ECE 1659H Assignment 1 Solutions

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Note: You are always free to use `MATLAB` for any calculation, as long as you understand what `MATLAB` is doing and provide sufficient commentary.

Problem 1 (Gain and Phase Margins in a SISO Loop)

Consider the standard feedback arrangement shown in Figure 1, where $L(s)$ is a SISO transfer function.

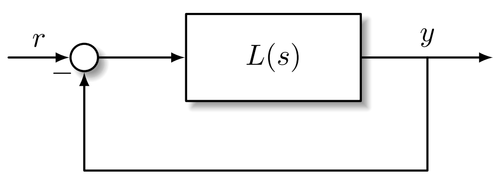

Figure 1: Feedback interconnection for Problem 5.

Gain and phase margins are the classical tools used to assess robustness of this feedback system; refresh yourself on these concepts if necessary, including their interpretation in terms of the Nyquist plot of $L(s)$. While sufficiently large gain and phase margins are *necessary* for a robust SISO design, they are in fact not sufficient. Consider the following loop gain given by Seiler, Packard, and Gahinet:

$$L(s) = \frac{-47.252s^7 - 20.234s^6 - 135.4086s^5 + 61.6166s^4 + 804.6454s^3 + 600.0611s^2 + 59.1451s + 1.888}{99.8696s^7 + 175.5045s^6 + 673.7378s^5 + 890.5109s^4 + 553.1742s^3 - 49.2268s^2 + 12.1448s + 1}$$

Using `MATLAB`, compute the gain and phase margins for this loop. Then, plot the Nyquist curve. Are the gain and phase margins representative of the actual ability of the system to tolerate perturbations in L? Explain.

Solution: We can begin by plotting the Nyquist plots, with a few different zooms so that we make sure we understand things.

In Figure 2 we get the overall picture. There are two counter-clockwise encirclements of $s = -1$, and you can check that $L(s)$ has two poles in the RHP, so we conclude via Nyquist's criteria that the closed-loop system is stable.

Importantly, note from Figure 3 that the curves are not touching around -0.75. From Figure 4, we can eye-ball the gain margins to be about $1/0.45 \approx 2$ and $45°$. As you know, the phase margin just considers adding phase uncertainty to $L(s)$, which means rotating the Nyquist plot, and the gain margin just considers adding gain uncertainty to $L(s)$, which means scaling the Nyquist plot. Note though that as we scale the Nyquist plot with a multiple from 1 (the nominal system) to 2 (the upper gain margin), there is an intermediate range where the curves in Figure 3 pass *very* close

