ii) with non-strict inequalities.

6. Signals and Systems for Stability and Performance Analysis

Problem setup for I/O performance

We will focus on FD CT-LTI systems

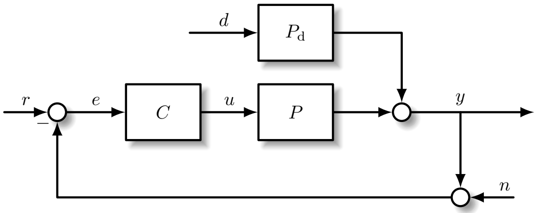
$$M : \left[\begin{array}{c} \dot{x} \\ \hline z \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c} x \\ \hline w \end{array}\right]$$



with $x(0) = \mathbb{0}_n$ and frequency response $\hat{M}(\mathbf{j}\omega) = C(\mathbf{j}\omega I_n - A)^{-1}B + D$.

- $w(t)$ is an ***exogenous input***; a vector of signals from the environment that drives the system. This could include (possibly, weighted) process disturbances, reference commands, and measurement noise.
- $z(t)$ is a ***performance output***; a vector of signals that should be (in some sense to be determined) kept small. Typically, z contains (possibly, weighted) tracking errors, states, and/or control signals.

Example: SISO control loop



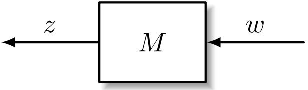
- Exogenous signals $w = (r, d, n)$, performance signals $z = (e, u)$
- Model M is easily described by, e.g., a 2×3 transfer matrix

$$\begin{bmatrix} e \\ u \end{bmatrix} = \begin{bmatrix} \frac{1}{1+PC} & -\frac{P_{\mathrm{d}}}{1+PC} & \frac{-1}{1+PC} \\ \frac{C}{1+PC} & -\frac{P_{\mathrm{d}}C}{1+PC} & \frac{-C}{1+PC} \end{bmatrix} \begin{bmatrix} r \\ d \\ n \end{bmatrix}$$

Disturbance types in I/O performance

Big question: how to quantify the effect of w on z?

$$M: \quad \left[\begin{array}{c} \dot{x} \\ \hline z \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c} x \\ \hline w \end{array}\right]$$



Any sensible answer ***must*** depend on the character of w. For instance

(i) ***Deterministic* w:** Compare the output energy to the input energy

(ii) ***Stochastic* w:** Look at the variance of the output

(iii) ***Impulsive* w:** Look at the energy in the impulse response

We need to look closer at signal modelling.

Modelling deterministic time-domain signals

- The simplest model of a time-domain signal is as a map from $\mathbb{R}$ into a vector space, often $\mathbb{F}^n$

$$\mathrm{Sig}(\mathbb{R};\mathbb{F}^n) \triangleq \{f \mid f : \mathbb{R} \to \mathbb{F}^n\}.$$

- The set $\mathrm{Sig}(\mathbb{R};\mathbb{F}^n)$ has an obvious vector space structure (***that's good***) but cannot be normed (***that's bad***).

- So, what else should go into a useful set of signals? Potentially ...
  - (i) Restrictions on support
  - (ii) Continuity, differentiability, ...
  - (iii) Boundedness
  - (iv) Integrability, square integrability, ...

Modelling deterministic time-domain signals

- **Recall:** the space $\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{F})$ of all *continuous* and *compactly supported* signals

$$\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{F}^n) = \{f \in \mathsf{C}^0(\mathbb{R};\mathbb{F}^n) \mid \exists\, T > 0 \text{ s.t. } f(t) = 0 \;\forall |t| \geq T\}.$$

- This is a normed vector space with any of

$$\|f\|_{\mathcal{L}_\mathrm{p}} = \left(\int_{-\infty}^{\infty} \|f(t)\|_2^p \,\mathrm{d}t\right)^{1/p}, \qquad p \in [1,\infty)$$
$$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(x)\|_2$$

but it is not complete with any of these norms.

Completing $\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{F}^n)$ leads to useful Banach spaces of signals.

Banach spaces of continuous-time signals

Theorem 6.1 (Banach spaces of CT signals).

(i) The completion of $\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{F}^n)$ in the norm $\|\cdot\|_\infty$ is

$$\mathsf{C}^0_0(\mathbb{R};\mathbb{F}^n) = \{f : \mathbb{R} \to \mathbb{F}^n \mid f \text{ continuous and } \lim_{t\to\pm\infty} \|f(t)\|_2 = 0\}$$

(ii) The completion of $\mathsf{C}^0_{\mathrm{cpt}}(\mathbb{R};\mathbb{F}^n)$ in the norm $\|\cdot\|_{\mathcal{L}_\mathrm{p}}$ for $p \in [1,\infty)$ is

$$\mathcal{L}_\mathrm{p}(\mathbb{R};\mathbb{F}^n) = \left\{f : \mathbb{R} \to \mathbb{F}^n \;\middle|\; \|f\|_{\mathcal{L}_\mathrm{p}} < \infty\right\}.$$

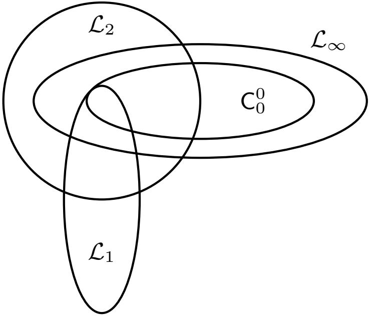
(iii) The space $\mathcal{L}_\infty(\mathbb{R};\mathbb{F}^n) = \{f : \mathbb{R} \to \mathbb{F}^n \mid \|f\|_{\mathcal{L}_\infty} < \infty\}$ is a Banach space, where

$$\|f\|_{\mathcal{L}_\infty} = \inf\{M \geq 0 \;\big|\; \|f(t)\|_2 \leq M \text{ almost everywhere}\}.$$

- **Note:** if $f \in \mathsf{C}^0(\mathbb{R};\mathbb{F}^n) \cap \mathcal{L}_\infty(\mathbb{R};\mathbb{F}^n)$, then $\|f\|_{\mathcal{L}_\infty} = \|f\|_\infty$.

Comments on complete signal spaces

- Within the $\mathcal{L}_\mathrm{p}/\mathcal{L}_\infty$ spaces, there are ***no continuity requirements***.
- The $\mathcal{L}_\mathrm{p}/\mathcal{L}_\infty$ norms do not care if you change the signal values at a point; two signals are considered as the same if they are equal almost everywhere.
- There are *some* relationships between the signal spaces, but few inclusions



Selected results:

(i) $\mathsf{C}_0^0(\mathbb{R};\mathbb{F}^n) \subset \mathcal{L}_\infty(\mathbb{R};\mathbb{F}^n)$.

(ii) If $f \in \mathcal{L}_1(\mathbb{R};\mathbb{F}^n) \cap \mathcal{L}_\infty(\mathbb{R};\mathbb{F}^n)$, then $f \in \mathcal{L}_2(\mathbb{R};\mathbb{F}^n)$.

(iii) If $f, g \in \mathcal{L}_2(\mathbb{R};\mathbb{F})$, then $fg \in \mathcal{L}_1(\mathbb{R};\mathbb{F})$.

Comments on complete signal spaces

- The $\mathcal{L}_\mathrm{p}$ spaces are defined in terms of integrals, so we are talking about "finite area under the curve". Surely then, the signals must be bounded and tend to 0? ***No.*** For instance, with $a > 0$ and

$$\mathsf{Box}_a(t) = \mathbb{1}(t + \tfrac{a}{2}) - \mathbb{1}(t - \tfrac{a}{2}), \quad f(t) = \sum_{n=2}^{\infty} n \cdot \mathsf{Box}_{\frac{1}{n^2}}(t - n)$$

  we have $\|f\|_{\mathcal{L}_2} < \infty$, but f is unbounded and never tends to 0.

- The major issue here ends up being a lack of bound on the derivative; with this additional assumption things become more intuitive.

Lemma 6.1. If $f \in \mathsf{C}^1(\mathbb{R};\mathbb{F}^n) \cap \mathcal{L}_\mathrm{p}(\mathbb{R};\mathbb{F}^n)$ and $\dot{f} \in \mathcal{L}_\infty(\mathbb{R};\mathbb{F}^n)$, then $f \in \mathsf{C}_0^1(\mathbb{R};\mathbb{F}^n) \cap \mathcal{L}_\mathrm{p}(\mathbb{R};\mathbb{F}^n)$.

Be careful interpreting these signal spaces! They are not intuitive.

The $\mathcal{L}_2$-space of signals

- For the $\mathcal{L}_\mathrm{p}$ spaces, the case $p = 2$ is extremely important

$$\|f\|_{\mathcal{L}_2} = \left(\int_{-\infty}^{\infty} \|f(t)\|_2^2 \,\mathrm{d}t\right)^{1/2}$$

- The norm $\|\cdot\|_{\mathcal{L}_2}$ can be seen to arise from the inner product

$$\langle f, g\rangle_{\mathcal{L}_2} \triangleq \int_{-\infty}^{\infty} \langle f(t), g(t)\rangle_2 \,\mathrm{d}t = \int_{-\infty}^{\infty} f(t)^* g(t) \,\mathrm{d}t.$$

  with associated Cauchy-Schwarz inequality

$$|\langle f, g\rangle_{\mathcal{L}_2}| \leq \|f\|_{\mathcal{L}_2} \|g\|_{\mathcal{L}_2}$$

- Thus, $\mathcal{L}_2(\mathbb{R}; \mathbb{F}^n)$ is a Hilbert space, and signals $f \in \mathcal{L}_2(\mathbb{R}; \mathbb{F}^n)$ are interpreted as having ***finite energy***

The $\mathcal{L}_1$ Fourier Transform

- The ***Fourier Transform*** $\mathscr{F}(f)$ of a signal $f \in \mathcal{L}_1(\mathbb{R};\mathbb{F}^n)$ is defined by

$$\mathscr{F}(f)(\mathbf{j}\omega) \triangleq \int_{-\infty}^{\infty} f(t)e^{-\mathbf{j}\omega t}\,\mathrm{d}t.$$

Proposition 6.1. Let $f \in \mathcal{L}_1(\mathbb{R};\mathbb{F}^n)$. The following statements hold:

(i) $\mathscr{F} : \mathcal{L}_1(\mathbb{R};\mathbb{F}^n) \to \mathsf{C}_0^0(\mathbf{j}\mathbb{R};\mathbb{F}^n)$, so $\hat{f} = \mathscr{F}(f)$ is continuous in ω, bounded, and tends to 0 as $\omega \to \pm\infty$;

(ii) $\mathscr{F}$ is a bounded linear operator, and $\|\mathscr{F}\|_{\mathcal{L}_\mathrm{p}\to\mathsf{C}_0^0} \leq 1$;

(iii) $\mathscr{F}$ is injective, and therefore possess a left inverse

$$\mathscr{F}^{-1} : \mathsf{C}_0^0(\mathbf{j}\mathbb{R};\mathbb{F}^n) \to \mathcal{L}_1(\mathbb{R};\mathbb{F}^n), \qquad \mathscr{F}^{-1} \circ \mathscr{F}(f) = f.$$

The $\mathcal{L}_2$ Fourier Transform

It is possible to extend the definition of the Fourier transform from $\mathcal{L}_1$ to $\mathcal{L}_2$ via a limiting procedure; we skip the details.

- For frequency-domain $\mathcal{L}_2$ signals we will use the inner product

$$\langle \hat{f}, \hat{g} \rangle_{\mathcal{L}_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\mathbf{j}\omega)^* \hat{g}(\mathbf{j}\omega) \, \mathrm{d}\omega.$$

Theorem 6.2. The $\mathcal{L}_2$ Fourier transform $\mathscr{F} : \mathcal{L}_2(\mathbb{R}; \mathbb{F}^n) \to \mathcal{L}_2(\mathbf{j}\mathbb{R}; \mathbb{F}^n)$ is a bounded and invertible linear operator satisfying $\langle f, g \rangle_{\mathcal{L}_2} = \langle \hat{f}, \hat{g} \rangle_{\mathcal{L}_2}$, or explicitly

$$\int_{0}^{\infty} f(t)^* g(t) \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\mathbf{j}\omega)^* \hat{g}(\mathbf{j}\omega) \, \mathrm{d}\omega.$$

for any $f, g \in \mathcal{L}_2(\mathbb{R}; \mathbb{F}^n)$ with transforms $\hat{f} = \mathscr{F}(f)$ and $\hat{g} = \mathscr{F}(g)$.

Truncation and right-sided signals

- A useful tool is ***truncation***: for $T \geq 0$, the ***T-truncation operator*** $\mathcal{T}_T : \mathsf{Sig}(\mathbb{R};\mathbb{F}^n) \to \mathsf{Sig}(\mathbb{R};\mathbb{F}^n)$ is defined as

$$\mathcal{T}_T(f)(t) \triangleq \begin{cases} f(t) & \text{if} \quad -T \leq t \leq T \\ \mathbb{0} & \text{else.} \end{cases}$$

  and we often write $f_T = \mathcal{T}_T(f)$.

- We will work mostly with ***right-sided*** time-domain signals, which are forced to equal zero for $t < 0$.

- Our notational convention indicating right-sidedness will be

$$\mathsf{Sig}[0,\infty), \quad \mathcal{L}_1[0,\infty), \quad \mathcal{L}_2[0,\infty), \quad \mathcal{L}_\infty[0,\infty), \quad \text{etc.}$$

Extended right-sided $\mathcal{L}_\mathrm{p}$-spaces of signals

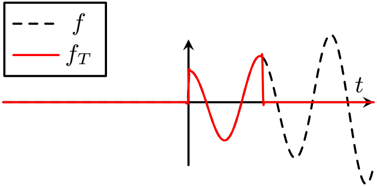
- The $\mathcal{L}_\mathrm{p}$ spaces are unfortunately missing some fairly benign signals that we would wish to work with for control applications

- **Example:** $t \mapsto 1(t)$ is in $\mathcal{L}_\infty$, but not in $\mathcal{L}_\mathrm{p}$ for any $\mathrm{p} \in [1, \infty)$, and $t \mapsto e^t 1(t)$ is not in $\mathcal{L}_\mathrm{p}$ or $\mathcal{L}_\infty$.

- For $\mathrm{p} \in [1, \infty]$ we define the ***extended*** $\mathcal{L}_\mathrm{p}[0, \infty)$ spaces

$$\mathcal{L}_\mathrm{pe}[0,\infty) \triangleq \left\{ f \in \mathsf{Sig}[0,\infty) \;\middle|\; f_T \in \mathcal{L}_\mathrm{p}[0,\infty) \quad \text{for all} \quad T \geq 0 \right\}$$

in which *truncated* signals must have finite $\mathcal{L}_\mathrm{p}$-norm.



Extended right-sided $\mathcal{L}_\mathrm{p}$-spaces of signals

- $\mathcal{L}_\mathrm{pe}[0,\infty)$ is a vector space, but is ***not*** a normed vector space.
- Nonetheless, we can still discuss convergence in $\mathcal{L}_\mathrm{pe}[0,\infty)$, and $\mathcal{L}_\mathrm{pe}[0,\infty)$ relates in a nice continuous way to $\mathcal{L}_\mathrm{p}[0,\infty)$

Proposition 6.2 (Facts about $\mathcal{L}_\mathrm{pe}$).

(i) $\mathcal{L}_\mathrm{p}[0,\infty)$ is a subspace of $\mathcal{L}_\mathrm{pe}[0,\infty)$;

(ii) If $f \in \mathcal{L}_\mathrm{pe}[0,\infty)$, then $T \mapsto \|f_T\|_{\mathcal{L}_\mathrm{p}}$ is a non-decreasing function;

(iii) If $f \in \mathcal{L}_\mathrm{pe}[0,\infty)$, then $f \in \mathcal{L}_\mathrm{p}[0,\infty)$ if and only if $\lim_{T\to\infty} \|f_T\|_{\mathcal{L}_\mathrm{p}} < \infty$, in which case $\lim_{T\to\infty} \|f_T\|_{\mathcal{L}_\mathrm{p}} = \|f\|_{\mathcal{L}_\mathrm{p}}$;

(iv) If $f \in \mathcal{L}_\mathrm{pe}[0,\infty)$, then $f_T \in \mathcal{L}_\mathrm{q}[0,\infty)$ for all $\mathrm{q} \in [1,\infty]$ satisfying $\mathrm{q} \geq \mathrm{p}$.

Final comments on signal modelling

- We have a nice collection of signal spaces to work with, norms, relationships between the spaces, and even an inner product on $\mathcal{L}_2$ ...
- For properly discussing all relevant aspects of linear systems theory, this discussion is not *quite* comprehensive enough, because
  (i) even $\mathcal{L}_{2\mathrm{e}}[0, \infty)$ does not contain the Dirac impulse signal δ, and
  (ii) we have no models of random signals.
- We will sidestep these issues by using "$\delta(t)$" when needed anyways, and we will not worry about making our (brief) stochastic arguments very rigorous

Systems as signal-space operators

- We think of a ***system*** M as a mapping between $\mathcal{L}_{\mathrm{pe}}[0,\infty)$ spaces, and most often, as a mapping between $\mathcal{L}_{2\mathrm{e}}[0,\infty)$ spaces

$$M : \mathcal{L}_{2\mathrm{e}}[0,\infty) \to \mathcal{L}_{2\mathrm{e}}[0,\infty), \qquad M(0) = 0$$

- M takes an input w and produces an output $z = M(w)$
- The operation M usually ***cannot*** be written out, as this would usually amount to explicitly solving the underlying (e.g., nonlinear differential) equations.
- That M actually maps $\mathcal{L}_{2\mathrm{e}}[0,\infty)$ to $\mathcal{L}_{2\mathrm{e}}[0,\infty)$ is a standing assumption which we term ***well-posedness***
- We will additionally ***impose*** causality as an assumption

Causality

Definition 6.1 (Causality). A system $M : \mathcal{L}_{2e}[0,\infty) \to \mathcal{L}_{2e}[0,\infty)$ is causal if for any $w, v \in \mathcal{L}_{2e}[0,\infty)$ and any $T \geq 0$

$$w(t) = v(t) \text{ for all } t \leq T \quad \Longrightarrow \quad M(w)(t) = M(v)(t) \text{ for all } t \leq T.$$

- If the inputs agree up to time T, the outputs must also agree

Proposition 6.3 (Causality). A system M is causal if and only if

$$\mathcal{T}_T \circ M = \mathcal{T}_T \circ M \circ \mathcal{T}_T, \qquad \text{for all } T \geq 0,$$

or equivalently if $M(w)_T = M(w_T)_T$ for all $w \in \mathcal{L}_{2e}[0,\infty)$ and all $T \geq 0$.

- The output at time T depends only on the input up to time T.

Sanity check for LTI systems

- For (CT-LTI), the signal-space mapping M is defined via convolution with the impulse response $m(t) = Ce^{At}B\,1(t) + D\delta(t)$, yielding $M(w)(t) = 0$ for $t < 0$ and

$$z(t) = M(w)(t) = Dw(t) + \int_0^t Ce^{A(t-\tau)}Bw(\tau)\mathrm{d}\tau, \qquad t \geq 0.$$

- Clearly $M(0) = 0$, and easy to show that M is causal
- **Proof on next slide:** If $w \in \mathcal{L}_{2\mathrm{e}}[0,\infty)$, then $z \in \mathcal{L}_{2\mathrm{e}}[0,\infty)$

Thus, our usual LTI model does indeed define a causal signal-space operator $M : \mathcal{L}_{2\mathrm{e}}[0,\infty) \to \mathcal{L}_{2\mathrm{e}}[0,\infty)$!

Proof that LTI maps $\mathcal{L}_{2\mathrm{e}}[0,\infty)$ to $\mathcal{L}_{2\mathrm{e}}[0,\infty)$

If $w \in \mathcal{L}_{2\mathrm{e}}[0,\infty)$, then $Dw \in \mathcal{L}_{2\mathrm{e}}[0,\infty)$, so we need only show that $f(t) \triangleq \int_0^t Ce^{A\tau}Bw(t-\tau)\mathrm{d}\tau$ belongs to $\mathcal{L}_{2\mathrm{e}}[0,\infty)$. We compute

$$\|f(t)\|_2 = \left\|\int_0^t Ce^{A\tau}Bw(t-\tau)\mathrm{d}\tau\right\|_2 \leq \int_0^t ce^{\gamma\tau}\|w(t-\tau)\|_2\mathrm{d}\tau$$

for some constants $c, \gamma > 0$. Therefore,

$$\|f_T\|_{\mathcal{L}_2}^2 = \int_0^T \|f(t)\|_2^2\,\mathrm{d}t \leq \int_0^t \int_0^t c^2 e^{\gamma\tau}e^{\gamma\sigma}\left[\int_0^T \|w(t-\tau)\|_2\|w(t-\sigma)\|_2\,\mathrm{d}t\right]\mathrm{d}\tau\mathrm{d}\sigma$$

Let $S_\tau : \mathcal{L}_{2\mathrm{e}}[0,\infty) \to \mathcal{L}_{2\mathrm{e}}[0,\infty)$ denote the shift operator $(S_\tau f)(t) = f(t-\tau)$. With $f(t) = \|w(t)\|_2$, the term in brackets is

$$\begin{aligned}\int_0^T \langle (S_\tau f)(t), (S_\sigma f)(t)\rangle_2\,\mathrm{d}t = \langle (S_\tau f)_T, (S_\sigma f)_T\rangle_{\mathcal{L}_2} &\leq \|(S_\tau f)_T\|_{\mathcal{L}_2}\|(S_\sigma f)_T\|_{\mathcal{L}_2} \\ &\leq \|f_T\|_{\mathcal{L}_2}\|f_T\|_{\mathcal{L}_2} = \|w_T\|_{\mathcal{L}_2}^2\end{aligned}$$

so $\|f_T\|_{\mathcal{L}_2} \leq \frac{c}{\gamma}(e^{\gamma T}-1)\|w_T\|_{\mathcal{L}_2} < \infty$ for all $T \geq 0$, so $f \in \mathcal{L}_{2\mathrm{e}}[0,\infty)$.

Stability of signal-space operators

Definition 6.2 (Stability). We say $M : \mathcal{L}_{2\mathrm{e}}[0,\infty) \to \mathcal{L}_{2\mathrm{e}}[0,\infty)$ is

(i) ***$\mathcal{L}_2$-stable*** if M maps $\mathcal{L}_2[0,\infty)$ to $\mathcal{L}_2[0,\infty)$.

(ii) ***$\mathcal{L}_2$-stable with finite gain*** if it is $\mathcal{L}_2$-stable and $\exists \gamma \geq 0$ s.t.

$$\|M(w)\|_{\mathcal{L}_2} \leq \gamma \|w\|_{\mathcal{L}_2}, \quad \forall w \in \mathcal{L}_2[0,\infty). \tag{6}$$

In this case $\|M\|_{\mathcal{L}_2 \to \mathcal{L}_2} = \inf\{\gamma \mid (6) \text{ holds}\}$ is the ***$\mathcal{L}_2$-gain*** of M.

(iii) ***$\mathcal{L}_{2\mathrm{e}}$-stable with finite gain*** if $\exists \gamma \geq 0$ s.t.

$$\|M(w)_T\|_{\mathcal{L}_2} \leq \gamma \|w_T\|_{\mathcal{L}_2}, \quad \forall T \geq 0,\ w \in \mathcal{L}_{2\mathrm{e}}[0,\infty). \tag{7}$$

In this case, $\gamma_{\mathrm{e}}(M) \triangleq \inf\{\gamma \mid (7) \text{ holds}\}$ is the ***$\mathcal{L}_{2\mathrm{e}}$-gain*** of M.

Comments on I/O stability definitions

- Item (i) says $\mathcal{L}_2$ inputs produce $\mathcal{L}_2$ outputs. This is a bit weak; the output can't be bounded in terms of the input.
- In (ii) and (iii), we try to bound the output in terms of the input, either using signals in $\mathcal{L}_2[0,\infty)$ or signals in $\mathcal{L}_{2\mathrm{e}}[0,\infty)$.
- Remarkably, (ii) and (iii) are ***equivalent***.

Proposition 6.4. A causal operator M is $\mathcal{L}_2$-stab. w/ finite gain if and only if M is $\mathcal{L}_{2\mathrm{e}}$-stab. w/ finite gain. In either case, $\gamma_e(M) = \|M\|_{\mathcal{L}_2 \to \mathcal{L}_2}$.