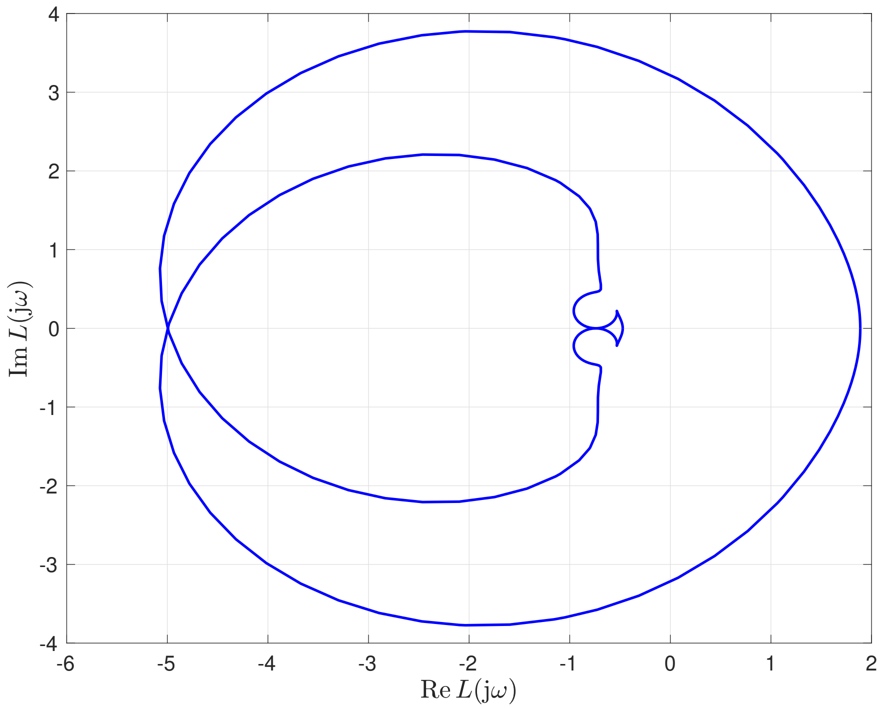

Figure 2

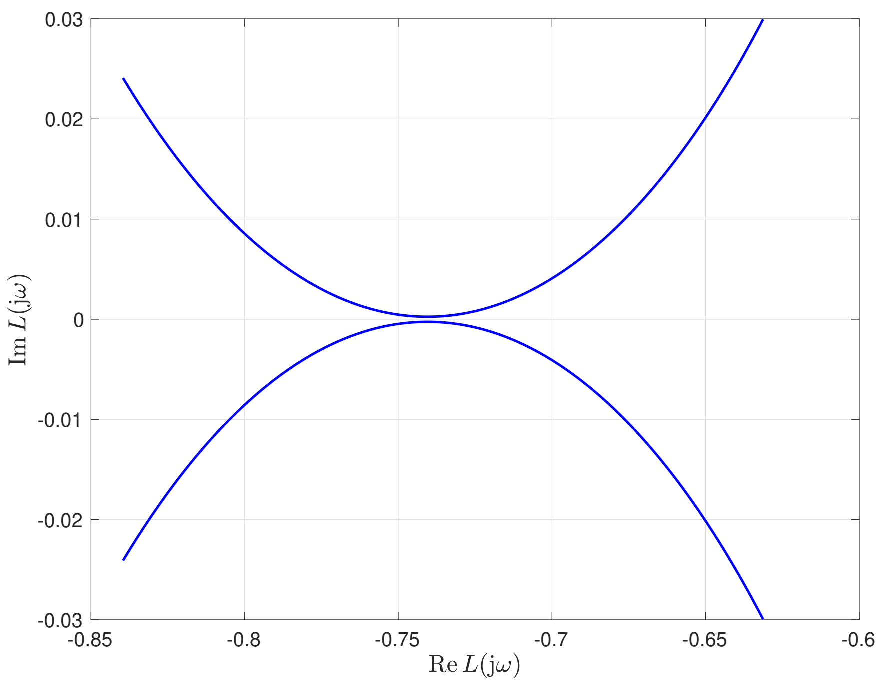

Figure 3

to the critical point. If we scaled the gain to say, 1.5, and then added some very small amount of phase uncertainty, we would change the number of encirclements and the closed-loop system would become unstable. We conclude that the system is substantially *less* robust than the gain and phase

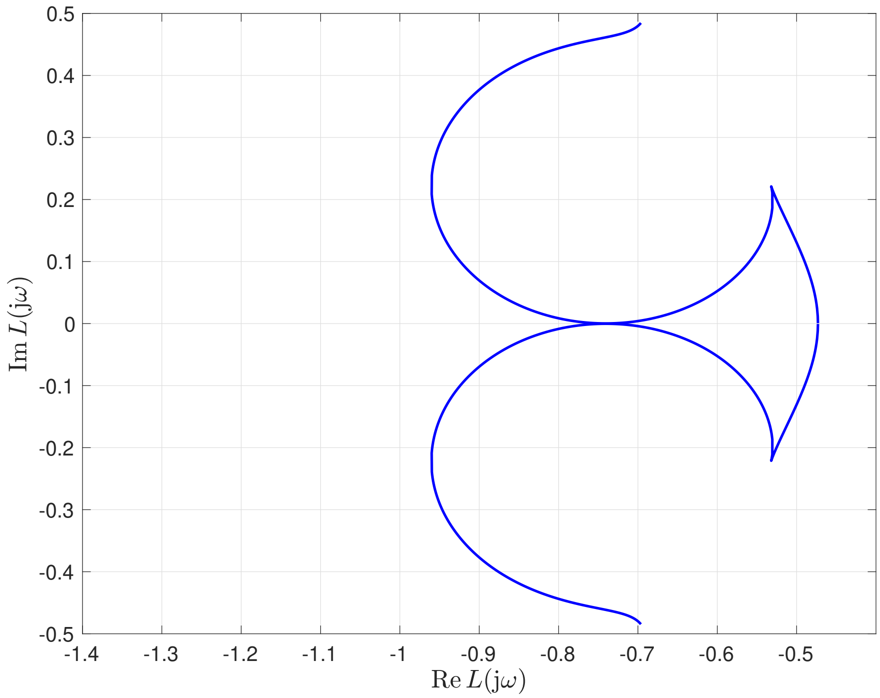

Figure 4

margins alone suggest, because these margins consider only independent one-at-a-time variations in gain and phase, and not joint variations.

Problem 2 (Robustness of MIMO Spinning Satellite)

This problem tugs at a similar thread to the first problem, but in a MIMO state-space model. A satellite spinning around one of its principal axes can be described after linearization by the minimal LTI state-space model with parameters

$$A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where $a = 10$.

(i) Verify that with the output feedback controller $u = -Ky$ where $K = I_2$, the closed-loop system is asymptotically stable.

(ii) Suppose now that there is uncertainty in how strongly the inputs enter the system, modeled as parametric uncertainty in the B matrix which now becomes

$$B' = \begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix}.$$

where $(\epsilon_1, \epsilon_2) \in \mathbb{R}^2$. Study the stability of the uncertain system (A, B', C, D) with the same controller as in part (i), for different values of (ϵ_1, ϵ_2) between -0.5 and 0.5. You could do this analytically, or you could make a numerical plot by colouring in regions of interest in the (ϵ_1, ϵ_2) plane. Describe what you find, and explain the implications of your results.

Solution: (i): With this controller the closed-loop A matrix is given by

$$A_{\mathrm{cl}} = A - BKC = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} - \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

The system has two eigenvalues at -1 and is therefore stable.

(ii): With this new B matrix the closed-loop system matrix becomes

$$A_{\mathrm{cl}} = A - B'KC = \begin{pmatrix} -\epsilon_1 - 1 & a - a\,(\epsilon_1 + 1) \\ a\,(\epsilon_2 + 1) - a & -\epsilon_2 - 1 \end{pmatrix}$$

To check stability of this system, we compute that

$$\det(sI_n - A_{\mathrm{cl}}) = s^2 + (2 + \epsilon_1 + \epsilon_2)s + (1 + \epsilon_1 + \epsilon_2 + (a^2+1)\epsilon_1\epsilon_2).$$

By the Routh-Hurwitz criteria for a second-order system, the system is stable if the coefficients of all powers of s are positive.

Let's look at some cases. If $\epsilon_1 = 0$ and ϵ_2 is non-zero, then we obtain the polynomial

$$\pi_1(s) = s^2 + (2 + \epsilon_2)s + (1 + \epsilon_2)$$

so the system is stable for $\epsilon_2 > -1$. Similarly, if $\epsilon_2 = 0$ and ϵ_1 is non-zero, we obtain the polynomial

$$\pi_2(s) = s^2 + (2 + \epsilon_1)s + (1 + \epsilon_1)$$

so we conclude that the system is stable for $\epsilon_1 > -1$. We conclude that if we change ϵ_1 and ϵ_2 individually, then the system is quite robust to changes in these variables. What happens if we change them *simultaneously*? As a particular case, consider $\epsilon_2 = -\epsilon_1 = \epsilon$, in which case we obtain the polynomial

$$\pi_3(s) = s^2 + 2s + (1 - (1 + a^2)\epsilon^2).$$

We conclude now that the system is stable only when

$$\epsilon < \frac{1}{\sqrt{1 + a^2}} \approx 0.1.$$

The system is therefore actually *very sensitive* to *simultaneous variation* in the uncertain parameters. This shows that multivariable stability robustness can be subtle; you can be robust when you vary each parameter individually, but this just corresponds to moving along the cardinal axes in parameter space. If you explore other directions in the parameter space, you may encounter instability very quickly, as we do here. In this example, we would go back and conclude that our original controller design $u = -y$ was not a very robust design.

Problem 3 (Vector concepts on the space $\mathbb{R}^{m\times n}$)

In this problem we let $(A)_{ij}$ denote the ijth element of a matrix A. Consider the vector space $\mathbb{R}^{m\times n}$ over the field $\mathbb{R}$ of real-valued $m \times n$ matrices, where addition and scalar multiplication of elements $A, B \in \mathbb{R}^{m\times n}$ are defined by

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij}, \qquad (\alpha A)_{ij} = \alpha(A)_{ij}, \qquad \alpha \in \mathbb{R}.$$

(i) Establish a basis for this vector space; what is its dimension?