($\Longrightarrow$): Trivial ($\Longleftarrow$): For any $v \in \mathcal{L}_{2\mathrm{e}}[0,\infty)$ we have that $v_T \in \mathcal{L}_2[0,\infty)$, and therefore $\|M(v_T)\|_{\mathcal{L}_2} \leq \gamma\|v_T\|_{\mathcal{L}_2}$. By causality, we have that $M(v_T)_T = M(v)_T$ for all $v \in \mathcal{L}_{2\mathrm{e}}[0,\infty)$, and we can therefore compute for any $T \geq 0$ that

$$\|M(v)_T\|_{\mathcal{L}_2} = \|M(v_T)_T\|_{\mathcal{L}_2} \leq \|M(v_T)\|_{\mathcal{L}_2} \leq \gamma\|v_T\|_{\mathcal{L}_2}$$

which shows that the desired finite $\mathcal{L}_{2\mathrm{e}}$-gain result. •

Stability of LTI signal-space operators

- Consider a causal finite-dimensional CT-LTI system M

$$m(t) = Ce^{At}B1(t) + D\delta(t)$$
$$\hat{M}(s) = C(sI_n - A)^{-1}B + D$$

Proposition 6.5 (Stability of FD-LTI Systems). The following statements are equivalent:

(i) All poles of all elements of $\hat{M}(s)$ are contained in $\mathbb{C}_{<0}$;

(ii) $t \mapsto Ce^{At}B1(t)$ belongs to $\mathcal{L}_1[0, \infty)$

(iii) M is $\mathcal{L}_2$-stable;

(iv) M is $\mathcal{L}_2$-stable with finite gain.

These stability concepts are identical for LTI systems.

Proof of Proposition 6.5

(i) $\Longleftrightarrow$ (ii): This equivalence is standard.

(i) $\Longrightarrow$ (iv): Let (A, B, C, D) be a minimal realization of M, with A Hurwitz. Let $Q \succ 0$ be such that $Q \succ C^\mathsf{T} C$. By Lyapunov theory for LTI systems, there exists $P \succ 0$ such that $A^\mathsf{T} P + PA = -Q \prec -C^\mathsf{T} C$, or simply $A^\mathsf{T} P + PA + C^\mathsf{T} C \prec 0$. Since the inequality is strict, there exists some sufficiently large $\gamma > 0$ such that

$$A^\mathsf{T} P + PA + C^\mathsf{T} C + \frac{1}{\gamma^2}(PB + C^\mathsf{T} D)^\mathsf{T}(PB + C^\mathsf{T} D) \prec \mathbb{0}$$

or equivalently, via Schur complements, that

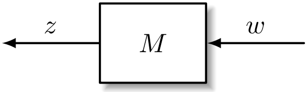
$$\begin{bmatrix} A^T P + PA & PB \\ B^\mathsf{T} P & \mathbb{0} \end{bmatrix} - \begin{bmatrix} C & D \\ 0 & I_m \end{bmatrix}^\mathsf{T} \begin{bmatrix} -I_p & 0 \\ 0 & \gamma^2 I_m \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I_m \end{bmatrix} \prec \mathbb{0}$$

The rest of the proof follows similar lines to that of Corollary 6.1, to follow.

(iii) $\Longrightarrow$ (i): Consider the SISO case. By contraposition, suppose that $\hat{M}(s)$ has at least one pole with nonnegative real part. Consider the input signal $w(t) = e^{-t} 1(t)$, which is obviously in $\mathcal{L}_2[0, \infty)$. Standard computation of $z(t)$ using partial fraction expansion will show that the response must contain a persistent or growing term, and hence will not be in $\mathcal{L}_2[0, \infty)$, so M is not $\mathcal{L}_2$-stable.

I/O performance: finite $\mathcal{L}_2$-gain

$$M: \left[\begin{array}{c} \dot{x} \\ \hline z \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c} x \\ \hline w \end{array}\right]$$



- We now understand that our state-space LTI system defines a causal signal-space operator

$$M : \mathcal{L}_{2\mathrm{e}}^m[0,\infty) \to \mathcal{L}_{2\mathrm{e}}^p[0,\infty), \quad M(w)(t) = Dw(t) + \int_0^t Ce^{A(t-\tau)}Bw(\tau)\mathrm{d}\tau$$

- If M is $\mathcal{L}_2$-stable with finite gain, then we know that we will have

$$\|z_T\|_{\mathcal{L}_2}^2 \leq \gamma^2 \|w_T\|_{\mathcal{L}_2}^2, \quad \forall\, T \geq 0,\, \forall w \in \mathcal{L}_{2\mathrm{e}}[0,\infty)$$

  which bounds output energy in terms of input energy.

- This looks *suspiciously like dissipativity* ...

Input-output $\mathcal{L}_2$-gain performance

Corollary 6.1 ("Bounded Real" Lemma). Assume that A is Hurwitz and let $\gamma > 0$. The following statements are equivalent:

(i) (CT-LTI) is i.s.d. with supply rate $s(w,z) = -\|z\|_2^2 + \gamma^2\|w\|_2^2$ and storage function $V(x) = x^\mathsf{T} P x$ with $P \succ 0$;

(ii) $\|M\|_{\mathcal{L}_2 \to \mathcal{L}_2} < \gamma$;

(iii) there exists $P \succ \mathbb{0}$ satisfying the strict LMI

$$\begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix} - \begin{bmatrix} C & D \\ \mathbb{0} & I_m \end{bmatrix}^\mathsf{T} \begin{bmatrix} -I_p & \mathbb{0} \\ \mathbb{0} & \gamma^2 I_m \end{bmatrix} \begin{bmatrix} C & D \\ \mathbb{0} & I_m \end{bmatrix} \prec \mathbb{0};$$

(iv) for all $\omega \in \mathbb{R} \cup \{\infty\}$ the frequency response $\hat{M}(\mathbf{j}\omega)$ satisfies

$$\hat{M}(\mathbf{j}\omega)^* \hat{M}(\mathbf{j}\omega) \prec \gamma^2 I_m \quad \Longleftrightarrow \quad \sigma_{\max}(\hat{M}(\mathbf{j}\omega)) < \gamma.$$

Proof of Corollary 6.1

(i) $\iff$ (iii) $\iff$ (iv): This is precisely the strict dissipativity theorem applied to the supply rate under consideration.

(i) $\implies$ (ii): Our dissipation inequality is that

$$\frac{\mathrm{d}V(x(t))}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} x(t)^{\mathsf{T}} P x(t) \leq -\|z(t)\|_2^2 + (\gamma^2 - \epsilon^2)\|w(t)\|_2^2$$

for some $\epsilon > 0$. Since $x(0) = 0$ we may integrate both sides over $[0, T]$ to obtain

$$x(T)^{\mathsf{T}} P x(T) \leq \int_0^T -\|z(t)\|_2^2 + (\gamma^2 - \epsilon^2)\|w(t)\|_2^2 \, \mathrm{d}t.$$

Since $P \succ 0$, the LHS is always nonnegative. We therefore find that

$$\|z_T\|_{\mathcal{L}_2}^2 \leq (\gamma^2 - \epsilon^2)\|w_T\|_{\mathcal{L}_2}^2 \quad \Longrightarrow \quad \|z_T\|_{\mathcal{L}_2} < \gamma \|w_T\|_{\mathcal{L}_2}$$

which shows that $\|M\|_{\mathcal{L}_2 \to \mathcal{L}_2} < \gamma$.

Comments on Corollary 6.1

- Given a desired level of performance $\gamma > 0$, the strict LMI in (iii) allows us to check if our system meets this performance level.

- We can do one better, and compute the ***best*** upper bound $\gamma^\star$ by setting $\rho = \gamma^2$ and solving the SDP

$$\underset{P \succ 0,\, \rho \geq 0}{\text{minimize}} \;\; \rho \qquad \text{subject to} \qquad \text{LMI in (iii)}$$

- The FDI in (iv) can be equivalently expressed as

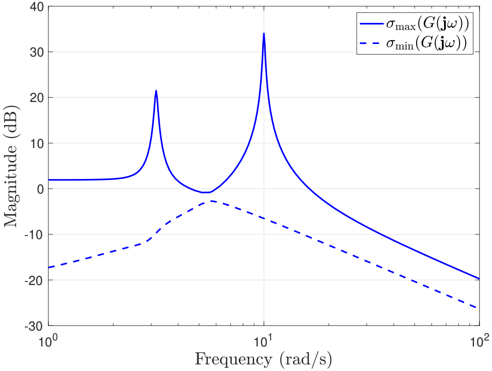
$$\sup_{\omega \in \mathbb{R}} \sigma_{\max}(\hat{M}(\mathbf{j}\omega)) < \gamma \quad \iff \quad \sup_{\omega \in \mathbb{R}} \|\hat{M}(\mathbf{j}\omega)\|_2 < \gamma.$$

  This quantity is known as the $\mathcal{H}_\infty$ ***norm*** of the associated transfer function $\hat{M}(s)$, denoted by $\|\hat{M}\|_{\mathcal{H}_\infty}$. Optimal $\mathcal{L}_2$-control of linear systems is therefore usually referred to as $\mathcal{H}_\infty$ ***control***.

- In fact: $\|\hat{M}\|_{\mathcal{H}_\infty} = \|M\|_{\mathcal{L}_2 \to \mathcal{L}_2}$, so these two quantities coincide.

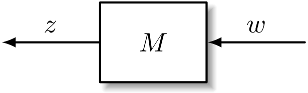
Example: Bode plot interpretation of $\|M\|_{\mathcal{L}_2 \to \mathcal{L}_2}$

$$G(s) = \begin{bmatrix} \dfrac{10(s+1)}{s^2+0.2s+100} & \dfrac{1}{s+1} \\ \dfrac{s+2}{s^2+0.1s+10} & \dfrac{5(s+1)}{(s+2)(s+3)} \end{bmatrix}$$



I/O performance: peak vs. area

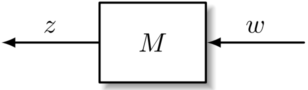
$$M : \left[\begin{array}{c} \dot{x} \\ \hline z \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c} x \\ \hline w \end{array}\right]$$



- We have seen that a system has $\mathcal{L}_2$-gain less than γ if the ***peak value*** of the Bode magnitude plot $\omega \mapsto \|\hat{M}(\mathbf{j}\omega)\|_2$ is less than γ.
- This seems intuitive from a classical control perspective; keeping the magnitude of the transfer function small is a good way to reduce the effect of inputs on the output.
- A different way to quantify this same idea is to instead look at the ***area*** under the Bode plot, as opposed to the ***peak*** value

I/O performance: the $\mathcal{H}_2$ norm

$$M: \quad \left[\begin{array}{c} \dot{x} \\ \hline z \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & \mathbf{0} \end{array}\right] \left[\begin{array}{c} x \\ \hline w \end{array}\right]$$



- We assume A Hurwitz, but now require $D = 0$. The associated transfer matrix $\hat{M}(s) = C(sI_n - A)^{-1}B$ belongs to the space

$$\mathcal{RH}_2 \triangleq \{\hat{M}(s) \mid \hat{M} \text{ strictly proper with all poles in } \mathbb{C}_{<0}\}.$$

- On this vector space of transfer matrices, define inner product

$$\begin{aligned} \langle \hat{M}, \hat{N} \rangle_{\mathcal{H}_2} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle \hat{M}(\mathbf{j}\omega), \hat{N}(\mathbf{j}\omega) \rangle_{\mathsf{F}} \, \mathrm{d}\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{trace}\left[\hat{M}(\mathbf{j}\omega)^* \hat{N}(\mathbf{j}\omega)\right] \mathrm{d}\omega \end{aligned}$$

The $\mathcal{H}_2$ norm of an LTI system

- The $\mathcal{H}_2$ norm of M is defined using this inner product

$$
\begin{aligned}
\|M\|_{\mathcal{H}_2} &\triangleq \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{trace}\left[\hat{M}(\mathbf{j}\omega)^* \hat{M}(\mathbf{j}\omega) \right] \mathrm{d}\omega \right)^{\frac{1}{2}} \\
&= \left(\frac{1}{2\pi} \sum_{k=1}^{m} \int_{-\infty}^{\infty} \sigma_k^2(\hat{M}(\mathbf{j}\omega))\, \mathrm{d}\omega \right)^{\frac{1}{2}}
\end{aligned}
$$

- The $\mathcal{H}_\infty$ norm is an ***induced norm*** — the induced norm from $\mathcal{L}_2[0, \infty)$ to $\mathcal{L}_2[0, \infty)$. The $\mathcal{H}_2$ norm is ***not***, and is defined using an inner product placed directly on the space of transfer functions.

- Both $\mathcal{H}_\infty$ and $\mathcal{H}_2$ measure the "gain" of an LTI system, but they are ***not*** equivalent norms; you cannot in bound one in terms of the other.

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are not equivalent

Consider the two transfer functions

$$M_1(s) = \frac{1}{\epsilon s + 1}, \qquad M_2(s) = \frac{\epsilon s}{s^2 + \epsilon s + 1}$$

As $\epsilon \to 0$ ...

- M_1 is a low-pass filter with very large bandwidth. The peak Bode plot value $\|M_1\|_{\mathcal{H}_\infty}$ equals 1, but the area under the Bode plot is infinite, so $\|M_1\|_{\mathcal{H}_2} = +\infty$.

- For M_2 we can compute that

$$\|M_2\|_{\mathcal{H}_\infty} = \sup_\omega \frac{\epsilon\omega}{\sqrt{(1-\omega^2)^2 + \epsilon^2\omega^2}} = 1$$

$$\|M_2\|_{\mathcal{H}_2}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\epsilon^2\omega^2}{(1-\omega^2)^2 + \epsilon^2\omega^2}\,\mathrm{d}\omega \to 0.$$

State-space formulas for the $\mathcal{H}_2$ norm

- The $\mathcal{H}_2$ norm admits a very simple characterization in terms of Lyapunov-like variables

Proposition 6.6 (Lyapunov Equations for $\mathcal{H}_2$ Norm). Consider (CT-LTI) and assume that A is Hurwitz and $D = 0$. Then

$$\|M\|_{\mathcal{H}_2}^2 = \mathrm{trace}(CXC^\mathsf{T}) = \mathrm{trace}(B^\mathsf{T}YB).$$

where $Y \succ 0$ and $X \succ 0$ are the unique solutions to

$$A^\mathsf{T}Y + YA + C^\mathsf{T}C = 0$$
$$AX + XA^\mathsf{T} + BB^\mathsf{T} = 0.$$

Proof of Proposition 6.6

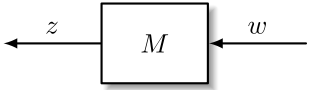
The Fourier-transform pair of the frequency response $\hat{M}(\mathbf{j}\omega)$ is the causal impulse response $t \mapsto Ce^{At}B\mathbb{1}(t)$. By Parseval's Theorem, we may equivalently write

$$\begin{aligned} \|M\|_{\mathcal{H}_2}^2 &= \frac{1}{2\pi} \operatorname{trace} \int_{-\infty}^{\infty} \hat{M}(\mathbf{j}\omega)^* \hat{M}(\mathbf{j}\omega) \, \mathrm{d}\omega \\ &= \operatorname{trace} \int_0^{\infty} (Ce^{At}B)^{\mathsf{T}} Ce^{At} B \, \mathrm{d}t \\ &= \operatorname{trace} B^{\mathsf{T}} \underbrace{\left[\int_0^{\infty} e^{A^{\mathsf{T}}t} C^{\mathsf{T}} C e^{At} \, \mathrm{d}t \right]}_{\triangleq Y} B \end{aligned}$$

It follows by calculations similar to those in the proof of Lyapunov's Theorem for LTI systems that Y as defined above is the unique positive definite solution to $A^{\mathsf{T}}Y + YA + C^{\mathsf{T}}C = 0$. The other formula can be similarly obtained after applying the cyclic property of the trace operation to the above expression for $\|M\|_{\mathcal{H}_2}$. •

Stochastic interpretation of the $\mathcal{H}_2$ norm

$$M : \left[\begin{array}{c} \dot{x} \\ \hline z \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & \mathbf{0} \end{array}\right] \left[\begin{array}{c} x \\ \hline w \end{array}\right]$$



Suppose that the input w to (CT-LTI) is a *random process* satisfying

(i) $\mathbb{E}\{w(t)\} = 0$ for all $t \geq 0$;

(ii) $\mathbb{E}\{w(t)w(\tau)^{\mathsf{T}}\} = I_m \delta(t - \tau)$;

(iii) $\mathbb{E}\{x_0 w(t)^{\mathsf{T}}\} = \mathbb{0}$ for all $t \geq 0$.

- The idea is that noise will cause $x(t)$ to randomly bounce around the origin; our goal is to quantify the variance of $z(t)$
- First, if $\bar{x} = \mathbb{E}\{x(t)\}$, then

$$\dot{\bar{x}} = \mathbb{E}\{Ax + Bw\} = A\mathbb{E}\{x\} + B\mathbb{E}\{w\} = A\bar{x}$$

so $\bar{x}(t) = e^{At}x_0 = 0$.

Stochastic interpretation of $\mathcal{H}_2$ norm

- Now define the covariance matrix of the state

$$P(t) \triangleq \mathbb{E}\{(x(t)-\bar{x}(t))(x(t)-\bar{x}(t))^{\mathsf{T}}\} = \mathbb{E}\{x(t)x(t)^{\mathsf{T}}\}$$

- Substituting for $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bw(\tau)\,\mathrm{d}\tau$, we have

$$P(t) = \mathbb{E}\{e^{At}x_0x_0^{\mathsf{T}}e^{A^{\mathsf{T}}t}\} + \mathbb{E}\left\{e^{At}x_0\int_0^t w(\tau)^{\mathsf{T}}B^{\mathsf{T}}e^{A^{\mathsf{T}}(t-\tau)}\,\mathrm{d}\tau\right\} + (\star)^{\mathsf{T}}$$
$$+\,\mathbb{E}\left\{\int_0^t\int_0^t e^{A(t-\tau)}Bw(\tau)w(\sigma)^{\mathsf{T}}B^{\mathsf{T}}e^{A^{\mathsf{T}}(t-\sigma)}\,\mathrm{d}\tau\mathrm{d}\sigma\right\}$$

- Using Assumptions (ii) and (iii), we obtain

$$P(t) = e^{At}P(0)e^{A^{\mathsf{T}}t} + \int_0^t e^{A(t-\tau)}BB^{\mathsf{T}}e^{A^{\mathsf{T}}(t-\tau)}\,\mathrm{d}\tau$$

Stochastic interpretation of $\mathcal{H}_2$ norm

- Direct calculation shows that $\dot{P} = AP + PA^\mathsf{T} + BB^\mathsf{T}$
- Since A is Hurwitz, $\lim_{t\to\infty} P(t)$ converges to unique pos.-def. solution of $AP + PA^\mathsf{T} + BB^\mathsf{T} = 0$, i.e., $\lim_{t\to\infty} P(t) = X$!

$$
\begin{aligned}
\lim_{t\to\infty} \mathbb{E}\{z(t)^\mathsf{T} z(t)\} &= \lim_{t\to\infty} \mathbb{E}\{x(t)^\mathsf{T} C^\mathsf{T} C x(t)\} \\
&= \lim_{t\to\infty} \mathbb{E}\{\mathrm{trace}(x(t)^\mathsf{T} C^\mathsf{T} C x(t))\} \\
&= \lim_{t\to\infty} \mathbb{E}\{\mathrm{trace}(C x(t) x(t)^\mathsf{T} C^\mathsf{T})\} \\
&= \mathrm{trace}\ C \left[\lim_{t\to\infty} \mathbb{E}\{x(t) x(t)^\mathsf{T}\}\right] C^\mathsf{T} \\
&= \mathrm{trace}\ C X C^\mathsf{T} = \|M\|_{\mathcal{H}_2}^2 .
\end{aligned}
$$

$\|M\|_{\mathcal{H}_2}^2$ is the ***asymptotic variance*** of the output $z(t)$

Impulse response interpretation of $\mathcal{H}_2$ norm

- For simplicity consider (CT-LTI) with a single scalar input, and let the input be $w = \delta(t)$, a unit impulse at $t = 0$.
- The corresponding output z is given by the impulse response $z(t) = Ce^{At}B1(t)$, and we compute that

$$
\begin{aligned}
\|z\|_{\mathcal{L}_2}^2 &= \int_0^\infty z(t)^\mathsf{T} z(t)\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{z}(\mathbf{j}\omega)^* \hat{z}(\mathbf{j}\omega)\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{M}(\mathbf{j}\omega)^* \hat{M}(\mathbf{j}\omega)\,\mathrm{d}\omega \\
&= \|M\|_{\mathcal{H}_2}^2.
\end{aligned}
$$

$\|M\|_{\mathcal{H}_2}$ is the ***output energy*** of the impulse response.

LMI conditions for $\mathcal{H}_2$ norm

Theorem 6.3 (LMI for $\mathcal{H}_2$ Performance). Consider (CT-LTI) with $D = 0$, and let $\gamma > 0$. The following statements are equivalent:

(i) A is Hurwitz and $\|M\|_{\mathcal{H}_2} < \gamma$;

(ii) $\exists X \succ \mathbb{0}$ satisfying $AX + XA^{\mathsf{T}} + BB^{\mathsf{T}} \prec \mathbb{0}$ and $\mathrm{trace}(CXC^{\mathsf{T}}) < \gamma^2$;

(iii) $\exists Y \succ \mathbb{0}$ satisfying $A^{\mathsf{T}}Y + YA + C^{\mathsf{T}}C \prec \mathbb{0}$ and $\mathrm{trace}(B^{\mathsf{T}}YB) < \gamma^2$;

(iv) $\exists P \succ 0$ and $W \in \mathbb{S}^p$ satisfying $\mathrm{trace}(W) < \gamma$ and

$$\begin{bmatrix} A^{\mathsf{T}}P + PA & PB \\ B^{\mathsf{T}}P & \mathbb{0} \end{bmatrix} - \begin{bmatrix} \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \gamma I_m \end{bmatrix} \prec \mathbb{0}\,, \quad \begin{bmatrix} P & C^{\mathsf{T}} \\ C & W \end{bmatrix} \succ \mathbb{0};$$

(v) $\exists L \succ 0$ and $W \in \mathbb{S}^m$ satisfying $\mathrm{trace}(W) < \gamma$ and

$$\begin{bmatrix} AL + LA^{\mathsf{T}} & LC^{\mathsf{T}} \\ CL & \mathbb{0} \end{bmatrix} - \begin{bmatrix} \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \gamma I_p \end{bmatrix} \prec \mathbb{0}\,, \quad \begin{bmatrix} L & B \\ B^{\mathsf{T}} & W \end{bmatrix} \succ \mathbb{0}.$$

Proof of Theorem 6.3

(i) $\Longrightarrow$ (iii): By Prop. 6.6 we have that for some $W \succ 0$

$$A^\mathsf{T} W + WA + C^\mathsf{T} C = 0, \quad \mathrm{trace}(B^\mathsf{T} W B) < \gamma^2.$$

Moreover, since A is Hurwitz, there exists $L \succ 0$ such that $A^\mathsf{T} L + LA \prec 0$. Due to the strictness of the inequality $\mathrm{trace}(B^\mathsf{T} W B) < \gamma^2$, there must exist $\epsilon > 0$ such that

$$\mathrm{trace}(B^\mathsf{T} (W + \epsilon L) B) < \gamma^2$$

Define $Y = W + \epsilon L$, which obviously satisfies $Y \succ 0$. We compute then that

$$A^\mathsf{T} Y + YA + C^\mathsf{T} C = A^\mathsf{T}(W + \epsilon L) + (W + \epsilon L)A + C^\mathsf{T} C = \epsilon(A^\mathsf{T} L + LA) \prec 0.$$

Proof of Theorem 6.3

(iii) $\implies$ (i): That A is Hurwitz follows by Lyapunov's theorem. Since $-(A^\mathsf{T}Y + YA + C^\mathsf{T}C) \succ 0$, there must exist a matrix C_0 such that $C_0^\mathsf{T}C_0 = -(A^\mathsf{T}Y + YA + C^\mathsf{T}C)$, and therefore

$$A^\mathsf{T}Y + YA + C^\mathsf{T}C + C_0^\mathsf{T}C = 0, \quad \mathrm{trace}(B^\mathsf{T}YB) < \gamma^2. \tag{8}$$

We augment the system M with an additional output $z_0 = C_0x$, so the overall output is now $(z, z_0) = \left[\begin{smallmatrix} C \\ C_0 \end{smallmatrix}\right] x$ and the overall transfer matrix is $\begin{bmatrix} M(s) \\ M_0(s) \end{bmatrix}$ where $M_0(s) = C_0(sI_n - A)^{-1}B$. It follows that (8) establishes

$$\left\| \left[\begin{smallmatrix} M \\ M_0 \end{smallmatrix}\right] \right\|_{\mathcal{H}_2}^2 = \|M\|_{\mathcal{H}_2}^2 + \|M_0\|_{\mathcal{H}_2}^2 < \gamma^2$$

which shows that $\|M\|_{\mathcal{H}_2} < \gamma$.