(ii) For the case $m = 2$, $n = 3$, express the element

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 3} \tag{1}$$

in your basis.

(iii) For the matrix in (1), compute its Frobenius norm $\|\cdot\|_{\mathsf{F}} = \sqrt{\langle\cdot,\cdot\rangle_{\mathsf{F}}}$ and its induced 1, 2, and ∞ norms.

(iv) Is the set $\{A \in \mathbb{R}^{m\times n} \mid \|A\|_{\mathsf{F}} \le 1\}$ a subspace of $\mathbb{R}^{m\times n}$?

(v) Prove that for any $A, B \in \mathbb{R}^{m\times n}$

$$\|A\|_{\mathsf{F}}^2 + \|B\|_{\mathsf{F}}^2 = \frac{1}{2}\left(\|A + B\|_{\mathsf{F}}^2 + \|A - B\|_{\mathsf{F}}^2\right). \tag{2}$$

(vi) Provide an example to show that (2) *does not* hold if the induced 2-norm is used in place of the Frobenius norm.

Solution:

(i): The space is finite-dimensional since the finite set of vectors defined component-wise by

$$(E_{ij})_{k\ell} = \begin{cases} 1 & \text{if } i = k \text{ and } j = \ell \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

for $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ form a basis. It therefore follows that $\dim(\mathbb{R}^{m\times n}) = nm$.

(ii): We can easily write

$$A = 1 \cdot E_{11} + 2 \cdot E_{12} + 3 \cdot E_{13} + 4 \cdot E_{21} + 5 \cdot E_{22} + 6 \cdot E_{23}.$$

(iii): Simple computations show that

$$\|A\|_{\mathsf{F}} = \sqrt{91} \approx 9.54, \quad \|A\|_1 = 9, \quad \|A\|_2 = \sqrt{\frac{91}{2} + \frac{\sqrt{8065}}{2}} \approx 9.51, \quad \|A\|_\infty = 15.$$

(iv): Let $m = n = 1$ and consider the matrix $A = 1$. Then $\|A\|_{\mathsf{F}} = 1$. But $A + A = 2$ which has Frobenius norm larger than one. So, we have summed two vectors in the set and obtained a vector outside the set. Therefore, the specified set is not a subspace.

(v): We compute that

$$\begin{aligned} \|A + B\|_{\mathsf{F}}^2 + \|A - B\|_{\mathsf{F}}^2 &= \langle A + B, A + B\rangle + \langle A - B, A - B\rangle \\ &= \langle A, A\rangle + \langle A, B\rangle + \langle B, A\rangle + \langle B, B\rangle + \langle A, A\rangle - \langle A, B\rangle - \langle B, A\rangle + \langle B, B\rangle \\ &= 2\left(\|A\|_{\mathsf{F}}^2 + \|B\|_{\mathsf{F}}^2\right) \end{aligned}$$

from which the result follows.

(vi): A randomly generated example which verifies the claim is

$$A = \begin{bmatrix} 5 & 2 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem 4 (Prove Lemma 2.1)

Solution: Let $A, B : \mathsf{V} \to \mathsf{V}$ be bounded operators, and compute that

$$\begin{aligned} \|AB\|_{\mathsf{V}\to\mathsf{V}} = \sup_{v\in\mathsf{V}\setminus\{0\}} \frac{\|A(B(v))\|_{\mathsf{V}}}{\|v\|_{\mathsf{V}}} &= \sup_{\substack{v\in\mathsf{V}\setminus\{0\} \\ B(v)\neq 0}} \frac{\|A(B(v))\|_{\mathsf{V}}}{\|B(v)\|_{\mathsf{V}}} \cdot \frac{\|B(v)\|_{\mathsf{V}}}{\|v\|_{\mathsf{V}}} \\ &\leq \sup_{w\in\mathsf{V}\setminus\{0\}} \frac{\|A(w)\|_{\mathsf{V}}}{\|w\|_{\mathsf{V}}} \cdot \sup_{v\in\mathsf{V}\setminus\{0\}} \frac{\|B(v)\|_{\mathsf{V}}}{\|v\|_{\mathsf{V}}} \\ &= \|A\| \cdot \|B\| \end{aligned}$$

which shows the result.