Proof of Theorem 6.3

(ii) $\implies$ (iv): Define $P \triangleq \gamma X^{-1} \succ 0$. We compute that

$$\begin{aligned} A^\mathsf{T} P + PA + \frac{1}{\gamma} PBB^\mathsf{T} P &= A^\mathsf{T}(\gamma X^{-1}) + (\gamma X^{-1})A + \frac{1}{\gamma}(\gamma X^{-1})BB^\mathsf{T}(\gamma X^{-1}) \\ &= \gamma X^{-1}\left[XA^\mathsf{T} + AX + BB^\mathsf{T}\right]\gamma X^{-1} \\ &\prec \mathbb{0} \end{aligned}$$

by congruence. By Schur complements we further obtain

$$\begin{bmatrix} A^\mathsf{T} P + PA & PB \\ B^\mathsf{T} P & -\gamma I_m \end{bmatrix} \prec \mathbb{0}.$$

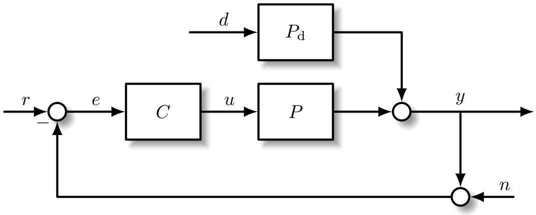
Now let $\epsilon > 0$ be sufficiently small such that $W \triangleq \frac{1}{\gamma} CXC^\mathsf{T} + \epsilon I$ satisfies $\mathrm{trace}(W) < \gamma$, and note that $W \succ C(\frac{1}{\gamma}X)C^\mathsf{T} = CP^{-1}C$. By Schur complements then

$$W - CP^{-1}C^\mathsf{T} \succ 0 \quad \iff \quad \begin{bmatrix} P & C^\mathsf{T} \\ C & W \end{bmatrix} \succ \mathbb{0}$$

which shows the result. $\bullet$

Performance weights

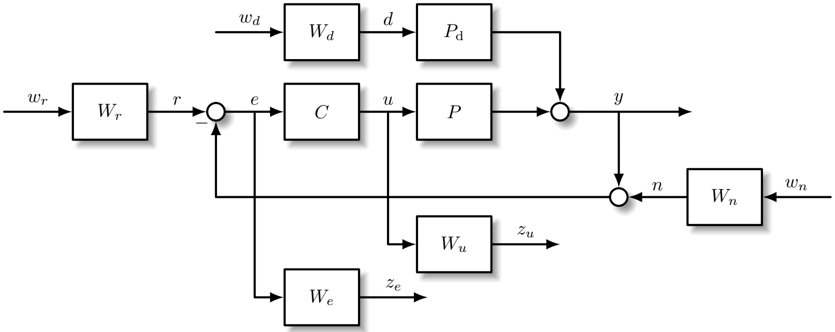
Let's now return to our SISO control example



- Exogenous signals $w = (r, d, n)$, performance signals $z = (e, u)$
- ***Problem:*** these signals are all very different! We expect $r(t)$ to be mostly low-frequency, $n(t)$ to be mostly high frequency, and so on. Lumping them all into one vector and quantifying performance using a norm such as $\mathcal{H}_2$ or $\mathcal{H}_\infty$ doesn't seem to make much sense ...

Performance weights

The solution is to add ***weighting filters*** to the model, which attempt to capture the relative importance and frequency content of the signals.



- Exogenous signals $w = (w_r, w_d, w_n)$, performance signals $z = (z_e, z_u)$
- ***For you to think about:*** how should the filters W be chosen?

Appendix: Stability of convolution operators

- Consider our causal CT-LTI system M with impulse response $m(t) = Ce^{At}B1(t)$; we consider the case $D = 0$ here.
- Recall the ***convolution operator*** Conv_m defined by

$$\mathsf{Conv}_m(u)(t) \triangleq \int_{-\infty}^{\infty} m(t-\tau)u(\tau)\,\mathrm{d}\tau.$$

Proposition 6.7 (BIBO Stability). If $m \in \mathcal{L}_1[0,\infty)$, then

(i) $\mathsf{Conv}_m : \mathcal{L}_2[0,\infty) \to \mathcal{L}_2[0,\infty)$ and

$$\|\mathsf{Conv}_m\|_{\mathcal{L}_2 \to \mathcal{L}_2} \leq \|m\|_{\mathcal{L}_1};$$

(ii) $\mathsf{Conv}_m : \mathcal{L}_\infty[0,\infty) \to \mathcal{L}_\infty[0,\infty)$ and

$$\|\mathsf{Conv}_m\|_{\mathcal{L}_\infty \to \mathcal{L}_\infty} \leq \|m\|_{\mathcal{L}_1}.$$

Appendix: Incremental stability

- The idea of finite $\mathcal{L}_2$-gain is in fact a bit weak, because it has no relationship to ***continuity*** of the operator M. Lipschitz continuity of M is referred to as ***incremental gain***
- An operator $M : \mathcal{L}_{2\mathrm{e}}[0,\infty) \to \mathcal{L}_{2\mathrm{e}}[0,\infty)$ is $\mathcal{L}_2$-stable with finite incremental gain if it is $\mathcal{L}_2$-stable and there exists $\gamma \geq 0$ s.t.

$$\|M(w) - M(w')\|_{\mathcal{L}_2} \leq \gamma \|w - w'\|_{\mathcal{L}_2}, \quad \forall\, w, w' \in \mathcal{L}_2[0,\infty)$$

- For LTI operators, stability and incremental stability are equivalent (try to prove it)
- Useful property for contraction mapping arguments (can guarantee existence/uniqueness of solutions)

Appendix: Discrete-time signal spaces

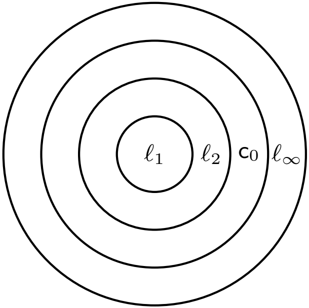
- The analogous discrete-time signal spaces are simpler than their continuous-time cousins
- **Recall:** the Banach spaces

$$\mathsf{c}_0(\mathbb{Z};\mathbb{F}^n) = \left\{ f : \mathbb{Z} \to \mathbb{F}^n \;\middle|\; \lim_{n\to\pm\infty} f(n) = 0 \right\}$$
$$\ell_{\mathrm{p}}(\mathbb{Z};\mathbb{F}^n) = \left\{ f : \mathbb{Z} \to \mathbb{F}^n \;\middle|\; \|f\|_{\ell_{\mathrm{p}}} < \infty \right\}$$
$$\ell_{\infty}(\mathbb{Z};\mathbb{F}^n) = \left\{ f : \mathbb{Z} \to \mathbb{F}^n \;\middle|\; \|f\|_{\ell_{\infty}} < \infty \right\}$$

with norms

$$\|f\|_{\ell_{\mathrm{p}}} = \left(\sum_{-\infty}^{\infty} \|f(n)\|_2^{\mathrm{p}} \right)^{1/\mathrm{p}}, \qquad \mathrm{p} \in [1,\infty)$$
$$\|f\|_{\ell_{\infty}} = \sup_{n\in\mathbb{Z}} \|f(n)\|_2$$

Appendix: Discrete-time signal spaces



- Very simple inclusions

$$\ell_1 \subset \ell_2 \subset \mathsf{c}_0 \subset \ell_\infty$$

- ℓ_2 is a Hilbert space

$$\langle f, g\rangle_{\ell_2} = \sum_{n=-\infty}^{\infty} f(n)^* g(n)$$

$$|\langle f, g\rangle_{\ell_2}| \leq \|f\|_{\ell_2}\|g\|_{\ell_2}$$

- We can have right-sided versions $\ell_1[0,\infty)$, $\ell_2[0,\infty)$, $\ell_\infty[0,\infty)$, ...
- The extended spaces $\ell_{1\mathrm{e}}[0,\infty)$, $\ell_{2\mathrm{e}}[0,\infty)$, etc. are all simply equal to the space of all right-sided DT signals $f : \mathbb{Z} \to \mathbb{F}^n$

Appendix: ℓ_1 and ℓ_2 Fourier Transforms

- The ***Fourier transform*** $\mathscr{F}(f)$ of a signal $f \in \ell_1(\mathbb{Z};\mathbb{F}^n)$ is defined by

$$\mathscr{F}(f)(e^{\mathbf{j}\omega}) \triangleq \sum_{-\infty}^{\infty} f(n)e^{-\mathbf{j}\omega n}.$$

- $\mathscr{F} : \ell_1(\mathbb{Z};\mathbb{F}^n) \to \mathsf{C}^0_{\mathrm{per},2\pi}(\mathbf{j}\mathbb{R};\mathbb{F}^n)$, where

$$\mathsf{C}^0_{\mathrm{per},2\pi}(\mathbf{j}\mathbb{R};\mathbb{F}^n) = \{f \in \mathsf{C}^0(\mathbf{j}\mathbb{R};\mathbb{F}^n) \mid f \text{ is } 2\pi \text{ periodic}\}$$

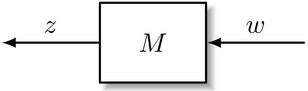
- $\mathscr{F}$ is injective with left inverse $\mathscr{F}^{-1} : \mathsf{C}^0_{\mathrm{per},2\pi}(\mathbf{j}\mathbb{R};\mathbb{F}^n) \to \mathsf{c}_0(\mathbb{Z};\mathbb{F}^n)$

$$\mathscr{F}^{-1}(\hat{f})(n) = \int_{-\pi}^{\pi} \hat{f}(e^{\mathbf{j}\omega})e^{\mathbf{j}\omega t}\,\mathrm{d}\omega.$$

- The transform admits an extension to $\ell_2(\mathbb{Z};\mathbb{F}^n)$ similar to the $\mathcal{L}_2$ case

Appendix: $\mathcal{H}_2$ norm of a discrete-time system

$$M : \left[\begin{array}{c} x^+ \\ \hline z \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c} x \\ \hline w \end{array}\right]$$



- We assume A Schur, and ***do not*** require $D = 0$. The associated transfer matrix is $\hat{M}(z) = C(zI_n - A)^{-1}B + D$
- The norm is defined as

$$\|M\|_{\mathcal{H}_2} \triangleq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{trace} \left[\hat{M}(e^{\mathbf{j}\omega})^* \hat{M}(e^{\mathbf{j}\omega}) \right] \mathrm{d}\omega \right)^{\frac{1}{2}}$$

- Can be shown that $\|M\|_{\mathcal{H}_2}^2 = \mathrm{trace}(D^\mathsf{T} D + CXC^\mathsf{T})$ and also equals $\mathrm{trace}(D^\mathsf{T} D + B^\mathsf{T} Y B)$ where $Y \succ 0$ and $X \succ 0$ are the unique solutions to

$$A^\mathsf{T} Y A - Y + C^\mathsf{T} C = 0, \qquad AXA^\mathsf{T} - A + BB^\mathsf{T} = 0.$$

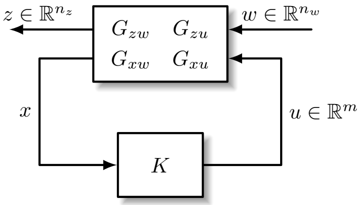
7. The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ State-Feedback Control Problems

Problem setup for $\mathcal{H}_2$ state feedback

We are now ready to design some controllers!

$$G : \begin{bmatrix} \dot{x} \\ \hline z \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & \mathbf{0} & D_{zu} \end{array}\right] \begin{bmatrix} x \\ \hline w \\ u \end{bmatrix}$$

$$u = Kx$$



Closed-loop system is:

$$M_{\text{cl}} : \qquad \dot{x} = (A + B_u K)x + B_w w, \quad z = (C_z + D_{zu}K)x.$$

Problem 7.1 (Suboptimal $\mathcal{H}_2$ state-feedback control). Given $\gamma > 0$, design (if possible) a state-feedback controller $u = Kx$ such that $(A + B_u K)$ is Hurwitz and $\|M_{\text{cl}}\|_{\mathcal{H}_2} < \gamma$.

Solution of $\mathcal{H}_2$ state-feedback problem

From Theorem 6.3, $A + B_u K$ is Hurwitz and the system meets the $\mathcal{H}_2$-norm constraint $\|M_{\mathrm{cl}}\|_{\mathcal{H}_2} < \gamma$ if and only if there exists $X \succ 0$ such that

$$(A + B_u K)X + X(A + B_u K)^{\mathsf{T}} + B_w B_w^{\mathsf{T}} \prec \mathbb{0}$$
$$\mathrm{trace}((C_z + D_{zu} K)X(C_z + D_{zu} K)^{\mathsf{T}}) < \gamma^2$$

If we define $Z = KX$, we can rewrite these inequalities as

$$(AX + BZ) + (AX + BZ)^{\mathsf{T}} + B_w B_w^{\mathsf{T}} \prec \mathbb{0}$$
$$\mathrm{trace}((C_z X + D_{zu} Z)X^{-1}(C_z X + D_{zu} Z)^{\mathsf{T}}) < \gamma^2$$

The second inequality is equivalent to the existence of $W \succ 0$ such that

$$(C_z X + D_{zu} Z)X^{-1}(C_z X + D_{zu} Z)^{\mathsf{T}} \prec W, \quad \mathrm{trace}(W) < \gamma^2.$$

Using Schur's Lemma to linearize the last inequality, we obtain

$$\begin{bmatrix} X & (C_z X + D_{zu} Z)^{\mathsf{T}} \\ (C_z X + D_{zu} Z) & W \end{bmatrix} \succ \mathbb{0}, \quad \mathrm{trace}(W) < \gamma^2.$$

Solution of $\mathcal{H}_2$ state-feedback problem

Theorem 7.1 (Optimal $\mathcal{H}_2$ state-feedback synthesis). The γ-suboptimal $\mathcal{H}_2$ state-feedback synthesis problem is solvable if and only if there exists $X \succ \mathbb{0}$, $Z \in \mathbb{R}^{m \times n}$ and $W \succ 0$ satisfying

$$\begin{bmatrix} A & B_u \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} X & Z^\mathsf{T} \end{bmatrix} \begin{bmatrix} A^\mathsf{T} \\ B_u^\mathsf{T} \end{bmatrix} + B_w B_w^\mathsf{T} \prec \mathbb{0}$$

$$\begin{bmatrix} X & (C_z X + D_{zu} Z)^\mathsf{T} \\ (C_z X + D_{zu} Z) & W \end{bmatrix} \succ \mathbb{0}$$

$$\mathrm{trace}(W) < \gamma^2$$

in which case the controller is reconstructed as $K = Z X^{-1}$.

To obtain the optimal controller, minimize over γ^2 s.t. LMIs.

$\mathcal{H}_2$ state-feedback control synthesis

```
X = sdpvar(n,n); Z = sdpvar(m,n,'full'); W = ...
    sdpvar(n_z,n_z);
small = 1e-6;
Constraints = [X ≥ small*eye(n), ...
               [A,Bu]*[X;Z] + ([A,Bu]*[X;Z])' + Bw*Bw'  ≤ ...
                   -small*eye(n), ...
               [X,(Cz*X+Dzu*Z)';(Cz*X+Dzu*Z),W] ≥ ...
                   small*eye(n+n_z)];
Cost = trace(W);
options = sdpsettings('solver','sdpt3','verbose',1);
sol = optimize(Constraints,Cost,options);
K_H2 = value(Z)*inv(value(X));
```

Relationship between $\mathcal{H}_2$ and LQR control

$\mathcal{H}_2$ control is a generalization of LQR control. In the classical static state-feedback LQR problem, one considers

$$\begin{aligned} \underset{K}{\text{minimize}} \qquad & J(x_0) \triangleq \int_0^\infty x(t)^\mathsf{T} Q x(t) + u(t)^\mathsf{T} R u(t)\,\mathrm{d}t \\ \text{subject to} \qquad & \dot{x}(t) = Ax(t) + B_u u(t) \\ & x(0) = x_0 \\ & u(t) = Kx(t). \end{aligned}$$

where $Q \succeq \mathbb{0}$ and $R \succ \mathbb{0}$.

- ***LQR:*** non-zero initial conditions, zero exogenous disturbances
- ***$\mathcal{H}_2$:*** zero initial conditions, non-zero exogenous disturbances

Relationship between $\mathcal{H}_2$ and LQR control

- Define the performance output

$$z = C_z x + D_{zu} u = \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} u = \begin{bmatrix} Q^{1/2} x \\ R^{1/2} u \end{bmatrix}$$

  in which case we see that

$$\|z\|_{\mathcal{L}_2}^2 = \int_0^\infty z(t)^\mathsf{T} z(t)\,\mathrm{d}t = \int_0^\infty [x^\mathsf{T} Q x + u^\mathsf{T} R u]\,\mathrm{d}t = J(x_0).$$

- The closed-loop LQR system is

$$M_{\mathrm{cl}} : \qquad \begin{aligned} \dot{x} &= (A + B_u K)x, \quad x(0) = x_0, \\ z &= (C_z + D_{zu} K)x \end{aligned}$$

  with Laplace-domain solution

$$\hat{z}(s) = \underbrace{(C_z + D_{zu} K)(sI_n - (A + B_u K))^{-1}}_{\triangleq \hat{M}_{\mathrm{cl}}(s)} x_0.$$

Relationship between $\mathcal{H}_2$ and LQR control

- We can exactly reproduce the effect of the initial condition through an impulse input applied to the fictitious system

$$M'_{\mathrm{cl}}: \quad \begin{aligned} \dot{x}' &= (A+BK)x' + x_0 w, \qquad x'(0) = \mathbb{0} \\ z' &= (C_z + D_{zu}K)x' \end{aligned}$$

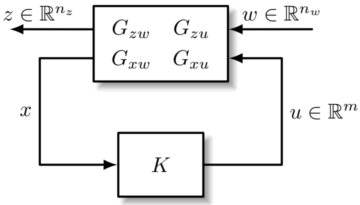
  with impulse input $w(t) = \delta(t) \Longrightarrow \hat{w}(s) = 1$. The Laplace solution $\hat{z}'(s) = \hat{M}_{\mathrm{cl}}(s)x_0\hat{w}(s) = \hat{M}_{\mathrm{cl}}(s)x_0$ is exactly the same as before.
- Therefore, we have

$$\begin{aligned} J(x_0) = \|z\|_{\mathcal{L}_2}^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{z}(\mathbf{j}\omega)^*\hat{z}(\mathbf{j}\omega)\,\mathrm{d}\omega \\ &= \|x_0\|_2^2\ \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname{trace}\hat{M}_{\mathrm{cl}}(\mathbf{j}\omega)^*\hat{M}_{\mathrm{cl}}(\mathbf{j}\omega)\,\mathrm{d}\omega \\ &= \|x_0\|_2^2\|M_{\mathrm{cl}}\|_{\mathcal{H}_2}^2 \end{aligned}$$

Problem setup for $\mathcal{H}_\infty$ state feedback

$$G : \left[\begin{array}{c} \dot{x} \\ \hline z \end{array}\right] = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \end{array}\right] \left[\begin{array}{c} x \\ \hline w \\ u \end{array}\right]$$

$$u = Kx$$



Closed-loop system is:

$$M_{\text{cl}} : \quad \begin{aligned} \dot{x} &= (A + B_u K)x + B_w w \\ z &= (C_z + D_{zu} K)x + D_{zw} w \end{aligned}$$

Problem 7.2 (Suboptimal $\mathcal{H}_\infty$ state-feedback control). Given $\gamma > 0$, design (if possible) a state-feedback controller $u = Kx$ such that $(A + B_u K)$ is Hurwitz and $\|M_{\text{cl}}\|_{\mathcal{H}_\infty} = \|M_{\text{cl}}\|_{\mathcal{L}_2 \to \mathcal{L}_2} < \gamma$.

Solution of the $\mathcal{H}_\infty$ state-feedback problem

From Corollary 6.1, ***if*** $(A + B_u K)$ is Hurwitz, then the closed-loop system meets the $\mathcal{H}_\infty$-norm constraint if and only if there exists $P \succ 0$ such that

$$\begin{bmatrix} (A + B_u K)^\mathsf{T} P + P(A + B_u K) & P B_w \\ B_w^\mathsf{T} P & 0 \end{bmatrix} - \begin{bmatrix} (C_z + D_{zu} K) & D_{zw} \\ \mathbb{0} & I_{n_w} \end{bmatrix}^\mathsf{T} \begin{bmatrix} -I_{n_z} & \mathbb{0} \\ \mathbb{0} & \gamma^2 I_{n_w} \end{bmatrix} \begin{bmatrix} (C_z + D_{zu} K) & D_{zw} \\ \mathbb{0} & I_{n_w} \end{bmatrix} \prec \mathbb{0}$$

The top left block of this LMI reads as

$$(A + B_u K)^\mathsf{T} P + P(A + B_u K) + \underbrace{(C_z + D_{zu} K)^\mathsf{T} (C_z + D_{zu} K)}_{\succeq 0} \prec 0$$

from which we conclude that

$$(A + B_u K)^\mathsf{T} P + P(A + B_u K) \prec 0.$$

Since $P \succ 0$, we conclude that $(A + B_u K)$ is Hurwitz, so stability comes automatically.

Solution of the $\mathcal{H}_\infty$ state-feedback problem

Defining $X = P^{-1} \succ 0$ and performing a congruence transformation with the matrix $\mathrm{diag}(X, I_{n_w})$ we obtain the equivalent LMI

$$\begin{bmatrix} X(A+B_uK)^\mathsf{T} + (A+B_uK)X & B_w \\ B_w^\mathsf{T} & \mathbb{0} \end{bmatrix} - \begin{bmatrix} (C_z + D_{zu}K)X & D_{zw} \\ \mathbb{0} & I_{n_w} \end{bmatrix}^\mathsf{T} \begin{bmatrix} -I_{n_z} & \mathbb{0} \\ \mathbb{0} & \gamma^2 I_{n_w} \end{bmatrix} \begin{bmatrix} (C_z + D_{zu}K)X & D_{zw} \\ \mathbb{0} & I_{n_w} \end{bmatrix} \prec \mathbb{0}$$

Now define $Z = KX$ to obtain

$$\begin{bmatrix} (AX+B_uZ)^\mathsf{T} + (AX+B_uZ) & B_w \\ B_w^\mathsf{T} & \mathbb{0} \end{bmatrix} - \begin{bmatrix} (C_zX + D_{zu}Z) & D_{zw} \\ \mathbb{0} & I_{n_w} \end{bmatrix}^\mathsf{T} \begin{bmatrix} -I_{n_z} & \mathbb{0} \\ \mathbb{0} & \gamma^2 I_{n_w} \end{bmatrix} \begin{bmatrix} (C_zX + D_{zu}Z) & D_{zw} \\ \mathbb{0} & I_{n_w} \end{bmatrix} \prec \mathbb{0}$$

Solution of the $\mathcal{H}_\infty$ state-feedback problem

This is still not an LMI, because the second term contains a product of decision variables. Rewriting this further as

$$\begin{bmatrix} (AX + B_u Z)^\mathsf{T} + (AX + B_u Z) & B_w \\ B_w^\mathsf{T} & -\gamma^2 I_{n_w} \end{bmatrix} + \begin{bmatrix} (C_z X + D_{zu} Z) & D_{zw} \end{bmatrix}^\mathsf{T} \begin{bmatrix} (C_z X + D_{zu} Z) & D_{zw} \end{bmatrix} \prec \mathbb{0}$$

We can linearize via Schur's Lemma to obtain a genuine LMI which is block 3×3:

$$\begin{bmatrix} (AX + B_u Z)^\mathsf{T} + (AX + B_u Z) & B_w & (C_z X + D_{zu} Z)^\mathsf{T} \\ B_w^\mathsf{T} & -\gamma^2 I_{n_w} & D_{zw}^\mathsf{T} \\ (C_z X + D_{zu} Z) & D_{zw} & -I_{n_z} \end{bmatrix} \prec 0$$

Solution of the $\mathcal{H}_\infty$ state-feedback problem

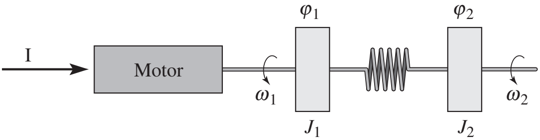
Theorem 7.2 (Optimal $\mathcal{H}_\infty$ state-feedback synthesis). The γ-suboptimal $\mathcal{H}_\infty$ state-feedback synthesis problem is solvable if and only if there exists $X \succ \mathbb{0}$ and $Z \in \mathbb{R}^{m \times n}$

$$\begin{bmatrix} (AX + B_u Z)^\mathsf{T} + (AX + B_u Z) & \star & \star \\ B_w^\mathsf{T} & -\gamma^2 I_{n_w} & \star \\ (C_z X + D_{zu} Z) & D_{zw} & -I_{n_z} \end{bmatrix} \prec 0$$

in which case the controller is reconstructed as $K = ZX^{-1}$.

To obtain the optimal controller, minimize over γ^2 s.t. LMIs.

Example: Two-inertia positioning system



- ***Goal:*** Maintain position $\varphi_2 = 0$
- ***Control:*** Motor torque u applied to J_1
- ***Disturbance:*** Load torque τ_{d} applied to J_2
- Integral state $\dot{\eta} = \varphi_2$ added to ensure steady-state regulation

$$\begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ J_1\dot{\omega}_1 \\ J_2\dot{\omega}_2 \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -k & k & -b & b & 0 \\ k & -k & b & -b & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \omega_1 \\ \omega_2 \\ \eta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tau_{\mathrm{d}} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u$$

Example: two-inertia positioning system

- To limit high-frequency controller response, introduce high-pass filter

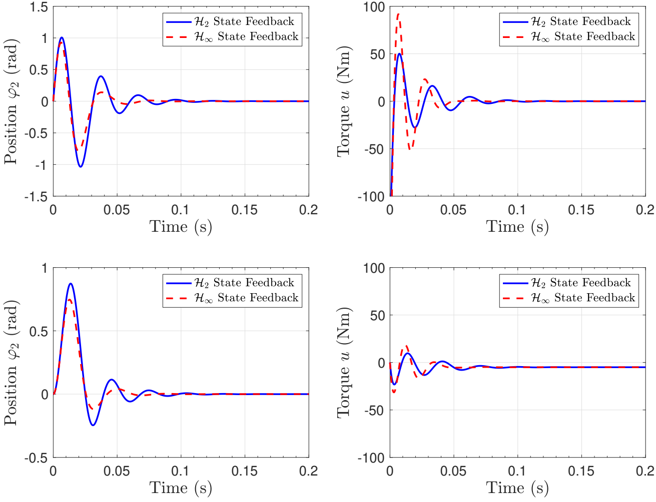
$$\tau_{\mathrm{c}}\dot{\xi} = -\xi + u, \quad u_{\mathrm{hp}} = -\xi + u$$

and use performance output $z = (\eta, \rho u_{\mathrm{hp}})$ for $\rho > 0$.

$$\begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ J_1\dot{\omega}_1 \\ J_2\dot{\omega}_2 \\ \dot{\eta} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -k & k & -b & b & 0 & 0 \\ k & -k & b & -b & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/\tau_{\mathrm{c}} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \omega_1 \\ \omega_2 \\ \eta \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tau_{\mathrm{d}} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1/\tau_{\mathrm{c}} \end{bmatrix} u$$

$$z = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\rho \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \omega_1 \\ \omega_2 \\ \eta \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tau_{\mathrm{d}} + \begin{bmatrix} 0 \\ \rho \end{bmatrix} u$$

Impulse and step on two-inertia positioning system

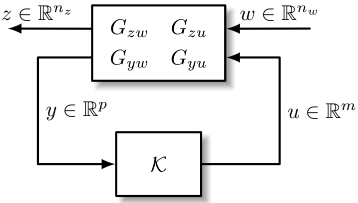


8. The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Output-Feedback Control Problems

Problem setup for output feedback control

We now replace state-feedback with measurement-feedback.

$$G: \quad \begin{bmatrix} \dot{x} \\ \hline z \\ y \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & \mathbf{0} \end{array}\right] \begin{bmatrix} x \\ \hline w \\ u \end{bmatrix}$$



- **Note:** $D_{yu} = 0$ ensures ***well-posedness***. Our goal is to design a ***dynamic*** feedback controller $\mathcal{K} : \mathcal{L}_{2\mathrm{e}}^{p}[0,\infty) \to \mathcal{L}_{2\mathrm{e}}^{m}[0,\infty)$ as

$$\mathcal{K}: \quad \begin{bmatrix} \dot{x}_{\mathrm{c}} \\ \hline u \end{bmatrix} = \left[\begin{array}{c|c} A_{\mathrm{c}} & B_{\mathrm{c}} \\ \hline C_{\mathrm{c}} & D_{\mathrm{c}} \end{array}\right] \begin{bmatrix} x_{\mathrm{c}} \\ \hline y \end{bmatrix}$$

with state $x_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}}}$ for some $n_{\mathrm{c}} \in \mathbb{Z}_{\geq 0}$ to be determined.

Problem setup for output feedback control

- Some simple algebra shows that the closed-loop system is

$$\left[\begin{array}{c}\dot{\xi} \\ \hline z\end{array}\right] = \left[\begin{array}{c|c}\mathcal{A} & \mathcal{B}_w \\ \hline \mathcal{C}_z & \mathcal{D}_{zw}\end{array}\right] \left[\begin{array}{c}\xi \\ \hline w\end{array}\right]. \tag{9}$$

  where

$$\left[\begin{array}{c|c}\mathcal{A} & \mathcal{B}_w \\ \hline \mathcal{C}_z & \mathcal{D}_{zw}\end{array}\right] = \left[\begin{array}{cc|c} A + B_u D_{\mathrm{c}} C_y & B_u C_{\mathrm{c}} & B_w + B_u D_{\mathrm{c}} D_{yw} \\ B_{\mathrm{c}} C_y & A_{\mathrm{c}} & B_{\mathrm{c}} D_{yw} \\ \hline C_z + D_{zu} D_{\mathrm{c}} C_y & D_{zu} C_{\mathrm{c}} & D_{zw} + D_{zu} D_{\mathrm{c}} D_{yw}\end{array}\right]$$

- Despite things being significantly more complicated, we will stick to our established principles and follow a similar sequence of steps:
  1. Write down a Lyapunov inequality capturing performance on $w \mapsto z$
  2. Find a smart change of variables which linearizes Lyapunov inequality
  3. Recover the controller

$\mathcal{H}_2$ output-feedback synthesis problem

By Theorem 6.3, $\mathcal{A}$ will be Hurwitz and the closed-loop system (9) will have an $\mathcal{H}_2$-norm less than γ iff there exists $\mathcal{P} \succ 0$ and $W \succ 0$ such that

$$\begin{bmatrix} \mathcal{A}^{\mathsf{T}}\mathcal{P} + \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B}_w \\ \mathcal{B}_w{}^{\mathsf{T}}\mathcal{P} & \mathbb{0} \end{bmatrix} - \begin{bmatrix} \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \gamma I_{n_w} \end{bmatrix} \prec \mathbb{0} \qquad \text{(nonlinear)} \qquad (10a)$$

$$\begin{bmatrix} \mathcal{P} & \mathcal{C}_z{}^{\mathsf{T}} \\ \mathcal{C}_z & W \end{bmatrix} \succ \mathbb{0} \qquad \text{(this is affine)} \qquad (10b)$$

$$\operatorname{trace}(W) < \gamma \qquad \text{(this is affine)} \qquad (10c)$$

$$\mathcal{D}_{zw} = D_{zw} + D_{zu}D_{\mathrm{c}}D_{yw} = 0 \qquad \text{(this is affine)} \qquad (10d)$$

It turns out that our previous tricks for state-feedback design will not work here. We need to develop a new ***linearization*** method which transforms (10) into a system of LMIs.

Linearization procedure for $\mathcal{H}_2$ synthesis

The inequality (10a) can be rewritten as

$$\left[\begin{array}{cc} I_{n+n_c} & 0 \\ \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B}_w \\ \hline 0 & I_{n_w} \end{array}\right]^{\mathsf{T}} \left[\begin{array}{cc|c} 0 & I_{n+n_c} & 0 \\ I_{n+n_c} & 0 & 0 \\ \hline 0 & 0 & -\gamma I_{n_w} \end{array}\right] \left[\begin{array}{cc} I_{n+n_c} & 0 \\ \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B}_w \\ \hline 0 & I_{n_w} \end{array}\right] \prec 0. \quad (11)$$

Let's take our matrix $\mathcal{P}$ and partition it and its inverse:

$$\mathcal{P} = \begin{bmatrix} X & U \\ U^{\mathsf{T}} & X_{\mathrm{o}} \end{bmatrix} \in \mathbb{S}^{n+n_c}, \qquad \mathcal{P}^{-1} = \begin{bmatrix} Y & V \\ V^{\mathsf{T}} & Y_{\mathrm{o}} \end{bmatrix} \in \mathbb{S}^{n+n_c}$$

from which it follows that $XY + UV^{\mathsf{T}} = I_n$ and $YU + VX_{\mathrm{o}} = 0$. Let's further define

$$\mathcal{Y} = \begin{bmatrix} Y & I_n \\ V^{\mathsf{T}} & 0_{n_c \times n} \end{bmatrix} \in \mathbb{R}^{(n+n_c)\times 2n}, \quad \mathcal{Z} = \begin{bmatrix} I_n & 0_{n \times n_c} \\ X & U \end{bmatrix}. \quad (12)$$

If we assume $n_c \geq n$, then we can always select $V \in \mathbb{R}^{n \times n_c}$ to have full row rank, and therefore $\mathcal{Y}$ will have full column rank. Note that

$$\mathcal{Y}^{\mathsf{T}}\mathcal{P} = \begin{bmatrix} Y & V \\ I_n & 0 \end{bmatrix} \begin{bmatrix} X & U \\ U^{\mathsf{T}} & X_{\mathrm{o}} \end{bmatrix} = \begin{bmatrix} YX + VU^{\mathsf{T}} & YU + VX_{\mathrm{o}} \\ X & U \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ X & U \end{bmatrix} = \mathcal{Z}$$

Linearization procedure for $\mathcal{H}_2$ synthesis

If (11) holds, then it also holds that

$$\begin{bmatrix} \mathcal{Y} & 0 \\ 0 & I \end{bmatrix}^{\mathsf{T}} \left[\begin{array}{cc} I & 0 \\ \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B}_w \\ \hline 0 & I \end{array}\right]^{\mathsf{T}} \left[\begin{array}{cc|c} 0 & I & 0 \\ I & 0 & 0 \\ \hline 0 & 0 & -\gamma I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B}_w \\ \hline 0 & I \end{array}\right] \begin{bmatrix} \mathcal{Y} & 0 \\ 0 & I \end{bmatrix} \prec 0. \quad (13)$$

The important piece here is the sub-block

$$\begin{bmatrix} \mathcal{Y} & 0 \\ 0 & I \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B}_w \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{Y} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \mathcal{Y}^{\mathsf{T}}\mathcal{P}\mathcal{A}\mathcal{Y} & \mathcal{Y}^{\mathsf{T}}\mathcal{P}\mathcal{B}_w \\ 0 & I \end{bmatrix}.$$

Working on the pieces, we first compute that

$$\begin{aligned} \mathcal{Y}^{\mathsf{T}}\mathcal{P}\mathcal{A}\mathcal{Y} &= \begin{bmatrix} I_n & 0 \\ X & U \end{bmatrix} \begin{bmatrix} A + B_u D_c C_y & B_u C_c \\ B_c C_y & A_c \end{bmatrix} \begin{bmatrix} Y & I_n \\ V^{\mathsf{T}} & 0 \end{bmatrix} \\ &= \begin{bmatrix} AY & A \\ 0 & XA \end{bmatrix} + \begin{bmatrix} 0 & B_u \\ I & 0 \end{bmatrix} \begin{bmatrix} K & L \\ M & N \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C_y \end{bmatrix}, \qquad \text{(affine!)} \end{aligned}$$

where

$$\begin{bmatrix} K & L \\ M & N \end{bmatrix} \triangleq \begin{bmatrix} U & XB_u \\ 0 & I_m \end{bmatrix} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} V^{\mathsf{T}} & 0 \\ C_y Y & I_p \end{bmatrix} + \begin{bmatrix} XAY & 0 \\ 0 & 0 \end{bmatrix} \quad (14)$$

Linearization procedure for $\mathcal{H}_2$ synthesis

Similarly we can calculate that

$$\begin{aligned}
\mathcal{Y}^{\mathsf{T}}\mathcal{P}\mathcal{B}_w &= \begin{bmatrix} I_n & 0 \\ X & U \end{bmatrix} \begin{bmatrix} B_w + B_u D_c D_{yw} \\ B_c D_{yw} \end{bmatrix} \\
&= \begin{bmatrix} B_w \\ X B_w \end{bmatrix} + \begin{bmatrix} B_u D_c D_{yw} \\ X B_u D_c D_{yw} + U B_c D_{yw} \end{bmatrix} \\
&= \begin{bmatrix} B_w \\ X B_w \end{bmatrix} + \begin{bmatrix} 0 & B_u \\ I & 0 \end{bmatrix} \begin{bmatrix} K & L \\ M & N \end{bmatrix} \begin{bmatrix} 0 \\ D_{yw} \end{bmatrix}, \qquad \text{(affine!)}
\end{aligned}$$

Putting things together, we find that

$$\begin{aligned}
\left[\begin{array}{c|c} \mathcal{Y}^{\mathsf{T}}\mathcal{P}\mathcal{A}\mathcal{Y} & \mathcal{Y}^{\mathsf{T}}\mathcal{P}\mathcal{B}_w \end{array}\right] &= \left[\begin{array}{cc|c} AY & A & B_w \\ 0 & XA & XB_w \end{array}\right] + \begin{bmatrix} 0 & B_u \\ I & 0 \end{bmatrix} \begin{bmatrix} K & L \\ M & N \end{bmatrix} \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & C_y & D_{yw} \end{array}\right] \\
&\triangleq \left[\begin{array}{c|c} \boldsymbol{A}(v) & \boldsymbol{B_w}(v) \end{array}\right]
\end{aligned}$$

where $v = (X, Y, K, L, M, N)$. ***This is affine in v!***

We can similarly compute that

$$\mathcal{P} \succ 0 \quad \Longrightarrow \quad \mathcal{Y}^{\mathsf{T}}\mathcal{P}\mathcal{Y} = \begin{bmatrix} Y & I_n \\ I_n & X \end{bmatrix} \triangleq \boldsymbol{P}(v) \succ 0,$$

which is also affine in v.

Linearization procedure for $\mathcal{H}_2$ synthesis

With these calculations, (13) simplifies to

$$\left[\begin{array}{cc} I & 0 \\ \boldsymbol{A}(v) & \boldsymbol{B_w}(v) \\ \hline 0 & I \end{array}\right]^\mathsf{T} \left[\begin{array}{cc|c} 0 & I & 0 \\ I & 0 & 0 \\ \hline 0 & 0 & -\gamma I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ \boldsymbol{A}(v) & \boldsymbol{B_w}(v) \\ \hline 0 & I \end{array}\right] \prec 0.$$

So (10a) implies the above. Similarly, if (10b) holds, then we have

$$\begin{bmatrix} \mathcal{Y} & 0 \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathcal{P} & \mathcal{C}_z{}^\mathsf{T} \\ \mathcal{C}_z & W \end{bmatrix} \begin{bmatrix} \mathcal{Y} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \mathcal{Y}^\mathsf{T}\mathcal{P}\mathcal{Y} & \mathcal{Y}^\mathsf{T}\mathcal{C}_z{}^\mathsf{T} \\ \mathcal{C}_z\mathcal{Y} & W \end{bmatrix} \succ 0. \tag{15}$$

We have calculated all blocks except $\mathcal{C}_z\mathcal{Y}$:

$$\begin{aligned} \mathcal{C}_z\mathcal{Y} &= \begin{bmatrix} C_z + D_{zu}D_\mathrm{c}C_y & D_{zu}C_\mathrm{c} \end{bmatrix} \begin{bmatrix} Y & I \\ V^\mathsf{T} & 0 \end{bmatrix} \\ &= \begin{bmatrix} C_zY & C_z \end{bmatrix} + \begin{bmatrix} 0 & D_{zu} \end{bmatrix} \begin{bmatrix} K & L \\ M & N \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C_y \end{bmatrix} \\ &\triangleq \boldsymbol{C_z}(v) \end{aligned}$$

Thus, (15) simplifies to

$$\begin{bmatrix} \boldsymbol{P}(v) & \boldsymbol{C_z}(v)^\mathsf{T} \\ \boldsymbol{C_z}(v) & W \end{bmatrix} \succ 0.$$

Linearization procedure for $\mathcal{H}_2$ synthesis

The inequality (10c) is already affine, and so is $\boldsymbol{D_{zw}}(v) \triangleq D_{zw} + D_{zu} N D_{yw} = 0$.

Summary so far (necessity): if $n_c \geq n$ and $\exists \mathcal{P}, W \succ 0$ satisfying (10), then one may define $v = (X, Y, K, L, M, N)$ satisfying the ***LMIs***

$$\boldsymbol{P}(v) \succ 0, \quad \mathrm{trace}(W) < \gamma, \quad \begin{bmatrix} \boldsymbol{P}(v) & \boldsymbol{C_z}(v)^\mathsf{T} \\ \boldsymbol{C_z}(v) & W \end{bmatrix} \succ 0, \quad \boldsymbol{D_{zw}}(v) = 0 \qquad (16)$$

and

$$\left[\begin{array}{cc} I & 0 \\ \boldsymbol{A}(v) & \boldsymbol{B_w}(v) \\ \hline 0 & I \end{array}\right]^\mathsf{T} \left[\begin{array}{cc|c} 0 & I & 0 \\ I & 0 & 0 \\ \hline 0 & 0 & -\gamma I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ \boldsymbol{A}(v) & \boldsymbol{B_w}(v) \\ \hline 0 & I \end{array}\right] \prec 0. \qquad (17)$$

In other words, feasibility of the ***nonlinear*** synthesis inequalities implies feasibility of this set of ***LMIs***. Thus, the above LMIs are *necessary* for solvability of the output feedback design problem. The key insight is that for $n_c = n$, we can actually invert all these transformations to obtain sufficiency.

Linearization procedure for $\mathcal{H}_2$ synthesis

Sufficiency: Set $n_c = n$. Suppose that (16)–(17) are feasible in W and v. Then since $\boldsymbol{P}(v) \succ 0$, it follows by Schur complements that $XY \succ I_n$, so $I_n - XY$ is nonsingular. We can always factor this as $I_n - XY = UV^\mathsf{T}$ for square invertible matrices U, V. This allows us to define the matrices $\mathcal{Y}$ and $\mathcal{Z}$ in (12), which are now square since $n_c = n$, and using the relationship $\mathcal{Y}^\mathsf{T}\mathcal{P} = \mathcal{Z}$, we can now immediately calculate $\mathcal{P}$. Since $v = (X, Y, K, L, M, N)$ is now known, we can use (14) to compute that

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} U & XB_u \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} K - XAY & L \\ M & N \end{bmatrix} \begin{bmatrix} V^\mathsf{T} & 0 \\ C_y Y & I \end{bmatrix}^{-1}. \tag{18}$$

Since $\mathcal{Y}$ is square, the previous congruence transformations are reversible, so one may work backwards from (16)–(17) to obtain the original inequalities (10).

Theorem 8.1 (LMI for $\mathcal{H}_2$ Output Feedback Synthesis). There exists a dynamic controller $\mathcal{K}$ such that $\mathcal{A}$ is Hurwitz and the closed-loop system achieves $\mathcal{H}_2$ performance at level $\gamma > 0$ ***if and only if*** there exists $v = (X, Y, K, L, M, N)$ and $W \succ 0$ satisfying (16)–(17).

Comments on $\mathcal{H}_2$ synthesis

- As sufficiency argument shows, controller order $n_c \geq n$ can always be chosen ***equal*** to the order of the plant; this is what you would expect based on ECE 557. If you instead impose that $n_c < n$, then arguments require non-convex ***rank constraints*** — reduced-order controller design is non-convex!
- You can enforce a strictly proper controller via constraint $N = 0$
- For factorization $I_n - XY = UV^{\mathsf{T}}$, simple choice is $U = I_n - XY$ and $V = I_n$.
- The number of variables can be reduced (*Elimination Lemma*); this can be important in large problems, but we will not pursue this here.

Numerical comments on $\mathcal{H}_2$ synthesis

Numerically, LMI-based synthesis requires some care. In particular, (i) the decision variables v can become quite large, and (ii) the matrix $I_n - XY$ may be close to singular. The following four step procedure usually produces reliable results:

1. Minimize γ subject to (16)–(17) to find optimal γ_{opt}
2. Fix some $\gamma > \gamma_{\text{opt}}$, introduce the additional bounding constraints

$$X \prec \alpha I_n, \qquad Y \prec \alpha I_n, \qquad \left[\begin{array}{cc|cc} \alpha I & 0 & K & L \\ 0 & \alpha I & M & N \\ \hline K^{\mathsf{T}} & M^{\mathsf{T}} & \alpha I & 0 \\ L^{\mathsf{T}} & N^{\mathsf{T}} & 0 & \alpha I \end{array}\right] \succ 0 \tag{19}$$

   and minimize over α subject to (16)–(17),(19).

3. Fix some $\alpha > \alpha_{\text{opt}}$, introduce additional constraint

$$\begin{bmatrix} Y & \beta I_n \\ \beta I_n & X \end{bmatrix} \succ 0 \tag{20}$$

   and maximize over β subject to (16)–(17),(19),(20).

4. Now reconstruct the controller parameters

$\mathcal{H}_\infty$ output-feedback synthesis problem

According to Corollary 6.1, $\mathcal{A}$ will be Hurwitz and the closed-loop system (9) will have an $\mathcal{H}_\infty$-norm less than $\gamma \geq 0$ iff there exists $\mathcal{P} \succ 0$ such that

$$\left[\begin{array}{cc} I_{n+n_c} & 0 \\ \mathcal{A} & \mathcal{B}_w \\ \hline \mathcal{C}_z & \mathcal{D}_{zw} \\ 0 & I_{n_w} \end{array}\right]^\mathsf{T} \left[\begin{array}{cc|cc} 0 & \mathcal{P} & 0 & 0 \\ \mathcal{P} & 0 & 0 & 0 \\ \hline 0 & 0 & I_{n_z} & 0 \\ 0 & 0 & 0 & -\gamma^2 I_{n_w} \end{array}\right] \left[\begin{array}{cc} I_{n+n_c} & 0 \\ \mathcal{A} & \mathcal{B}_w \\ \hline \mathcal{C}_z & \mathcal{D}_{zw} \\ 0 & I_{n_w} \end{array}\right] \prec 0;$$

An identical linearization procedure can be applied to this problem! In fact, the linearization procedure extends to a variety of other situations, including other performance objectives and to multi-objective synthesis ...

Solution of $\mathcal{H}_\infty$ output-feedback synthesis problem

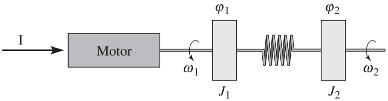
Theorem 8.2 (LMI for $\mathcal{H}_\infty$ Output Feedback Synthesis). There exists a dynamic controller $\mathcal{K}$ such that the closed-loop system is exponentially stable and achieves $\mathcal{H}_\infty$ performance at level $\gamma > 0$ ***if and only if*** there exists $v = (X, Y, K, L, M, N)$ satisfying $\boldsymbol{P}(v) \succ 0$ and

$$\left[\begin{array}{cc} I & 0 \\ \boldsymbol{A}(v) & \boldsymbol{B_w}(v) \\ \hline \boldsymbol{C_z}(v) & \boldsymbol{D_{zw}}(v) \\ 0 & I \end{array}\right]^{\mathsf{T}} \left[\begin{array}{cc|cc} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ \boldsymbol{A}(v) & \boldsymbol{B_w}(v) \\ \hline \boldsymbol{C_z}(v) & \boldsymbol{D_{zw}}(v) \\ 0 & I \end{array}\right] \prec 0.$$

In this case, $I_n - XY$ is nonsingular, and for any square nonsingular matrices U, V satisfying $I_n - XY = UV^{\mathsf{T}}$, the controller may be reconstructed via (18).

- This is still technically nonlinear; there are quadratic terms in $[\boldsymbol{C_z}(v), \boldsymbol{D}_{zw}(v)]$. However, you can quickly use Schur's Lemma to obtain a genuine LMI.

Example: two-inertia positioning system

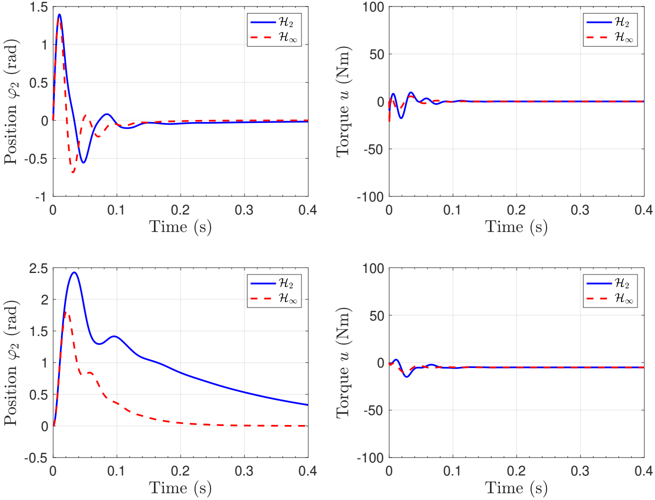


- ***Measurements:*** Only second position φ_2 is measurable; we are also free to take the integral state η and filter state ξ

$$\begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ J_1\dot{\omega}_1 \\ J_2\dot{\omega}_2 \\ \dot{\eta} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -k & k & -b & b & 0 & 0 \\ k & -k & b & -b & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/\tau_{\rm c} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \omega_1 \\ \omega_2 \\ \eta \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tau_{\rm d} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1/\tau_{\rm c} \end{bmatrix} u$$

$$\left[\begin{array}{c} z \\ \hline y \end{array}\right] = \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\rho \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \omega_1 \\ \omega_2 \\ \eta \\ \xi \end{bmatrix} + \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right] \tau_{\rm d} + \left[\begin{array}{c} 0 \\ \rho \\ \hline 0 \\ 0 \\ 0 \end{array}\right] u$$

Impulse and step disturbance test



9. Stability and Performance of Uncertain Systems

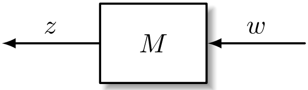
Sources of model uncertainty

- Ubiquitous sources of model uncertainty:
  (i) unmodelled (or unmodellable) higher-order dynamics,
  (ii) uncertain parameters and nonlinearities,
  (iii) imperfections in actuators and sensors, and
  (iv) deliberate simplification of a more complex model.

- It may also be the case that some elements of the system are known, but are "trouble-making" in the sense that their presence complicates our analysis or design (e.g., nonlinear infinite-dimensional components). It could then be advantageous to treat these known trouble-making components as being uncertain, if the uncertain model is ***easier to work with*** than the original model.

Uncertainty in I/O mappings

- Consider our usual input-output picture of a system



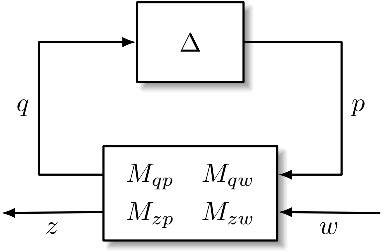
where $M : \mathcal{L}_{2\mathrm{e}}[0, \infty) \to \mathcal{L}_{2\mathrm{e}}[0, \infty)$. That is, associated with each input w is exactly one output $M(w)$.

- When we say M is "uncertain", we mean that there is ***ambiguity*** in what the output will be, even if the input is specified.

- Put differently, we are not dealing with *one* mapping M, but with a ***set*** of mappings, which is parameterized by the uncertainty.

We will focus on one parameterization, called the LFR.

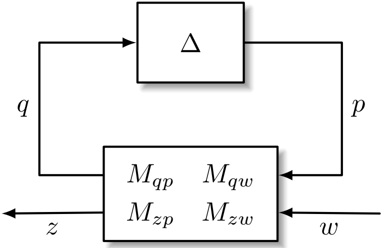
The LFR framework for uncertain systems



In a ***linear fractional representation (LFR)*** of uncertainty, an uncertain input-output system $z = \mathcal{F}(M, \Delta)(w)$ is described using a feedback interconnection between (i) a causal LTI system M, and (ii) a causal system Δ.

- If we let Δ range over a ***known set*** $\boldsymbol{\Delta}$, we obtain a ***set*** of input-output mappings $\{\mathcal{F}(M, \Delta) \mid \Delta \in \boldsymbol{\Delta}\}$.
- For $\Delta = 0$, we obtain a ***nominal*** I/O mapping $z = M_{zw}(w)$, which we are presumably already satisfied with. The question of interest is whether the closed-loop system is stable and performs well for ***all possible*** values of uncertainty $\Delta \in \boldsymbol{\Delta}$.

The LFR framework for uncertain systems



$$q = M_{qp}p + M_{qw}w$$
$$z = M_{zp}p + M_{zw}w$$
$$p = \Delta(q)$$

- Why is it called a *linear fractional* representation? Suppose that Δ is linear and all system blocks were actually just *fixed scalars*. Then

$$z = \frac{(M_{zp}M_{qw} - M_{zw}M_{qp})\Delta + M_{zw}}{(-M_{qp})\Delta + 1}w = \frac{a\Delta + b}{c\Delta + d}w$$

so $\mathcal{F}(M, \Delta)$ is a linear-fractional function of Δ.

LFR Example #1: parametric uncertainty

- Consider the uncertain scalar model

$$\begin{aligned} \dot{x} &= ax + bw, \qquad a_{\min} \leq a \leq a_{\max} \\ z &= x \end{aligned}$$

- If we define the average $\bar{a}$ and the relative spread W_a as

$$\bar{a} = \frac{a_{\min} + a_{\max}}{2}, \quad W_a = \frac{1}{\bar{a}} \frac{a_{\max} - a_{\min}}{2}$$

  then we can write $a = \bar{a}(1 + W_a \Delta)$ where $\Delta \in [-1, 1]$, so

$$\dot{x} = \bar{a}x + \bar{a}W_a \Delta x + bw, \qquad z = x$$

- We can therefore obtain the LFR model

$$\begin{aligned} \dot{x} &= \bar{a}x + \bar{a}W_a p + bw \\ q &= x \\ z &= x \end{aligned}$$

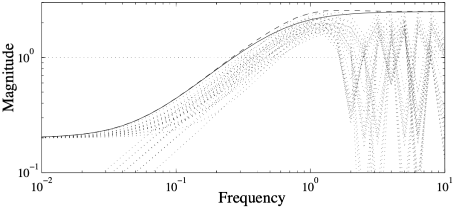
  with $p = \Delta q$.

LFR Example #2: uncertain SISO plant

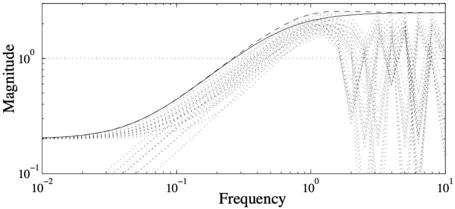
- Suppose that we have a plant G we want to model.
- From a set of n experiments, we are able to fit transfer functions $G_1(s), \ldots, G_n(s)$ describing the system. This gives us a ***nominal model*** $G_{\text{nom}}(s) = \frac{1}{n}\sum_{i=1}^{n} G_i(s)$
- To quantify the error in this choice, we can plot each relative error

$$|E_i(\mathbf{j}\omega)| = \left|\frac{G_i(\mathbf{j}\omega) - G_{\text{nom}}(\mathbf{j}\omega)}{G_{\text{nom}}(\mathbf{j}\omega)}\right|$$

over all frequencies. You will get a plot that looks something like this:



LFR Example #2: uncertain SISO plant



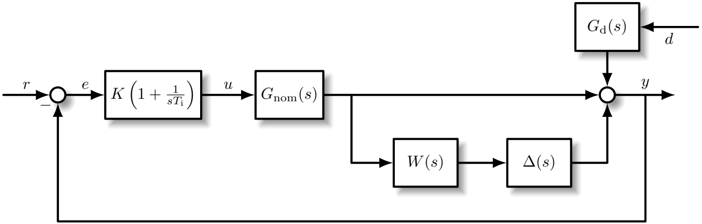
- ***Main idea:*** Find a (stable) weighting function $W(s)$ that upper bounds all relative errors: $|E_i(\mathbf{j}\omega)| \leq |W(\mathbf{j}\omega)|$ for all $i \in \{1, \ldots, n\}$.
- We can then model G using the uncertain transfer function model

$$G(s) = G_{\text{nom}}(s)[1 + W(s)\Delta(s)]$$

where $\Delta(s)$ is ***any*** stable proper TF with $\|\Delta\|_{\mathcal{H}_\infty} \leq 1$.

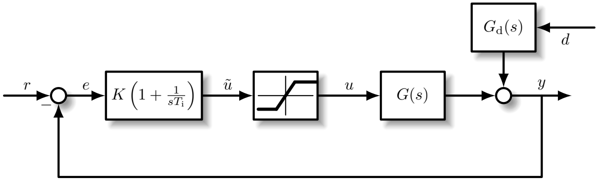
LFR Example #2: uncertain SISO plant

$$G(s) = G_{\text{nom}}(s)[1 + W(s)\Delta(s)]$$

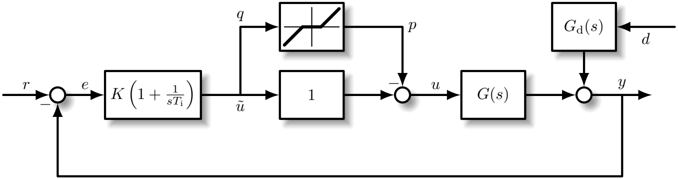


This is called an ***unstructured multiplicative*** representation of plant uncertainty.

LFR Example #3: actuator saturation in a SISO loop

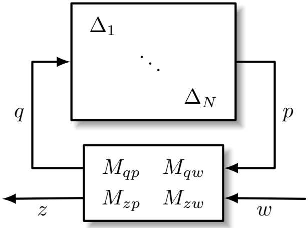


- The nominal model M should model the case *without* saturation. To do this, we note that $\mathrm{sat}(\tilde{u}) = \tilde{u} - \mathrm{deadzone}(\tilde{u})$.



Structured uncertainty

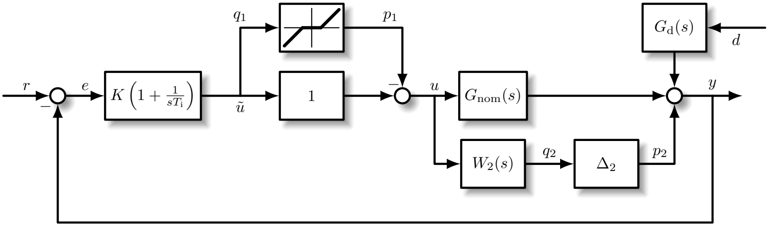
Often our uncertain operator Δ is not just one big operator, but is a collection of several smaller operators $\Delta_1, \ldots, \Delta_N$ which each act on individual sub-signals $q_1, \ldots, q_N$. This is called ***structured uncertainty***, and is the norm rather than the exception.



$$\begin{bmatrix} p_1 \\ \vdots \\ p_N \end{bmatrix} = \begin{bmatrix} \Delta_1(q_1) \\ \vdots \\ \Delta_N(q_N) \end{bmatrix}$$

Note: the ***exact same*** block Δ_k might appear multiple times.

LFR Example #4: two-block uncertainty



$$p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \quad q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad \Delta(q) = \begin{bmatrix} \mathsf{deadzone}(q_1) \\ \Delta_2(q_2) \end{bmatrix}$$

LFR Example #5: repeated uncertainty (h/t Scherer)

Consider the uncertain dynamics

$$\dot{x} = \begin{bmatrix} -1 & 2\delta_1 \\ -\frac{1}{2+\delta_1} & -4 \end{bmatrix} x, \qquad \delta_1 \in [-1, 1].$$

You can verify by direct calculation that an LFR for this is

$$\left[\begin{array}{c} \dot{x} \\ \hline q \end{array}\right] = \left[\begin{array}{cc|cc} -1 & 0 & 0 & 2 \\ -1/2 & -4 & -1/2 & -2 \\ \hline -1/2 & -4 & -1/2 & -2 \\ 0 & 1 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x \\ \hline p \end{array}\right], \quad p = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_1 \end{bmatrix} q$$

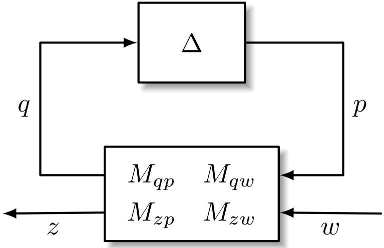
Repeated uncertain blocks will frequently occur when the same uncertain parameter appears in more than one place in your equations.

Uncertainty modelling and conservatism

- A very generic model for $\Delta : \mathcal{L}_{2\mathrm{e}}^{n_q}[0,\infty) \to \mathcal{L}_{2\mathrm{e}}^{n_p}[0,\infty)$ is as a causal operator with finite $\mathcal{L}_2$-gain bounded (without loss of generality) by 1.
- The set of all such operators is ***extremely large***; it contains, for instance, nonlinear time-varying infinite-dimensional dynamic systems.
- We may thus desire to restrict our attention to smaller uncertainty classes, by assuming other properties such as (i) linearity, (ii) time-invariance, (iii) memoryless-ness, and more ...

The general principle though is that large crude uncertainty classes are easy to describe and lead to simple computational tests, while smaller more nuanced classes are more difficult to describe and result in higher computational burden $\Longrightarrow$ trade-off between conservatism and problem complexity.

Robust stability and performance



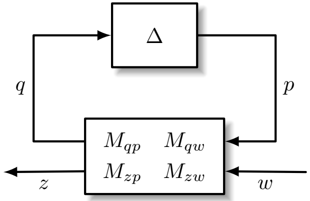
Questions we must answer:

(i) When is this loop ***stable*** (in some sense ...) for all $\Delta \in \boldsymbol{\Delta}$?

(ii) How can we bound the ***worst-case performance***

$$\sup_{\Delta \in \boldsymbol{\Delta}} \|\mathcal{F}(M,\Delta)\|_{\mathcal{H}_2} \quad \text{or} \quad \sup_{\Delta \in \boldsymbol{\Delta}} \|\mathcal{F}(M,\Delta)\|_{\mathcal{L}_2 \to \mathcal{L}_2}$$

Building intuition: The case of transfer functions



If Δ was described by a ***causal stable*** LTI system, then the whole system is LTI, and we would know how to calculate the closed-loop response . . .

$$q = M_{qp}p + M_{qw}w$$
$$z = M_{zp}p + M_{zw}w \qquad \Longrightarrow \qquad q = (I - M_{qp}\Delta)^{-1}M_{qw}w$$
$$p = \Delta q \qquad\qquad z = M_{zp}\Delta q + M_{zw}w$$

$$z = \mathcal{F}(M,\Delta)w = \left[M_{zw} + M_{zp}\Delta(I - M_{qp}\Delta)^{-1}M_{qw}\right]w$$

$\mathcal{F}(M,\Delta)$ causal & stable $\iff$ $(I - M_{qp}\Delta)^{-1}$ causal & stable

Building intuition: the case of constant matrices

- So we can now guess that $(I - M_{qp}\Delta)$ is very important. As Δ ranges over a set $\boldsymbol{\Delta}$, we would first want to ensure that $(I - M_{qp}\Delta)$ is invertible. This alone still seems like a hard question ...
- Maybe we can first try to answer this for the case of ***constant matrices***, before coming back to dynamic systems ...

Problem 9.1 (Robust matrix invertibility problem). Given a matrix $M \in \mathbb{C}^{n_q \times n_p}$ and a set $\boldsymbol{\Delta} \subset \mathbb{C}^{n_p \times n_q}$ of matrices, decide if $(I_{n_q} - M\Delta)$ is invertible for all $\Delta \in \boldsymbol{\Delta}$.

$$\text{Invertible} \quad \Longleftrightarrow \quad \det(I_{n_q} - M\Delta) \neq 0 \quad \Longleftrightarrow \quad \det \begin{bmatrix} I_{n_q} & M \\ \Delta & I_{n_p} \end{bmatrix} \neq 0$$

Robust invertibility of matrix families

$$\det \begin{bmatrix} I_{n_q} & M \\ \Delta & I_{n_p} \end{bmatrix} \neq 0 \quad \iff \quad \mathrm{range} \begin{bmatrix} I_{n_q} \\ \Delta \end{bmatrix} \cap \mathrm{range} \begin{bmatrix} M \\ I_{n_p} \end{bmatrix} = \{\mathbb{0}\}.$$

- If we interpret M and Δ as defining linear operators, this has an interpretation in terms of the ***graphs*** of M and Δ

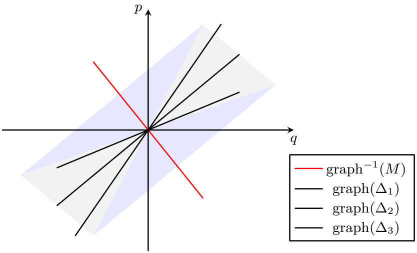
$$\mathrm{graph}(\Delta) \triangleq \{(q, \Delta q) \mid q \in \mathbb{C}^{n_q}\} = \mathrm{range} \begin{bmatrix} I_{n_q} \\ \Delta \end{bmatrix} \qquad \text{“Set of I/O pairs”}$$

$$\mathrm{graph}^{-1}(M) \triangleq \{(Mp, p) \mid p \in \mathbb{C}^{n_p}\} = \mathrm{range} \begin{bmatrix} M \\ I_p \end{bmatrix} \qquad \text{“Set of O/I pairs”}$$

Graph separation principle: $(I_{n_q} - M\Delta)$ invertible $\forall \Delta \in \boldsymbol{\Delta}$ if and only if $\mathrm{graph}(\Delta) \cap \mathrm{graph}^{-1}(M) = \{0\}$ $\forall \Delta \in \boldsymbol{\Delta}$.

Robust invertibility of matrix families

How can we guarantee this separation condition? Look for ***cones***.



If we knew that $\forall \Delta \in \mathbf{\Delta}$, $\mathrm{graph}(\Delta)$ was contained in the grey-shaded cone region, then we just need to make sure that $\mathrm{graph}^{-1}(M)$ is contained in the complementary blue-shaded cone!

- Parameterize this idea by introducing a ***quadratic form***

$$\pi : \mathbb{C}^{n_q+n_p} \to \mathbb{R}, \qquad \pi(\xi) = \xi^*\Pi\xi, \qquad \Pi = \Pi^* \in \mathbb{C}^{(n_q+n_p)\times(n_q+n_p)}$$

$$\{\text{grey shaded region}\} = \{\xi \in \mathbb{C}^{n_q \times n_p} \mid \pi(\xi) \geq 0\}$$

$$\{\text{blue shaded region}\} = \{\xi \in \mathbb{C}^{n_q \times n_p} \mid \pi(\xi) < 0\}.$$

Robust invertibility of matrix families

Putting things together, what we want is

$$\pi(\xi) = \xi^* \Pi \xi \geq 0 \quad \text{for all} \quad \xi = (q,p) \in \mathrm{graph}(\Delta) \quad \text{and all} \quad \Delta \in \boldsymbol{\Delta},$$
$$\pi(\xi) = \xi^* \Pi \xi < 0 \quad \text{for all} \quad \xi = (q,p) \in \mathrm{graph}^{-1}(M).$$

Proposition 9.1 (Invertibility of Matrix Families). Let $M \in \mathbb{C}^{n_q \times n_p}$ and let $\boldsymbol{\Delta} \subset \mathbb{C}^{n_p \times n_q}$ be a set of matrices. Suppose that there exists a Hermitian matrix $\Pi = \Pi^* \in \mathbb{C}^{(n_q+n_p)\times(n_q+n_p)}$ such that

$$\begin{bmatrix} I_{n_q} \\ \Delta \end{bmatrix}^* \Pi \begin{bmatrix} I_{n_q} \\ \Delta \end{bmatrix} \succeq \mathbb{0} \quad \text{for all} \quad \Delta \in \boldsymbol{\Delta}$$
$$\begin{bmatrix} M \\ I_{n_p} \end{bmatrix}^* \Pi \begin{bmatrix} M \\ I_{n_p} \end{bmatrix} \prec \mathbb{0}.$$

Then $(I_{n_q} - M\Delta)$ is invertible for all $\Delta \in \boldsymbol{\Delta}$.

Proof of Proposition 9.1

By contradiction, suppose there exists some element $\Delta \in \boldsymbol{\Delta}$ for which the conclusion fails. Then by our determinant conditions, there exists a non-zero vector $\mathrm{col}(q,p) \in \mathbb{C}^{n_q+n_p}$ such that

$$\mathbb{0}_{q+p} = \begin{bmatrix} I_{n_q} & M \\ \Delta & I_{n_p} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} I_{n_q} \\ \Delta \end{bmatrix} q + \begin{bmatrix} M \\ I_{n_p} \end{bmatrix} p \quad \Longleftrightarrow \quad \begin{bmatrix} I_{n_q} \\ \Delta \end{bmatrix} q = - \begin{bmatrix} M \\ I_{n_p} \end{bmatrix} p.$$

Since $q = -Mp$ and $p = -\Delta q$, this further implies that q and p are *individually* also non-zero. From the inequality conditions then, we find that

$$0 \leq q^* \begin{bmatrix} I_{n_q} \\ \Delta \end{bmatrix}^* \Pi \begin{bmatrix} I_{n_q} \\ \Delta \end{bmatrix} q = p^* \begin{bmatrix} M \\ I_{n_p} \end{bmatrix} \Pi \begin{bmatrix} M \\ I_{n_p} \end{bmatrix} p \leq -\varepsilon \|p\|_2^2.$$

for some sufficiently small $\varepsilon > 0$, which implies that $\|p\|_2 \leq 0$. This can hold only if $p = \mathbb{0}_{n_p}$, which is a contradiction. Hence there exists no element $\Delta \in \boldsymbol{\Delta}$ for which $\det(I_{n_q} - M\Delta) = 0$, which establishes the claim. •

Example

For $\beta \in \mathbb{R}$ consider the matrix and uncertainty set

$$M_\beta = \begin{bmatrix} 1/2 & \sqrt{3}\beta \\ \sqrt{3}\beta & 1/3 \end{bmatrix}, \qquad \mathbf{\Delta} = \left\{ \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \;\middle|\; \delta_1, \delta_2 \in [-1, 1] \right\}$$

If we define

$$\mathbf{\Pi} = \left\{ \left[\begin{array}{cc|cc} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ \hline 0 & 0 & -q_1 & 0 \\ 0 & 0 & 0 & -q_2 \end{array} \right] \;\middle|\; q_1, q_2 \geq 0 \right\}.$$

Then for any $\Pi \in \mathbf{\Pi}$ we have

$$\begin{bmatrix} I_2 \\ \Delta \end{bmatrix}^* \Pi \begin{bmatrix} I_2 \\ \Delta \end{bmatrix} = \begin{bmatrix} q_1(1-\delta_1^2) & 0 \\ 0 & q_2(1-\delta_2^2) \end{bmatrix} \succeq \mathbb{0}\,, \qquad \Delta \in \mathbf{\Delta}.$$

For any fixed value of β, we can try to solve the LMI problem

$$\text{find} \qquad \Pi \in \mathbf{\Pi} \quad \text{such that} \quad \begin{bmatrix} M_\beta \\ I_2 \end{bmatrix}^* \Pi \begin{bmatrix} M_\beta \\ I_2 \end{bmatrix} \prec 0$$

In this case, turns out LMI is feasible for $\beta \in (-\frac{1}{3}, \frac{1}{3})$.

Key insights from matrix invertibility problem

- The recipe for checking invertibility seems to be the following:
  1. Find a set $\mathbf{\Pi}$ of matrices such that

     $$\begin{bmatrix} I_{n_q} \\ \Delta \end{bmatrix}^* \Pi \begin{bmatrix} I_{n_q} \\ \Delta \end{bmatrix} \succeq 0 \qquad \forall \Delta \in \mathbf{\Delta}, \Pi \in \mathbf{\Pi}$$

     This is a ***quadratic constraint*** description of the uncertainty set $\mathbf{\Delta}$.
  2. Find *any particular* $\Pi \in \mathbf{\Pi}$ to satisfy the inequality

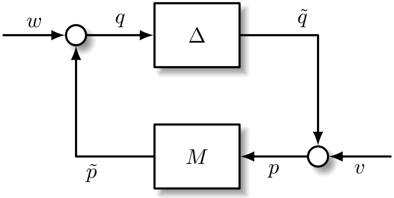
     $$\begin{bmatrix} M \\ I_{n_p} \end{bmatrix}^* \Pi \begin{bmatrix} M \\ I_{n_p} \end{bmatrix} \prec \mathbb{0}$$

- **Note:** The richer the set $\mathbf{\Pi}$ is in Step 1, the easier it will be to find *one particular* $\Pi \in \mathbf{\Pi}$ that works for Step 2!

Our goal is now to translate these ideas back to systems.

Robust feedback stability setup

We now consider the following prototypical feedback diagram



$$\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} p - \Delta(q) \\ q - Mp \end{bmatrix} \triangleq \Sigma_\Delta(p, q)$$

- M is a causal stable LTI system
- $\Delta \in \boldsymbol{\Delta}$, a set of causal operators which are ***uniformly*** $\mathcal{L}_2$-stable with finite gain (i.e., a uniform gain bound for all Δ)
- w and v are exogenous signals which excite the interconnection

As you learned in undergrad, the proper stability notion for a feedback interconnection is "bounded input–bounded output"

Robust feedback stability setup

We first need some basic conditions for this setup to make much sense:

(i) if we fix (v, w), there should exist a unique solution for (p, q), and

(ii) the solution (p, q) should depend *causally* on (v, w).

In other words, for all Δ, the operator Σ_Δ should be invertible and the inverse should be causal; we call this ***well-posedness*** of the interconnection, and we assume this going forward

Definition 9.1 (Robust Feedback Stability). Assume the interconnection is well-posed. Then it is ***robustly $\mathcal{L}_2$-stable with finite gain*** if there exists $\gamma \geq 0$ such that $\|\Sigma_\Delta^{-1}\|_{\mathcal{L}_2 \to \mathcal{L}_2} \leq \gamma$ for all $\Delta \in \boldsymbol{\Delta}$.

Note: this is the same as saying that $\|p\|_{\mathcal{L}_2}^2 + \|q\|_{\mathcal{L}_2}^2 \leq \gamma^2(\|v\|_{\mathcal{L}_2}^2 + \|w\|_{\mathcal{L}_2}^2)$ for some $\gamma \geq 0$ and all $(v, w) \in \mathcal{L}_2[0, \infty)$: ***BIBO Stability***!

Reduced problem for robust stability

It turns out the problem is simpler than it looks. As notation, we let I denote the identity operator on $\mathcal{L}_{2\mathrm{e}}[0,\infty)$.

Proposition 9.2 (Reduction to Stability of $(I - M\Delta)^{-1}$). The following statements are equivalent:

(i) the interconnection is robustly $\mathcal{L}_2$-stable with finite gain;

(ii) the operator $I - M\Delta : \mathcal{L}_{2\mathrm{e}}[0,\infty) \to \mathcal{L}_{2\mathrm{e}}[0,\infty)$ has a causal inverse $(I - M\Delta)^{-1}$ which is robustly $\mathcal{L}_2$-stable with finite gain.

- Stability of the overall interconnection is equivalent to stability of the much simpler mapping $(I - M\Delta)^{-1}$.
- We now know where to focus our attention
- Looks ***suspiciously*** like our robust matrix invertibility problem!

Proof of Proposition 9.2

(ii) $\Rightarrow$ (i): Using the feedback interconnection equations, we can eliminate $p = v + \Delta(q)$ and use linearity of M to find that

$$w = q - M(v + \Delta(q)) = q - Mv - M\Delta(q)$$

or simply $w + Mv = q - M\Delta(q)$. Since $(I - M\Delta)$ has a causal inverse, it follows that $q = (I - M\Delta)^{-1}(w + Mv)$ depends causally on (v, w), and hence so does $p = v + \Delta(q)$, so the interconnection is well-posed. By assumption M, Δ, and $(I - M\Delta)^{-1}$ are $\mathcal{L}_2$-stable with finite gain; call the gains $\gamma_1, \gamma_2, \gamma_3$. Then

$$\|q\|_{\mathcal{L}_2} \leq \gamma_3(\|w\|_{\mathcal{L}_2} + \gamma_1\|v\|_{\mathcal{L}_2})$$

and

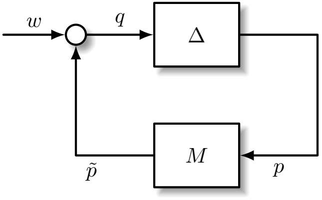
$$\begin{aligned}\|p\|_{\mathcal{L}_2} &\leq \|v\|_{\mathcal{L}_2} + \gamma_2\|q\|_{\mathcal{L}_2} \\ &\leq \|v\|_{\mathcal{L}_2} + \gamma_2\gamma_3(\|w\|_{\mathcal{L}_2} + \gamma_1\|v\|_{\mathcal{L}_2}).\end{aligned}$$

From here simple manipulations show (i).

(i) $\Rightarrow$ (ii): If the overall interconnection is well-posed and $\mathcal{L}_2$-stable with finite gain, then in particular so is the mapping $q = (I - M\Delta)^{-1}(w + Mv)$ from (v, w) to q, so $(I - M\Delta)^{-1}$ must be causal and $\mathcal{L}_2$-stable with finite gain. •

Reduced problem for robust stability

Proposition 9.2 allows us to study the simpler block diagram



$$q = w + M\Delta(q)$$

$$\iff \quad q = (I - M\Delta)^{-1}(w)$$

- As before, we say this system is robustly $\mathcal{L}_2$-stable with finite-gain if it is well-posed and if the $\mathcal{L}_2$ norms of the internal signals (p, q) can be bounded in terms of the $\mathcal{L}_2$ norm of w, uniformly with respect to Δ.

- We now come to what additional assumptions to place on Δ

Introduction to quadratic constraints

- Let's try to generalize our graph separation idea to systems
- For example, examine the class of causal operators with finite $\mathcal{L}_2$-gain

$$\boldsymbol{\Delta}_\gamma \triangleq \{\Delta : \mathcal{L}_{2\mathrm{e}}[0,\infty) \to \mathcal{L}_{2\mathrm{e}}[0,\infty) \mid \Delta \text{ causal and } \|\Delta\|_{\mathcal{L}_2 \to \mathcal{L}_2} \leq \gamma\}$$

- For $\Delta \in \boldsymbol{\Delta}_\gamma$ and $q \in \mathcal{L}_{2\mathrm{e}}[0,\infty)$ with $p = \Delta(q)$, we have

$$\|p_T\|_{\mathcal{L}_2} \leq \gamma \|q_T\|_{\mathcal{L}_2} \quad \Longleftrightarrow \quad \sigma\|p_T\|_{\mathcal{L}_2}^2 \leq \gamma^2 \sigma \|q_T\|_{\mathcal{L}_2}^2$$

  for all $T \geq 0$ and any $\sigma > 0$. Rearranging, this the same as saying

$$\int_0^T \begin{bmatrix} q(t) \\ p(t) \end{bmatrix}^{\mathsf{T}} \Pi(\sigma) \begin{bmatrix} q(t) \\ p(t) \end{bmatrix} \mathrm{d}t \geq 0, \quad \Pi = \sigma \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix}. \qquad (21)$$

- All I/O pairs satisfy a ***quadratic constraint***.

Example: Q.C. for parametric uncertainty

Scalar parametric uncertainty is defined by the class of operators

$$\boldsymbol{\Delta}_{\text{par}} = \{\Delta \mid \Delta \text{ memoryless, scalar, LTI, and } \|\Delta\|_{\mathcal{L}_2 \to \mathcal{L}_2} \leq \gamma\}$$

$$\Delta(q)(t) = \delta q(t) \qquad \text{where } |\delta| \leq \gamma.$$

For any $\sigma > 0$ with $p(t) = \Delta(q)(t)$ we obviously have for all $t \geq 0$ that

$$\sigma |p(t)|^2 \leq \sigma \gamma^2 |q(t)|^2 \quad \Longleftrightarrow \quad \begin{bmatrix} q(t) \\ p(t) \end{bmatrix}^{\mathsf{T}} \Pi(\sigma) \begin{bmatrix} q(t) \\ p(t) \end{bmatrix} \geq 0. \tag{22}$$

- In contrast to the ***integral*** quadratic constraint in (21), this is a stronger ***point-wise*** constraint, which holds at all points in time.
- A point-wise constraint like (22) will always imply (21) — just integrate (22) over $[0, T]$ — but the converse is false.

Example: Q.C. for repeated parametric uncertainty

What if we have ***repeated real*** parametric uncertainty

$$\Delta(q)(t) = \delta I_n \cdot q(t), \qquad \delta \in [-\gamma, \gamma].$$

If we take any $Q \succeq 0$ and any S such that $S = -S^*$, then

$$\begin{aligned}
\begin{bmatrix} q(t) \\ \delta I_n q(t) \end{bmatrix}^* \begin{bmatrix} \gamma^2 Q & S \\ S^* & -Q \end{bmatrix} \begin{bmatrix} q(t) \\ \delta I_n q(t) \end{bmatrix} &= (\gamma^2 - \delta^2) q(t)^* Q q(t) \\
&\quad + \delta q(t)^* S q(t) + \delta q(t)^* S^* q(t) \\
&= (\gamma^2 - \delta^2) q(t)^* Q q(t) \\
&\geq 0
\end{aligned}$$

so we again have a ***point-wise*** quadratic constraint. Note now though that we have ***much more freedom***, because we can choose Q and R as opposed to just one scalar σ.

Robust stability via dissipativity theory

Theorem 9.1 (Robust Stability). Consider the previously described feedback interconnection. Assume that there exists a set of Hermitian matrices $\mathbf{\Pi} \subseteq \mathbb{H}^{n_q+n_p}$ such that

$$\int_0^T \begin{bmatrix} q(t) \\ \Delta(q)(t) \end{bmatrix}^{\mathsf{T}} \Pi \begin{bmatrix} q(t) \\ \Delta(q)(t) \end{bmatrix} \mathrm{d}t \geq 0 \qquad \forall q \in \mathcal{L}_{2\mathrm{e}}[0,\infty),\ T \geq 0,$$

for all $\Pi \in \mathbf{\Pi}$ and all $\Delta \in \mathbf{\Delta}$. If there exists $\Pi \in \mathbf{\Pi}$ such that

$$\begin{bmatrix} \hat{M}(\mathbf{j}\omega) \\ I_m \end{bmatrix}^* \Pi \begin{bmatrix} \hat{M}(\mathbf{j}\omega) \\ I_m \end{bmatrix} \prec \mathbb{0}, \qquad \omega \in \mathbb{R} \cup \{\infty\},$$

then the interconnection is robustly $\mathcal{L}_2$-stable with finite gain.

- This is a version of the "hard IQC theorem"; more later on this
- Can specialize to recover many standard(ish) feedback stability results

Proof of Theorem 9.1

Let (A, B, C, D) be a minimal realization of M; since M is stable, A is Hurwitz. By the strict dissipativity theorem, the stated FDI involving $\hat{M}(\mathbf{j}\omega)$ is equivalent to the system being input-strictly dissipative with supply rate $s(p, \tilde{p}) = -\left[\begin{smallmatrix}\tilde{p}\\p\end{smallmatrix}\right]^{\mathsf{T}} \Pi \left[\begin{smallmatrix}\tilde{p}\\p\end{smallmatrix}\right]$ with storage function $V(x) = x^{\mathsf{T}}Px$ with $P \succ 0$. We compute along trajectories that

$$
\begin{aligned}
\dot{V}(x(t)) &\leq -\left[\begin{smallmatrix}\tilde{p}\\p\end{smallmatrix}\right]^{\mathsf{T}} \Pi \left[\begin{smallmatrix}\tilde{p}\\p\end{smallmatrix}\right] - \varepsilon^2\|p\|_2^2\\
&= -\left[\begin{smallmatrix}q-w\\p\end{smallmatrix}\right]^{\mathsf{T}} \Pi \left[\begin{smallmatrix}q-w\\p\end{smallmatrix}\right] - \varepsilon^2\|p\|_2^2\\
&= -\left[\begin{smallmatrix}q\\p\end{smallmatrix}\right]^{\mathsf{T}} \Pi \left[\begin{smallmatrix}q\\p\end{smallmatrix}\right] - \left[\begin{smallmatrix}w\\0\end{smallmatrix}\right]^{\mathsf{T}} \Pi \left[\begin{smallmatrix}w\\0\end{smallmatrix}\right] + 2\left[\begin{smallmatrix}w\\0\end{smallmatrix}\right]^{\mathsf{T}} \Pi \left[\begin{smallmatrix}q\\p\end{smallmatrix}\right] - \varepsilon^2\|p\|_2^2\\
&\leq -\left[\begin{smallmatrix}q\\p\end{smallmatrix}\right]^{\mathsf{T}} \Pi \left[\begin{smallmatrix}q\\p\end{smallmatrix}\right] + c_1\|w\|_2^2 + c_2\|w\|_2 \left\|\left[\begin{smallmatrix}q\\p\end{smallmatrix}\right]\right\|_2 - \epsilon^2\|p\|_2^2
\end{aligned}
$$

for some $c_1, c_2 \geq 0$ which depend only on Π. Integrating over $[0, T]$ and using $V(x(T)) \geq 0$ and $x(0) = 0$ we obtain

$$
0 \leq -\underbrace{\int_0^T \left[\begin{smallmatrix}q\\p\end{smallmatrix}\right]^{\mathsf{T}} \Pi \left[\begin{smallmatrix}q\\p\end{smallmatrix}\right]\,\mathrm{d}t}_{\geq 0} + \int_0^T c_1\|w\|_2^2 + c_2\|w\|_2 \left\|\left[\begin{smallmatrix}q\\p\end{smallmatrix}\right]\right\|_2 - \epsilon^2\|p\|_2^2\,\mathrm{d}t
$$

Proof of Theorem 9.1

So we arrive at the inequality

$$0 \leq \int_0^T c_1 \|w\|_2^2 + c_2 \|w\|_2 \left\| \begin{bmatrix} q \\ p \end{bmatrix} \right\|_2 - \epsilon^2 \|p\|_2^2 \,\mathrm{d}t, \qquad T \geq 0.$$

For any $a, b \in \mathbb{R}$ the so-called Peter-Paul inequality is $ab \leq \frac{a^2}{2\delta} + \frac{\delta b^2}{2}$ for $\delta > 0$. Using this to upper bound the cross-term, we find that

$$0 \leq \int_0^T \left(c_1 + \frac{c_2}{2\delta}\right) \|w\|_2^2 + \frac{c_2 \delta}{2}(\|q\|_2^2 + \|p\|_2^2) - \epsilon^2 \|p\|_2^2 \,\mathrm{d}t$$

or using the notation of truncated $\mathcal{L}_2$ signals, we have more simply that

$$0 \leq \left(c_1 + \frac{c_2}{2\delta}\right) \|w_T\|_{\mathcal{L}_2}^2 + \frac{c_2 \delta}{2}(\|q_T\|_{\mathcal{L}_2}^2 + \|p_T\|_{\mathcal{L}_2}^2) - \epsilon^2 \|p_T\|_{\mathcal{L}_2}^2$$

Since $q = w + Mp$ and M has finite gain, we further have that

$$\begin{aligned} \|q_T\|_{\mathcal{L}_2} &\leq \|w_T\|_{\mathcal{L}_2} + c_3 \|p_T\|_{\mathcal{L}_2} \\ \|q_T\|_{\mathcal{L}_2}^2 &\leq \|w_T\|_{\mathcal{L}_2}^2 + c_3^2 \|p_T\|_{\mathcal{L}_2}^2 + 2c_3 \|w_T\|_{\mathcal{L}_2} \|p_T\|_{\mathcal{L}_2} \\ &\leq (1 + c_3)\|w_T\|_2^2 + (c_3^2 + c_3)\|p_T\|_2^2 \end{aligned}$$

Proof of Theorem 9.1

Combining these inequalities and rearraging, we find that

$$\left(\epsilon^2 - \frac{c_2\delta}{2}(1 + c_3^2 + c_3)\right) \|p_T\|_{\mathcal{L}_2}^2 \leq \left(c_1 + \frac{c_2}{2\delta} + \frac{c_2\delta}{2}(1 + c_3)\right) \|w_T\|_2^2.$$

Selecting δ sufficiently small, we therefore find that $\|p_T\|_{\mathcal{L}_2}^2 \leq \gamma^2 \|w_T\|_{\mathcal{L}_2}^2$ for some $\gamma \geq 0$ and all $T \geq 0$; by our previous calculations, a similar inequality holds for $\|q_T\|_{\mathcal{L}_2}$. We conclude that all internal signals are bounded in terms of w, and therefore the interconnection is robustly $\mathcal{L}_2$-stable with finite gain. •

If the constraint $\Pi \in \mathbf{\Pi}$ admits an LMI description, then FDI condition ***equivalent*** to LMI feasibility problem! Find $P \succ 0$ and $\Pi \in \mathbf{\Pi}$ s.t.

$$\begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \mathbb{0} & P \\ P & \mathbb{0} \end{bmatrix} \begin{bmatrix} I_n & \mathbb{0} \\ A & B \end{bmatrix} + \begin{bmatrix} C & D \\ \mathbb{0} & I_m \end{bmatrix}^{\mathsf{T}} \Pi \begin{bmatrix} C & D \\ \mathbb{0} & I_m \end{bmatrix} \prec \mathbb{0}. \tag{23}$$

The small-gain theorem

Consider again our example uncertainty set

$$\mathbf{\Delta}_\gamma \triangleq \{\Delta \mid \Delta \text{ causal and } \|\Delta\|_{\mathcal{L}_2 \to \mathcal{L}_2} \leq \gamma\}$$

Corollary 9.1 (Small-Gain Theorem). If $\mathbf{\Delta} = \mathbf{\Delta}_\gamma$ and $\|M\|_{\mathcal{L}_2 \to \mathcal{L}_2} < \frac{1}{\gamma}$, then the interconn. is robustly $\mathcal{L}_2$-stable with finite gain.

Proof: With $\Pi = \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix}$ the FDI reduces to $\gamma^2 \hat{M}(\mathbf{j}\omega)^* \hat{M}(\mathbf{j}\omega) \prec I$ which by our previous results is precisely the specified gain condition $\|M\|_{\mathcal{H}_\infty} < 1/\gamma$ •

The product of the gains around the loop should be less than 1.

- ***SISO Interpretation***: the Nyquist plot of $\hat{M}$ is strictly contained within the circle of radius $1/\gamma$ in the complex plane.

A strong passivity theorem

An operator $\Delta : \mathcal{L}_2^m[0,\infty) \to \mathcal{L}_2^m[0,\infty)$ is ***passive*** if

$$\langle q_T, \Delta(q)_T \rangle_{\mathcal{L}_2} \geq 0 \qquad \text{for all } q \in \mathcal{L}_{2\mathrm{e}}[0,\infty),\ T \geq 0.$$

Consider now the uncertainty set

$$\boldsymbol{\Delta}_{\mathrm{p}} \triangleq \{\Delta \mid \Delta \text{ causal, finite-gain, and passive}\}$$

Corollary 9.2 (Strong SPR Theorem). If $\boldsymbol{\Delta} = \boldsymbol{\Delta}_{\mathrm{p}}$ and $\hat{M}(\mathbf{j}\omega)^* + \hat{M}(\mathbf{j}\omega) \prec 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$, then the interconnection is robustly $\mathcal{L}_2$-stable with finite gain.

- This is called a ***strong strictly positive real (SPR)*** condition on $-M$, or equivalently, that $-M$ is an input-strictly passive system
- ***SISO Interpretation:*** $\mathrm{Re}(\hat{M}(\mathbf{j}\omega)) \leq -\epsilon$ for some $\epsilon > 0$ and for all $\omega \in \mathbb{R} \cup \{\infty\}$; the Nyquist plot of $\hat{M}$ is strictly contained in $\mathbb{C}_{<0}$.

The circle criterion (SISO)

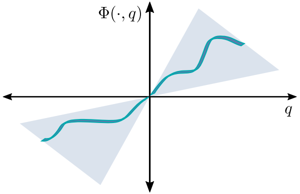
It's often of interest just to consider Δ blocks defined by memoryless nonlinear functions such as saturation, deadband, etc. Given a nonlinear (possibly time-varying) function $\Phi : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ satisfying $\Phi(t, 0) = 0$, we can define an associated operator $\Delta_\Phi(q)(t) = \Phi(t, q(t))$. We say Δ_Φ is ***sector bounded*** if

$$\begin{bmatrix} q \\ \Phi(t,q) \end{bmatrix}^\mathsf{T} \begin{bmatrix} -2\alpha\beta & (\alpha+\beta) \\ (\alpha+\beta) & -2 \end{bmatrix} \begin{bmatrix} q \\ \Phi(t,q) \end{bmatrix} \geq 0, \qquad t \geq 0, \quad q \in \mathbb{R}$$

for some $\alpha, \beta \in \mathbb{R}$ with $\beta \geq \alpha \geq 0$; we let $\boldsymbol{\Delta}_{\alpha\beta}$ denote the uncertainty set.

(Other cases for α, β are similarly treated.)

Interpretation: The function is bounded between the lines $p = \alpha q$ and $p = \beta q$.



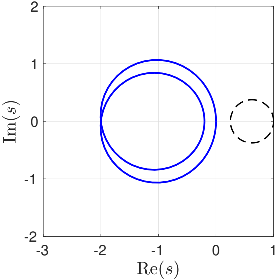
The circle criterion (SISO, $\alpha > 0$)

Corollary 9.3 (Circle Criterion). If $\mathbf{\Delta} = \mathbf{\Delta}_{\alpha\beta}$ and $\alpha > 0$, then the interconnection is robustly $\mathcal{L}_2$-stable with finite gain if

$$|\hat{M}(\mathbf{j}\omega) - c|^2 > r^2, \qquad \text{where } c = \frac{\alpha+\beta}{2\alpha\beta},\ r = \frac{\beta-\alpha}{2\alpha\beta}.$$

Proof: Follows by direct manipulation of the FDI. •

The Nyquist plot of the transfer function does not enter the closed disk of radius $r > 0$ centred at $s = c$.



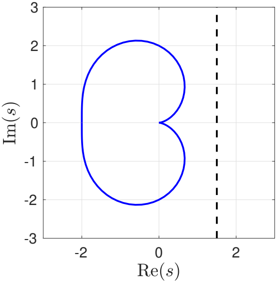
The circle criterion (SISO, $\alpha = 0$)

Corollary 9.4 (Circle Criterion). If $\mathbf{\Delta} = \mathbf{\Delta}_{\alpha\beta}$ with $\alpha = 0$ and $\beta > 0$, then the interconnection is robustly $\mathcal{L}_2$-stable with finite gain if

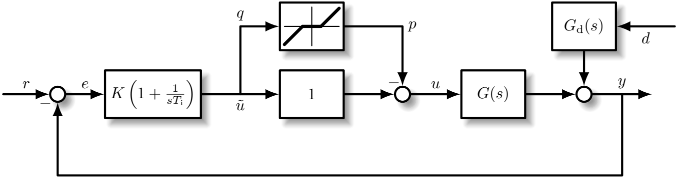
$$\mathrm{Re}(\hat{M}(\mathbf{j}\omega)) < \frac{1}{\beta}.$$

Proof: Follows by direct manipulation of the FDI. •

The Nyquist plot of the transfer function lies to the left of the vertical line $\mathrm{Re}\{s\} = 1/\beta$.

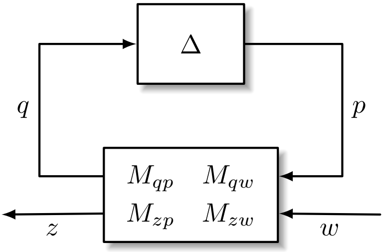


Example: saturation in a SISO loop



- The deadzone nonlinearity lies within the sector $[0,1]$, and therefore belongs to the set $\mathbf{\Delta}_{01}$. We can use the circle criterion to test robust stability of the interconnection as a function of K, T_{i}, either using the FDI or the LMI.
- Feasibility of the FDI/LMI is only a ***sufficient*** condition for stability of this loop, because the sector $[0,1]$ captures a much larger set of operators than just the deadzone nonlinearity. To reduce conservatism, you need a tighter description of the deadzone nonlinearity via IQC theory.

Robust $\mathcal{L}_2$ performance



With $z = \mathcal{F}(M, \Delta)(w)$, how can we bound ***worst-case performance***

$$\sup_{\Delta \in \boldsymbol{\Delta}} \|\mathcal{F}(M, \Delta)\|_{\mathcal{L}_2 \to \mathcal{L}_2}$$

- **Note:** The induced $\mathcal{L}_2$-gain is well-defined whether Δ is linear or nonlinear ... no problem.
- State-space realization for M:

$$\begin{bmatrix} \dot{x} \\ \hline q \\ z \end{bmatrix} = \left[\begin{array}{c|cc} A & B_p & B_w \\ \hline C_q & D_{qp} & D_{qw} \\ C_z & D_{zp} & D_{zw} \end{array}\right] \begin{bmatrix} x \\ p \\ w \end{bmatrix}$$

Robust $\mathcal{L}_2$ performance

- ***Our goal:*** Formulate an LMI for robust performance.
- To ensure that $\|\mathcal{F}(M,\Delta)\|_{\mathcal{L}_2\to\mathcal{L}_2} < \gamma$ we want $\exists \varepsilon > 0$ s.t.

$$\int_0^T \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \underbrace{\begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}}_{\triangleq \Pi_\gamma^{\mathrm{p}}} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \mathrm{d}t \leq -\varepsilon^2 \int_0^T w(t)^{\mathsf{T}} w(t)\,\mathrm{d}t \qquad \forall T \geq 0.$$

- With this notation, our robust stability LMI reads as: find $P \succ 0$, $\Pi \in \mathbf{\Pi}$ s.t.

$$\left[\begin{array}{cc} I_n & \mathbb{0} \\ A & B_p \\ \hline C_q & D_{qp} \\ \mathbb{0} & I_{n_p} \end{array}\right]^{\mathsf{T}} \left[\begin{array}{cc|c} \mathbb{0} & P & \mathbb{0} \\ P & \mathbb{0} & \mathbb{0} \\ \hline \mathbb{0} & \mathbb{0} & \Pi \end{array}\right] \left[\begin{array}{cc} I_n & \mathbb{0} \\ A & B_p \\ \hline C_q & D_{qp} \\ \mathbb{0} & I_{n_p} \end{array}\right] \prec \mathbb{0}$$

- First column corresponds to signal x ...second column corresponds to signal p ...can we just append another column for the signal w?

LMI for robust $\mathcal{L}_2$ performance

Theorem 9.2 (Robust $\mathcal{L}_2$-Performance). Consider the previously described feedback interconnection. Assume that there exists a set of Hermitian matrices $\mathbf{\Pi} \subseteq \mathbb{H}^{n_q+n_p}$ such that

$$\int_0^T \begin{bmatrix} q(t) \\ \Delta(q)(t) \end{bmatrix}^\mathsf{T} \Pi \begin{bmatrix} q(t) \\ \Delta(q)(t) \end{bmatrix} \mathrm{d}t \geq 0 \qquad \forall q \in \mathcal{L}_{2\mathrm{e}}[0,\infty),\ T \geq 0,$$

and all $\Pi \in \mathbf{\Pi}$ and all $\Delta \in \mathbf{\Delta}$. If there exists $P \succ 0$ and $\Pi \in \mathbf{\Pi}$ such that

$$\left[\begin{array}{ccc} I_n & 0 & 0 \\ A & B_p & B_w \\ \hline C_q & D_{qp} & D_{qw} \\ 0 & I_{n_p} & 0 \\ \hline C_z & D_{zp} & D_{zw} \\ 0 & 0 & I_{n_w} \end{array}\right]^\mathsf{T} \left[\begin{array}{cc|c|c} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ \hline 0 & 0 & \Pi & 0 \\ \hline 0 & 0 & 0 & \Pi_\gamma^{\mathrm{p}} \end{array}\right] \left[\begin{array}{ccc} I_n & 0 & 0 \\ A & B_p & B_w \\ \hline C_q & D_{qp} & D_{qw} \\ 0 & I_{n_p} & 0 \\ \hline C_z & D_{zp} & D_{zw} \\ 0 & 0 & I_{n_w} \end{array}\right] \prec 0$$

then the closed-loop system is robustly $\mathcal{L}_2$-stable with finite gain and $\sup_{\Delta \in \mathbf{\Delta}} \|\mathcal{F}(M,\Delta)\|_{\mathcal{L}_2 \to \mathcal{L}_2} < \gamma$.

Proof of Theorem 9.2

The upper two-by-two block of this LMI reads as

$$\begin{bmatrix} I_n & 0 \\ A & B_p \\ \hline C_q & D_{qp} \\ 0 & I_{n_p} \end{bmatrix}^{\mathsf{T}} \left[\begin{array}{cc|c} 0 & P & 0 \\ P & 0 & 0 \\ \hline 0 & 0 & \Pi \end{array}\right] \begin{bmatrix} I_n & 0 \\ A & B_p \\ \hline C_q & D_{qp} \\ 0 & I_{n_p} \end{bmatrix} + \underbrace{\begin{bmatrix} C_z & D_{zp} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} C_z & D_{zp} \end{bmatrix}}_{\succeq 0} \prec 0$$

from which we conclude that the first term is negative definite. This is precisely our robust stability LMI (23), so we conclude that the interconnection is robustly $\mathcal{L}_2$-stable with finite gain.

Now let $w \in \mathcal{L}_{2\mathrm{e}}[0, \infty)$ be the input, with corresponding unique trajectory trajectory $(x, p, q, z) \in \mathcal{L}_{2\mathrm{e}}[0, \infty)$. Left and right multiplying the LMI by (x, p, w) and we obtain

$$\begin{bmatrix} x \\ \dot{x} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} q \\ p \end{bmatrix}^{\mathsf{T}} \Pi \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} z \\ w \end{bmatrix} \Pi_\gamma^{\mathrm{p}} \begin{bmatrix} z \\ w \end{bmatrix} \leq -\varepsilon^2 \|w\|_2^2$$

for some small $\varepsilon > 0$.

Proof of Theorem 9.2

Defining $V(x) = x^\mathsf{T} P x$ the previous inequality states that

$$\dot{V}(x(t)) + \begin{bmatrix} q(t) \\ p(t) \end{bmatrix}^\mathsf{T} \Pi \begin{bmatrix} q(t) \\ p(t) \end{bmatrix} + \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \Pi_\gamma^\mathrm{p} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \leq -\varepsilon^2 \|w(t)\|_2^2$$

Integrating over $[0, T]$ and using $x(0) = 0$ we obtain

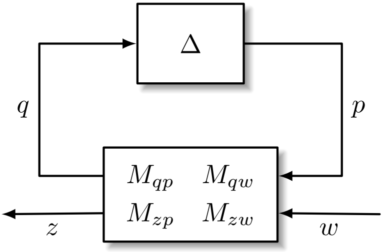
$$V(x(T)) + \underbrace{\int_0^T \begin{bmatrix} q(t) \\ p(t) \end{bmatrix}^\mathsf{T} \Pi \begin{bmatrix} q(t) \\ p(t) \end{bmatrix} \, \mathrm{d}t}_{\geq 0} + \int_0^T \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \Pi_\gamma^\mathrm{p} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} + \varepsilon^2 \|w(t)\|_2^2 \, \mathrm{d}t \leq 0$$

Since $V(x(T)) \geq 0$, we conclude that

$$\int_0^T \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \Pi_\gamma^\mathrm{p} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} \, \mathrm{d}t \leq -\int_0^T \varepsilon^2 \|w(t)\|_2^2 \, \mathrm{d}t$$

which completes the proof. •

Robust $\mathcal{H}_2$ performance



With $z = \mathcal{F}(M, \Delta)w$, how can we bound ***worst-case performance***

$$\sup_{\Delta \in \boldsymbol{\Delta}} \|\mathcal{F}(M, \Delta)\|_{\mathcal{H}_2}$$

- **Problem:** The $\mathcal{H}_2$ norm is defined for ***LTI*** systems; if Δ is nonlinear, the above analysis problem makes no sense!
- In fact, ***there is no unique generalization*** of the $\mathcal{H}_2$-norm to nonlinear systems. We will discuss *one* generalization based on the stochastic input interpretation.

Robust $\mathcal{H}_2$ performance

We consider the state-space realization for M:

$$M: \qquad \begin{bmatrix} \dot{x} \\ \hline q \\ z \end{bmatrix} = \left[\begin{array}{c|cc} A & B_p & B_w \\ \hline C_q & D_{qp} & \mathbf{0} \\ C_z & D_{zp} & \mathbf{0} \end{array}\right] \begin{bmatrix} x \\ p \\ w \end{bmatrix}$$

with $p = \Delta(q)$.

As we did before when studying the $\mathcal{H}_2$ norm, consider a white noise input w. We ***define*** the 2-norm of the mapping $\mathcal{F}(M, \Delta)$ to be the ***average asymptotic variance*** of the output z:

$$\|\mathcal{F}(M, \Delta)\|_2^2 \triangleq \limsup_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}\left\{ z(t)^{\mathsf{T}} z(t) \right\} \, \mathrm{d}t,$$

Robust $\mathcal{H}_2$ performance

Theorem 9.3 (Robust $\mathcal{H}_2$ Performance). Consider the previously described feedback interconnection. Assume that there exists a set of Hermitian matrices $\mathbf{\Pi} \subseteq \mathbb{H}^{n_q+n_p}$ such that

$$\int_0^T \begin{bmatrix} q(t) \\ \Delta(q)(t) \end{bmatrix}^\mathsf{T} \Pi \begin{bmatrix} q(t) \\ \Delta(q)(t) \end{bmatrix} \mathrm{d}t \geq 0 \qquad \forall q \in \mathcal{L}_{2\mathrm{e}}[0,\infty),\ T \geq 0,$$

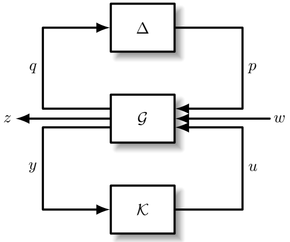
and all $\Pi \in \mathbf{\Pi}$ and all $\Delta \in \mathbf{\Delta}$. If there exists $Y \succ 0$ and $\Pi \in \mathbf{\Pi}$ such that

$$\left[\begin{array}{cc} I_n & 0 \\ A & B_p \\ \hline C_q & D_{qp} \\ 0 & I_{n_p} \\ \hline C_z & D_{zp} \end{array}\right]^\mathsf{T} \left[\begin{array}{cc|c|c} 0 & Y & 0 & 0 \\ Y & 0 & 0 & 0 \\ \hline 0 & 0 & \Pi & 0 \\ \hline 0 & 0 & 0 & I_{n_z} \end{array}\right] \left[\begin{array}{cc} I_n & 0 \\ A & B_p \\ \hline C_q & D_{qp} \\ 0 & I_{n_p} \\ \hline C_z & D_{zp} \end{array}\right] \prec 0$$

$$\mathrm{trace}(B_w^\mathsf{T} Y B_w) < \gamma^2$$

then the interconnection is robustly $\mathcal{L}_2$-stable with finite gain and $\sup_{\Delta \in \mathbf{\Delta}} \|\mathcal{F}(M,\Delta)\|_2 < \gamma$.

Robust synthesis of controllers



$$\mathcal{G}: \quad \begin{bmatrix} \dot{x} \\ \hline q \\ z \\ y \end{bmatrix} = \left[\begin{array}{c|ccc} A & B_p & B_w & B_u \\ \hline C_q & D_{qp} & D_{qw} & D_{qu} \\ C_z & D_{zp} & D_{zw} & D_{zu} \\ C_y & D_{yp} & D_{yw} & \mathbf{0} \end{array}\right] \begin{bmatrix} x \\ \hline p \\ w \\ u \end{bmatrix}$$

Problem: Design a (dynamic) feedback controller $\mathcal{K}$ such that the closed-loop system achieves ***robust performance*** on the channel $w \mapsto z$.

We combine our nominal synthesis and robust performance procedures:

1. close the loop with $\mathcal{K}$
2. write down the LMI for robust performance, and
3. change of variables to $v = (X, Y, \ldots)$

Robust synthesis of controllers

For example, for $\mathcal{L}_2$-performance: find v, $\Pi \in \mathbf{\Pi}$ such that $\boldsymbol{P}(v) \succ 0$ and

$$(\star)^{\mathsf{T}} \left[\begin{array}{cc|c|c} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & 0 & \Pi & 0 \\ \hline 0 & 0 & 0 & \Pi_\gamma^{\mathrm{p}} \end{array}\right] \left[\begin{array}{ccc} I & 0 & 0 \\ \boldsymbol{A}(v) & \boldsymbol{B_p}(v) & \boldsymbol{B_w}(v) \\ \hline \boldsymbol{C_q}(v) & \boldsymbol{D_{qp}}(v) & \boldsymbol{D_{qw}}(v) \\ 0 & I & 0 \\ \hline \boldsymbol{C_z}(v) & \boldsymbol{D_{zp}}(v) & \boldsymbol{D_{zw}}(v) \\ 0 & 0 & I \end{array}\right] \prec 0.$$

This problem is ***non-convex***, and no convexifying transformation has ever been found.

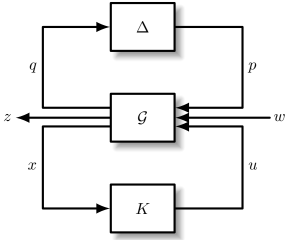
Observe however that if $\Pi_{11} \succeq 0$, then

1. For fixed $\Pi \in \mathbf{\Pi}$, the above is an LMI in v
2. For fixed v, the above is an LMI in $\Pi \in \mathbf{\Pi}$

This idea can be further developed into an iterative numerical method for solving robust synthesis problems; no guarantees, but often works well.

Robust state-feedback synthesis

While robust output feedback design is generally non-convex, the robust state feedback design problem can be convexified.



$$\mathcal{G}: \quad \begin{bmatrix} \dot{x} \\ \hline q \\ z \end{bmatrix} = \left[\begin{array}{c|ccc} A & B_p & B_w & B_u \\ \hline C_q & D_{qp} & D_{qw} & D_{qu} \\ C_z & D_{zp} & D_{zw} & D_{zu} \end{array}\right] \begin{bmatrix} x \\ \hline p \\ w \\ u \end{bmatrix}$$

$$u = Kx$$

$$p = \Delta(q)$$

$$M_{\text{cl}}: \quad \begin{aligned} \dot{x} &= (A + B_u K)x + B_p p + B_w w \\ q &= (C_q + D_{qu} K)x + D_{qp} p + D_{qw} w \\ z &= (C_z + D_{zu} K)x + D_{zp} p + D_{zw} w \end{aligned}$$

Robust state-feedback synthesis

The loop achieves robust performance on $w \mapsto z$ if $\exists P \succ 0$, $\Pi \in \mathbf{\Pi}$ s.t.

$$
(\star)^{\mathsf{T}} \left[\begin{array}{cc|c|c} 0 & P & 0 & 0 \\ P & 0 & 0 & 0 \\ \hline 0 & 0 & \Pi & 0 \\ \hline 0 & 0 & 0 & \Pi^{\mathrm{p}} \end{array}\right] \left[\begin{array}{ccc} I_n & 0 & 0 \\ A + B_u K & B_p & B_w \\ \hline C_q + D_{qu} K & D_{qp} & D_{qw} \\ 0 & I_{n_p} & 0 \\ \hline C_z + D_{zu} K & D_{zp} & D_{zw} \\ 0 & 0 & I_{n_w} \end{array}\right] \prec 0
$$

With $Y = P^{-1}$, $Z = KY$, congruence transformation $\mathrm{diag}(Y, I, I)$ yields

$$
(\star)^{\mathsf{T}} \left[\begin{array}{cc|c|c} 0 & I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ \hline 0 & 0 & \Pi & 0 \\ \hline 0 & 0 & 0 & \Pi^{\mathrm{p}} \end{array}\right] \left[\begin{array}{ccc} I_n & 0 & 0 \\ AY + B_u Z & B_p & B_w \\ \hline C_q Y + D_{qu} Z & D_{qp} & D_{qw} \\ 0 & I_{n_p} & 0 \\ \hline C_z Y + D_{zu} Z & D_{zp} & D_{zw} \\ 0 & 0 & I_{n_w} \end{array}\right] \prec 0
$$

Still non-convex! Products of Π, Π^{p} with Y, Z.

The "dualization" lemma

Lemma 9.1. Let $X \in \mathbb{S}^n$ be nonsingular. Then

$$\begin{bmatrix} 0 \\ I \end{bmatrix}^{\mathsf{T}} X \begin{bmatrix} 0 \\ I \end{bmatrix} \succeq 0 \quad \text{and} \quad \begin{bmatrix} I \\ W \end{bmatrix}^{\mathsf{T}} X \begin{bmatrix} I \\ W \end{bmatrix} \prec 0$$

if and only if

$$\begin{bmatrix} I \\ 0 \end{bmatrix}^{\mathsf{T}} X^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} \preceq 0 \quad \text{and} \quad \begin{bmatrix} W^{\mathsf{T}} \\ -I \end{bmatrix}^{\mathsf{T}} X^{-1} \begin{bmatrix} W^{\mathsf{T}} \\ -I \end{bmatrix} \succ 0.$$

- **First line**: X p.s.d. on $\mathcal{V} = \mathrm{Im}\left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$ and neg. def. on the complementary subspace $\mathcal{W} = \mathrm{Im}\left[\begin{smallmatrix} I \\ W \end{smallmatrix}\right]$
- **Second line:** X^{-1} n.s.d. on $\mathcal{V}^{\perp}$ and X^{-1} pos. def. on $\mathcal{W}^{\perp}$

Robust state-feedback synthesis

Shuffling the rows and columns, our inequality can be written as

$$(\star)^{\mathsf{T}} \underbrace{\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & I_n & 0 & 0 \\ 0 & \Pi_{22} & 0 & 0 & \Pi_{12} & 0 \\ 0 & 0 & \Pi_{22}^{\mathrm{p}} & 0 & 0 & \Pi_{12}^{\mathrm{p}} \\ \hline I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & \Pi_{12}^{\mathsf{T}} & 0 & 0 & \Pi_{11} & 0 \\ 0 & 0 & (\Pi_{12}^{\mathrm{p}})^{\mathsf{T}} & 0 & 0 & \Pi_{11}^{\mathrm{p}} \end{array}\right]}_{\triangleq X} \underbrace{\left[\begin{array}{ccc} I_n & 0 & 0 \\ 0 & I_{n_p} & 0 \\ 0 & 0 & I_{n_w} \\ \hline AY + B_u Z & B_p & B_w \\ C_q Y + D_{qu} Z & D_{qp} & D_{qw} \\ C_z Y + D_{zu} Z & D_{zp} & D_{zw} \end{array}\right]}_{\triangleq \left[\begin{smallmatrix} I \\ W \end{smallmatrix}\right]} \prec 0$$

We need to assume: Π is nonsingular and that

$$\Pi_{11} \succeq 0, \qquad \Pi^{-1} = \tilde{\Pi} = \begin{bmatrix} \tilde{\Pi}_{11} & \tilde{\Pi}_{12} \\ \tilde{\Pi}_{12}^{\mathsf{T}} & \tilde{\Pi}_{22} \end{bmatrix} \text{ satisfies } \tilde{\Pi}_{22} \preceq 0$$

We already have

$$\Pi^{\mathrm{p}} = \begin{bmatrix} \frac{1}{\gamma^2} I_{n_z} & 0 \\ 0 & -I_{n_w} \end{bmatrix} \text{ with } \Pi_{11}^{\mathrm{p}} \succeq 0, \quad (\Pi^{\mathrm{p}})^{-1} = \begin{bmatrix} \gamma^2 I_{n_z} & 0 \\ 0 & -I_{n_w} \end{bmatrix} \text{ with } (\Pi^{\mathrm{p}})_{22}^{-1} \preceq 0$$

Robust state-feedback synthesis

With

$$A(v) \triangleq AY + B_u Z, \qquad C_1(v) \triangleq C_q Y + D_{qu} Z, \qquad C_2(v) \triangleq C_z Y + D_{zu} Z$$

the dualization lemma yields the ***convex*** inequality

$$(\star)^{\mathsf{T}} \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & I_n & 0 & 0 \\ 0 & \tilde{\Pi}_{22} & 0 & 0 & \tilde{\Pi}_{12} & 0 \\ 0 & 0 & -I_{n_w} & 0 & 0 & 0 \\ \hline I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{\Pi}_{12}^{\mathsf{T}} & 0 & 0 & \tilde{\Pi}_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma^2 I_{n_z} \end{array}\right] \left[\begin{array}{ccc} A(v)^{\mathsf{T}} & C_1(v)^{\mathsf{T}} & C_2(v)^{\mathsf{T}} \\ B_p^{\mathsf{T}} & D_{qp}^{\mathsf{T}} & D_{zp}^{\mathsf{T}} \\ B_w^{\mathsf{T}} & D_{qw}^{\mathsf{T}} & D_{zw}^{\mathsf{T}} \\ \hline -I_n & 0 & 0 \\ 0 & -I_{n_q} & 0 \\ 0 & 0 & -I_{n_z} \end{array}\right] \succ 0$$

or (again, reshuffling rows and columns)

$$(\star)^{\mathsf{T}} \left[\begin{array}{cc|cc|cc} 0 & I_n & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{\Pi}_{11} & \tilde{\Pi}_{12} & 0 & 0 \\ 0 & 0 & \tilde{\Pi}_{12}^{\mathsf{T}} & \tilde{\Pi}_{22} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -I_{n_w} & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma^2 I_{n_z} \end{array}\right] \left[\begin{array}{ccc} A(v)^{\mathsf{T}} & C_1(v)^{\mathsf{T}} & C_2(v)^{\mathsf{T}} \\ -I_n & 0 & 0 \\ \hline B_p^{\mathsf{T}} & D_{qp}^{\mathsf{T}} & D_{zp}^{\mathsf{T}} \\ 0 & -I_{n_q} & 0 \\ \hline B_w^{\mathsf{T}} & D_{qw}^{\mathsf{T}} & D_{zw}^{\mathsf{T}} \\ 0 & 0 & -I_{n_z} \end{array}\right] \succ 0$$

10. Introduction to Integral Quadratic Constraints

Introduction to integral quadratic constraints

Recall: Scalar parametric uncertainty

$$\boldsymbol{\Delta}_{\mathrm{par}} = \{\Delta \mid \Delta \text{ memoryless, scalar, LTI, and } \|\Delta\|_{\mathcal{L}_2 \to \mathcal{L}_2} \leq \gamma\}.$$

or more simply $p(t) = \delta q(t)$ with $\delta \in [-\gamma, \gamma]$. We know this satisfies the point-wise quadratic constraint

$$\begin{bmatrix} q(t) \\ p(t) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \sigma\gamma^2 & 0 \\ 0 & -\sigma \end{bmatrix} \begin{bmatrix} q(t) \\ p(t) \end{bmatrix} \geq 0, \qquad \forall\, \sigma > 0,\ t \geq 0.$$

We also know that ***if*** $q \in \mathcal{L}_2[0, \infty)$, then $p = \delta q \in \mathcal{L}_2[0, \infty)$. In this case, we could take Fourier transforms and write $\hat{p}(\mathbf{j}\omega) = \delta \hat{q}(\mathbf{j}\omega)$, leading to

$$\begin{bmatrix} \hat{q}(\mathbf{j}\omega) \\ \hat{p}(\mathbf{j}\omega) \end{bmatrix}^{*} \begin{bmatrix} \sigma\gamma^2 & 0 \\ 0 & -\sigma \end{bmatrix} \begin{bmatrix} \hat{q}(\mathbf{j}\omega) \\ \hat{p}(\mathbf{j}\omega) \end{bmatrix} \geq 0, \qquad \sigma > 0,\, \omega \in \mathbb{R}.$$

Introduction to integral quadratic constraints

Idea: If we are allowed to work in the frequency domain, can we add ***even more*** flexibility by making σ frequency dependent? Yes!

Roughly, we could replace σ by $\sigma(\mathbf{j}\omega)$, as long as $\sigma(\mathbf{j}\omega) > 0$, yielding

$$\begin{bmatrix} \hat{q}(\mathbf{j}\omega) \\ \hat{p}(\mathbf{j}\omega) \end{bmatrix}^* \begin{bmatrix} \sigma(\mathbf{j}\omega)\gamma^2 & 0 \\ 0 & -\sigma(\mathbf{j}\omega) \end{bmatrix} \begin{bmatrix} \hat{q}(\mathbf{j}\omega) \\ \hat{p}(\mathbf{j}\omega) \end{bmatrix} \geq 0.$$

- Instead of just a scalar $\sigma > 0$, we can now search over a whole set of of ***transfer functions*** $\sigma(\mathbf{j}\omega)$ when we want to satisfy the stability conditions derived earlier!

As some helpful frequency-domain notation, we let

$$\mathcal{R}\hat{\mathcal{L}}_\infty \triangleq \{\hat{\Pi} : \mathbb{C} \to \mathbb{C} \mid \hat{\Pi}(s) \text{ is rational, proper, and has no poles on } \mathbf{j}\mathbb{R}\}$$

$$\mathcal{R}\mathcal{H}_\infty \triangleq \{\hat{\Pi} : \mathbb{C} \to \mathbb{C} \mid \hat{\Pi}(s) \text{ is rational, proper, and has no poles in } \mathbb{C}_{\geq 0}\}$$

Definition of an IQC

Definition 10.1 (Frequency-Domain IQC). Let $\hat{\Pi} \in \mathcal{R}\hat{\mathcal{L}}_{\infty}^{(q+p)\times(q+p)}$ be a Hermitian ***IQC multiplier***, and let $\Delta : \mathcal{L}_{2e}^{q}[0,\infty) \to \mathcal{L}_{2e}^{p}[0,\infty)$ be a causal operator with finite $\mathcal{L}_2$-gain. We say Δ satisfies the ***integral quadratic constraint (IQC)*** defined by $\hat{\Pi}$ if

$$\left\langle \begin{bmatrix}\hat{q}\\ \hat{p}\end{bmatrix}, \hat{\Pi}\begin{bmatrix}\hat{q}\\ \hat{p}\end{bmatrix}\right\rangle_{\hat{\mathcal{L}}_2} \geq 0 \quad \Longleftrightarrow \quad \int_{-\infty}^{\infty} \begin{bmatrix}\hat{q}(\mathbf{j}\omega)\\ \hat{p}(\mathbf{j}\omega)\end{bmatrix}^* \hat{\Pi}(\mathbf{j}\omega) \begin{bmatrix}\hat{q}(\mathbf{j}\omega)\\ \hat{p}(\mathbf{j}\omega)\end{bmatrix} \mathrm{d}\omega \geq 0,$$

for all $q \in \mathcal{L}_2^{q}[0,\infty)$ with corresponding outputs $p = \Delta(q) \in \mathcal{L}_2^{p}[0,\infty)$. Notationally, we write that $\Delta \in \mathsf{IQC}(\hat{\Pi})$.

- A quadratic relationship between all possible I/O pairs
- **Note**: the restriction that $q, p \in \mathcal{L}_2[0,\infty)$ is ***crucial***. If $q \in \mathcal{L}_{2e}[0,\infty)$, the above generally makes no sense, because the Fourier transform may not be defined.

Example: IQC for parametric uncertainty

Scalar parametric uncertainty is defined by the class of operators

$$\boldsymbol{\Delta}_{\mathrm{par}} = \{\Delta \mid \Delta \text{ memoryless, scalar, LTI, and } \|\Delta\|_{\mathcal{L}_2 \to \mathcal{L}_2} \leq \gamma\}.$$

Then $\Delta \in \mathsf{IQC}(\hat{\Pi}_{\mathrm{par}})$ where

$$\hat{\boldsymbol{\Pi}}_{\mathrm{par}} \triangleq \left\{ \hat{\Pi} \;\middle|\; \hat{\Pi}(\mathbf{j}\omega) = \hat{\sigma}(\mathbf{j}\omega) \begin{bmatrix} \gamma^2 & \mathbb{0} \\ \mathbb{0} & -1 \end{bmatrix}, \ \hat{\sigma} = \hat{\sigma}^* \in \mathcal{R}\hat{\mathcal{L}}_\infty, \ \hat{\sigma}(\mathbf{j}\omega) > 0 \text{ for all } \omega \in \mathbb{R} \cup \{\infty\} \right\}.$$

- In practice, one just looks at a finite-dimensional subspace of $\mathcal{R}\hat{\mathcal{L}}_\infty$, expands $\hat{\sigma}$ in a basis for that subspace, and then you just have a set of scalar coefficients which describe $\hat{\sigma}$.

Example: monotone and slope-Restricted nonlinearity

- Memoryless nonlinear functions such as saturation, deadband, etc. are not just sector-bounded, but have bounded slopes.
- A function $\Phi : \mathbb{R} \to \mathbb{R}$ satisfying $\Phi(0) = 0$ is ***slope-restricted*** if

$$\begin{bmatrix} q - q' \\ \Phi(q) - \Phi(q') \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} -2\alpha\beta & (\alpha+\beta) \\ (\alpha+\beta) & -2 \end{bmatrix} \begin{bmatrix} q - q' \\ \Phi(q) - \Phi(q') \end{bmatrix} \geq 0, \quad \forall\, q, q' \in \mathbb{R}$$

  where $\beta \geq \alpha$.
- Φ is slope-restricted $\implies$ Φ is sector bounded.
- If $\beta = +\infty$ and $\alpha = 0$, then Φ is ***monotone*** and we can divide through by β to obtain

$$\begin{bmatrix} q - q' \\ \Phi(q) - \Phi(q') \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q - q' \\ \Phi(q) - \Phi(q') \end{bmatrix} \geq 0, \quad \forall\, q, q' \in \mathbb{R}$$

Example: the Zames-Falb IQC

- A ***huge*** class of IQCs for slope-restricted and monotone nonlinearities
- Any slope-restricted Φ satisfies the IQC defined by

$$\hat{\Pi}_{\mathrm{ZF}}(\mathbf{j}\omega) = \begin{bmatrix} -\alpha & 1 \\ \beta & -1 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & \hat{m}(\mathbf{j}\omega) \\ \hat{m}^*(\mathbf{j}\omega) & 0 \end{bmatrix} \begin{bmatrix} -\alpha & 1 \\ \beta & -1 \end{bmatrix}$$

  where m is an impulse response constructed as

$$m(t) = h_0\delta(t) - h(t), \quad h_0 > 0,\ h(t) \geq 0,\ \|h\|_{\mathcal{L}_1} < h_0.$$

- This is not obvious! Example: $\hat{m}(s) = \frac{s+2}{s+1}$ is in this class.
- In the MIMO case, can be used for describing ***gradients of convex functions***; lots of interesting research going on in this area.

The philosophy of IQC analysis

- An IQC tells you that the possible input-output pairs of a given operator are constrained; it provides a (possibly, very coarse) description of the operator
- If your operator satisfies ***two*** IQCs $\hat{\Pi}_1, \hat{\Pi}_2$, they probably each tell you something useful about the operator, and you can combine them as

$$\hat{\Pi}(\mathbf{j}\omega) = \sigma_1 \hat{\Pi}_1(\mathbf{j}\omega) + \sigma_2 \hat{\Pi}_2(\mathbf{j}\omega), \quad \sigma_1, \sigma_2 \geq 0$$

  and the operator will satisfy the IQC defined by $\hat{\Pi}$. You can then optimize over the combination. This idea even extends to infinite combinations ...
- The more IQCs you can find, the better! Just add them up. We will go over some basic ones soon ...

IQCs in the time-domain

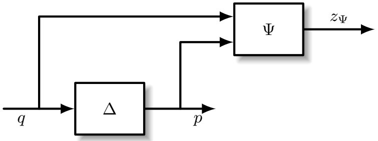
We can use Plancherel's Theorem to translate our definition back to the time-domain; we need the following simple result first.

Lemma 10.1. Every Hermitian $\hat{\Pi} \in \mathcal{R}\hat{\mathcal{L}}_\infty^{(q+p)\times(q+p)}$ can be factored as

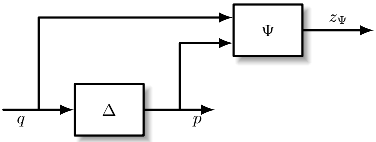
$$\hat{\Pi}(\mathbf{j}\omega) = \hat{\Psi}^*(\mathbf{j}\omega) X \hat{\Psi}(\mathbf{j}\omega)$$

where $X = X^\mathsf{T} \in \mathbb{R}^{\bullet\times\bullet}$ is a symmetric matrix and $\hat{\Psi} \in \mathcal{RH}_\infty^{\bullet\times(q+p)}$

- $\hat{\Psi}$ is typically a "tall" transfer matrix; we think of Ψ as filter the input/output pairs (q,p) of Δ to produce a new signal z_Ψ



IQCs in the time-domain



$$\hat{\Pi}(\mathbf{j}\omega) = \hat{\Psi}^*(\mathbf{j}\omega) X \hat{\Psi}(\mathbf{j}\omega)$$

We can now express the IQC in the time-domain as

$$\begin{aligned}
\int_{-\infty}^{\infty} \begin{bmatrix} \hat{q}(\mathbf{j}\omega) \\ \hat{p}(\mathbf{j}\omega) \end{bmatrix}^* \hat{\Pi}(\mathbf{j}\omega) \begin{bmatrix} \hat{q}(\mathbf{j}\omega) \\ \hat{p}(\mathbf{j}\omega) \end{bmatrix} \mathrm{d}\omega &= \int_{-\infty}^{\infty} \begin{bmatrix} \hat{q}(\mathbf{j}\omega) \\ \hat{p}(\mathbf{j}\omega) \end{bmatrix}^* \hat{\Psi}(\mathbf{j}\omega)^* X \hat{\Psi}(\mathbf{j}\omega) \begin{bmatrix} \hat{q}(\mathbf{j}\omega) \\ \hat{p}(\mathbf{j}\omega) \end{bmatrix} \mathrm{d}\omega \\
&= \int_{-\infty}^{\infty} \hat{z}_\psi(\mathbf{j}\omega)^* X \hat{z}_\Psi(\mathbf{j}\omega)\, \mathrm{d}\omega \\
&= \int_{0}^{\infty} z_\Psi(t)^\mathsf{T} X z_\Psi(t)\, \mathrm{d}t \\
&= \langle z_\Psi, X z_\Psi \rangle_{\mathcal{L}_2}
\end{aligned}$$

The soft IQC theorem: a graph separation result

Theorem 10.1 (Soft IQC Theorem). Consider the previously discussed feedback interconnection, and assume additionally that

(i) the interconnection of M and $\tau\boldsymbol{\Delta}$ is well-posed for all $\tau \in [0,1]$;

(ii) $\tau\Delta \in \boldsymbol{\Delta}$ for all $\tau \in [0,1]$;

(iii) there exists a set of multipliers $\hat{\boldsymbol{\Pi}} \subset \mathcal{R}\hat{\mathcal{L}}_\infty^{(q+p)\times(q+p)}$ such that $\tau\Delta \in \mathsf{IQC}(\hat{\Pi})$ for all $\hat{\Pi} \in \hat{\boldsymbol{\Pi}}$ and all $\tau \in [0,1]$.

Under these conditions, if there exists a $\hat{\Pi} \in \hat{\boldsymbol{\Pi}}$ such that

$$\begin{bmatrix} \hat{M}(\mathbf{j}\omega) \\ I_m \end{bmatrix}^* \hat{\Pi}(\mathbf{j}\omega) \begin{bmatrix} \hat{M}(\mathbf{j}\omega) \\ I_m \end{bmatrix} \prec \mathbb{0}, \qquad \forall\, \omega \in \mathbb{R} \cup \{\infty\},$$

then the interconnection is robustly $\mathcal{L}_2$-stable with finite gain.

Using the soft IQC theorem

To check the main FDI condition, one factorizes the multiplier set

$$\hat{\mathbf{\Pi}} = \left\{ \hat{\Psi}^* X \hat{\Psi} \mid X \in \boldsymbol{X} \right\},$$

where $\hat{\Psi} \in \mathcal{RH}_\infty^{\bullet \times (q+p)}$ and the set $\boldsymbol{X}$ can be represented as the feasible set of an LMI. The FDI condition becomes

$$\begin{bmatrix} \hat{M}(\mathbf{j}\omega) \\ I_m \end{bmatrix}^* \hat{\Psi}(\mathbf{j}\omega)^* X \hat{\Psi}(\mathbf{j}\omega) \begin{bmatrix} \hat{M}(\mathbf{j}\omega) \\ I_m \end{bmatrix} \prec \mathbb{0}, \qquad \forall\, \omega \in \mathbb{R} \cup \{\infty\}.$$

Let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ be a realization for the system $\hat{\Psi} \left[\begin{smallmatrix} \hat{M} \\ I_m \end{smallmatrix} \right]$. Applying the KYP Lemma, an equivalent LMI test: find $P \in \mathbb{S}^\bullet$ and $X \in \boldsymbol{X}$ such that

$$\left[\begin{array}{cc} I_n & \mathbb{0} \\ \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right]^\mathsf{T} \left[\begin{array}{cc|c} \mathbb{0} & P & \mathbb{0} \\ P & \mathbb{0} & \mathbb{0} \\ \hline \mathbb{0} & \mathbb{0} & X \end{array}\right] \left[\begin{array}{cc} I_n & \mathbb{0} \\ \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] \prec \mathbb{0}.$$

Comments on soft IQC theorem

Similarities with Theorem 9.1:

- We assume the loop is well-posed
- We assume Δ satisfies some quadratic constraints
- The main condition on M is again an FDI

Differences with Theorem 9.1:

- The quadratic constraint matrix Π is now ***frequency-dependent***
- In the time-domain, the quadratic constraints *need only hold* for $T \to \infty$ ("soft") as opposed to for all $T \geq 0$ ("hard")
- Some minor but important technical changes involving parameter τ; these are what allows the relaxation from "hard" to "soft" constraints.

Proof of Theorem 10.1

For notational simplicity, we let π denote the quadratic form defined by the IQC, and we write the conditions on Δ and M as

$$\pi(q, \tau\Delta(q)) \geq 0 \tag{24a}$$

$$\pi(Mp, p) \leq -\epsilon\|p\|^2_{\mathcal{L}_2} \tag{24b}$$

for all $p, q \in \mathcal{L}_2[0, \infty)$ and all $\tau \in [0, 1]$. The proof proceeds by induction. Fix $\tau \in [0, 1]$ and assume that $(I - \tau M\Delta)^{-1}$ is is $\mathcal{L}_2$-stable, i.e., maps $\mathcal{L}_2^q[0, \infty)$ to $\mathcal{L}_2^q[0, \infty)$. In other words, for any $w \in \mathcal{L}_2^q[0, \infty)$ there exists a unique solution $q \in \mathcal{L}_2^q[0, \infty)$ to the equation

$$q - \tau M\Delta(q) = w. \tag{25}$$

Let $q \in \mathcal{L}_2^q[0, \infty)$ be arbitrary and set $p = \tau\Delta(q)$; note that $p \in \mathcal{L}_2^p[0, \infty)$ since Δ is bounded. From (24a) we have

$$\begin{aligned} 0 \leq \pi(q, p) &= \pi(Mp, p) + \pi(q, p) - \pi(Mp, p) \\ &= \pi(Mp, p) + [\pi(Mp + q - Mp, p) - \pi(Mp, p)] \\ &\leq -\epsilon\|p\|^2_{\mathcal{L}_2} + [\pi(Mp + q - Mp, p) - \pi(Mp, p)], \end{aligned}$$

where we have used (24b).

Proof of Theorem 10.1

In the above inequality, note that since p and q are in $\mathcal{L}_2[0,\infty)$ and M is $\mathcal{L}_2$-stable with finite gain, all signals belong to $\mathcal{L}_2[0,\infty)$. Since π is defined by $\hat{\Pi}$ which is bounded, there exist constants $c_1, c_2 \geq 0$ depending only on Π such that

$$\epsilon\|p\|_{\mathcal{L}_2}^2 \leq c_1 \left\| \begin{pmatrix} Mp \\ p \end{pmatrix} \right\|_{\mathcal{L}_2} \cdot \left\| \begin{pmatrix} q - Mp \\ 0 \end{pmatrix} \right\|_{\mathcal{L}_2} + c_2 \left\| \begin{pmatrix} q - Mp \\ 0 \end{pmatrix} \right\|_{\mathcal{L}_2}^2 . \qquad (26)$$

Using the Peter–Paul inequality, we can further bound the cross term as

$$\begin{aligned} c_1 \left\| \begin{pmatrix} Mp \\ p \end{pmatrix} \right\|_{\mathcal{L}_2} \cdot \left\| \begin{pmatrix} q - Mp \\ 0 \end{pmatrix} \right\|_{\mathcal{L}_2} &\leq \frac{c_1\delta}{2} \left\| \begin{pmatrix} Mp \\ p \end{pmatrix} \right\|_{\mathcal{L}_2}^2 + \frac{c_1}{2\delta} \left\| \begin{pmatrix} q - Mp \\ 0 \end{pmatrix} \right\|_{\mathcal{L}_2}^2 \\ &= \frac{c_1\delta}{2} \left[\|Mp\|_{\mathcal{L}_2}^2 + \|p\|_{\mathcal{L}_2}^2 \right] + \frac{c_1}{2\delta} \|q - Mp\|_{\mathcal{L}_2}^2 \\ &\leq \frac{c_1\delta(1 + \|M\|^2)}{2} \|p\|_{\mathcal{L}_2}^2 + \frac{c_1}{2\delta} \|q - Mp\|_{\mathcal{L}_2}^2 . \end{aligned}$$

for any $\delta > 0$. Inserting this into our inequality (26), we find that

$$\left(\epsilon - \frac{c_1\delta(1 + \|M\|^2)}{2} \right) \|p\|_{\mathcal{L}_2}^2 \leq \left(\frac{c_1}{2\delta} + c_2 \right) \|q - Mp\|_{\mathcal{L}_2}^2 .$$

Proof of Theorem 10.1

Selecting $\delta < \frac{2\epsilon}{c_1(1+\|M\|^2)}$, the term in brackets is strictly positive and we find that

$$\alpha\|p\|_{\mathcal{L}_2} \leq \|q - Mp\|_{\mathcal{L}_2} \tag{27}$$

where $\alpha^2 \triangleq \left(\frac{c_1}{2\delta} + c_2\right)^{-1}\left(\epsilon - \frac{c_1\delta(1+\|M\|^2)}{2}\right) > 0$ does not depend on Δ or on τ.
Using (27), linearity of M, and boundedness of M, we may compute that

$$\begin{aligned}
\|q\|_{\mathcal{L}_2} = \|q - Mp + Mp\|_{\mathcal{L}_2} &\leq \|q - Mp\|_{\mathcal{L}_2} + \|Mp\|_{\mathcal{L}_2}\\
&\leq \|q - Mp\|_{\mathcal{L}_2} + \|M\|\,\|p\|_{\mathcal{L}_2}\\
&= (1 + \|M\|\alpha^{-1})\|q - Mp\|_{\mathcal{L}_2}\\
&= \underbrace{(1 + \|M\|\alpha^{-1})}_{\triangleq\gamma^{-1}}\|q - \tau M\Delta(q))\|_{\mathcal{L}_2}
\end{aligned}$$

and we therefore conclude that

$$\|q - \tau M\Delta(q)\|_{\mathcal{L}_2} \geq \gamma\|q\|_{\mathcal{L}_2}, \qquad q \in \mathcal{L}_2[0,\infty).$$

In words, the operator $I - \tau M\Delta : \mathcal{L}_2^q[0,\infty) \to \mathcal{L}_2^q[0,\infty)$ is ***bounded below*** on $\mathcal{L}_2$.

Proof of Theorem 10.1

Since (by assumption at this point) $(I - \tau M\Delta)^{-1}$ is $\mathcal{L}_2$-stable, we have for any, we have for any $w \in \mathcal{L}_2[0,\infty)$ that

$$\begin{aligned} w &= (I - \tau M\Delta)((I - \tau M\Delta)^{-1}(w)) \\ &\geq \gamma (I - \tau M\Delta)^{-1}(w) \end{aligned}$$

which shows that $(I - \tau M\Delta)^{-1}$ is robustly $\mathcal{L}_2$-stable with finite gain less than or equal to γ. To summarize, we have established that if $\tau \in [0,1]$ is such that $(I - \tau M\Delta)^{-1}$ maps $\mathcal{L}_2^q[0,\infty)$ into $\mathcal{L}_2^q[0,\infty)$, then the assumptions guarantee the finite-gain bound

$$\|(I - \tau M\Delta)^{-1}\|_{\mathcal{L}_2 \to \mathcal{L}_2} \leq \gamma. \tag{28}$$

Now observe that since M and Δ are bounded, so is $M\Delta$, and we may define

$$\rho_{\text{crit}} \triangleq \frac{1}{\|M\Delta\|\gamma}$$

and choose $\rho \in (0, \rho_{\text{crit}})$. Proceeding inductively, we now claim that if $\tau \in [0,1]$ is such that $(I - \tau M\Delta)^{-1}$ maps $\mathcal{L}_2^q[0,\infty)$ into $\mathcal{L}_2^q[0,\infty)$, and $\tau + \rho \in [0,1]$, then $(I - (\tau+\rho)M\Delta)^{-1}$ also maps $\mathcal{L}_2[0,\infty)$ into $\mathcal{L}_2[0,\infty)$, and hence by the above argument, we have that $\|(I - (\tau+\rho)M\Delta)^{-1}\| \leq \gamma$.

Proof of Theorem 10.1

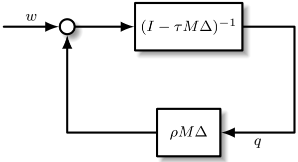
To show this, we wish to establish that for any $w \in \mathcal{L}_2^q[0,\infty)$, the equation

$$q - (\tau + \rho)M\Delta(q) = w \qquad \Longleftrightarrow \qquad q - \tau M\Delta(q) = \rho M\Delta(q) + w$$

is uniquely solvable for a finite-energy solution $q \in \mathcal{L}_2^q[0,\infty)$. This equation is in turn equivalent to

$$q = (I - \tau M\Delta)^{-1}(\rho M\Delta(q) + w). \tag{29}$$

We interpret equation (29) in terms of the block diagram below, where both blocks define bounded operators on $\mathcal{L}_2[0,\infty)$.



Proof of Theorem 10.1

Letting $z \in \mathcal{L}_2^q[0,\infty)$, we can bound the composition of these two operators as

$$\begin{aligned} \|(I-\tau M\Delta)^{-1}(\rho M\Delta(z))\| &\leq \|(I-\tau M\Delta)^{-1}\| \cdot \|\rho M\Delta(z)\| \\ &\leq \gamma\rho\|M\Delta\|\,\|z\| \\ &= \underbrace{(\rho/\rho_{\text{crit}})}_{<1}\|z\| \end{aligned}$$

and therefore the composition has induced norm strictly less than one. It follows from the small-gain theorem then and from the above equivalences that the operator $(I-(\tau+\rho)M\Delta)^{-1}$ is bounded, and hence maps $\mathcal{L}_2[0,\infty)$ into $\mathcal{L}_2[0,\infty)$, so we conclude that

$$\|(I-(\tau+\rho)M\Delta)^{-1}\| \leq \gamma.$$

Proof of Theorem 10.1

To complete the proof, note that with $\tau = 0$, $(I - \tau M\Delta)^{-1} = I$ obviously maps $\mathcal{L}_2[0, \infty)$ into $\mathcal{L}_2[0, \infty)$. We can apply the previous argument to conclude then that $(I - \rho M\Delta)^{-1}$ is bounded (uniformly in Δ) for any $\rho \in (0, \rho_{\text{crit}})$. Repeating the process from our new starting point at $\tau = \rho$, we can conclude that $(I - 2\rho M\Delta)^{-1}$ is bounded (uniformly in Δ) for any $\rho \in (0, \rho_{\text{crit}})$. Since ρ_{crit} is independent of τ, we repeat this process until we have covered the interval $[0, 1]$, and thereby conclude that $(I - M\Delta)^{-1}$ is bounded uniformly in Δ. •