Problem 5 (Trace Operator)

Let $\mathcal{M}_{n,n}(\mathbb{C})$ denote the Hilbert space of complex $n \times n$ matrices. The *trace* operator is defined as the map $\text{trace} : \mathcal{M}_{n,n}(\mathbb{C}) \to \mathbb{C}$ given by $\text{trace}(A) = \sum_{i=1}^{n} A_{ii}$.

(i) Show that trace is a linear operator.

(ii) Show that $\text{trace}(AB) = \text{trace}(BA)$ for compatible A, B.

Solution: (i): First note that $\text{trace}(0) = 0$. Let $A, B \in \mathcal{M}_{n,n}(\mathbb{C})$ and let $\alpha_1, \alpha_2 \in \mathbb{C}$. Then

$$\begin{aligned}\text{trace}(\alpha_1 A + \alpha_2 B) &= \sum\nolimits_{i=1}^{n} \alpha_1 A_{ii} + \alpha_2 B_{ii} \\ &= \alpha_1 \sum\nolimits_{i=1}^{n} A_{ii} + \alpha_2 \sum\nolimits_{i=1}^{n} B_{ii} \\ &= \alpha_1 \text{trace}(A) + \alpha_2 \text{trace}(B)\end{aligned}$$

which shows linearity. (ii): Let $A, B \in \mathcal{M}_{n,n}(\mathbb{C})$. Note that $(AB)_{ik} = \sum_{j=1}^{n} A_{ij} B_{jk}$. Then

$$\begin{aligned}\text{trace}(AB) &= \sum\nolimits_{i=1}^{n} \sum\nolimits_{j=1}^{n} A_{ij} B_{ji} \\ &= \sum\nolimits_{i=1}^{n} \sum\nolimits_{j=1}^{n} B_{ji} A_{ij} \\ &= \sum\nolimits_{j=1}^{n} \sum\nolimits_{i=1}^{n} B_{ji} A_{ij} \\ &= \text{trace}(BA).\end{aligned}$$

Problem 6 (A convolution operator)

Let $\ell(\mathbb{Z}_{\geq 0}; \mathbb{R})$ denote the vector space of all real-valued discrete-time signals $(\ldots, h(-1), h(0), h(1), \ldots)$ which equal zero for all times $n < 0$. Let $h \in \ell(\mathbb{Z}_{\geq 0}; \mathbb{R})$ be such a discrete-time signal, and consider the operator

$$\mathsf{Conv}_h : \ell(\mathbb{Z}_{\geq 0}; \mathbb{R}) \to \ell(\mathbb{Z}_{\geq 0}; \mathbb{R}), \qquad \mathsf{Conv}_h(u)(n) = \sum_{k=-\infty}^{\infty} h(n-k) u(k) = \sum_{k=0}^{n} h(n-k) u(k).$$

It is obvious that Conv_h defines a linear operator. Show that if $h \in \ell_1(\mathbb{Z}_{\geq 0}; \mathbb{R})$, then $\mathsf{Conv}_h : \ell_\infty(\mathbb{Z}_{\geq 0}; \mathbb{R}) \to \ell_\infty(\mathbb{Z}_{\geq 0}; \mathbb{R})$ is a bounded linear operator satisfying $\|\mathsf{Conv}_h\|_{\infty \to \infty} \leq \|h\|_1$.

Solution: For $u \in \ell_\infty(\mathbb{Z}_{\geq 0}; \mathbb{R})$ we compute that

$$\begin{aligned}\|\mathsf{Conv}_h(u)\|_\infty &= \sup_{n \geq 0} \left| \sum_{k=0}^{n} h(n-k) u(k) \right| \\ &\leq \sup_{n \geq 0} \sum_{k=0}^{n} |h(n-k)| |u(k)| \\ &\leq \sup_{n \geq 0} \sum_{k=0}^{n} |h(n-k)| \cdot \sup_{k \geq 0} |u(k)| \\ &= \|u\|_\infty \sum_{k=0}^{\infty} |h(k)| \\ &= \|h\|_1 \|u\|_\infty.\end{aligned}$$

from which the result follows.

Problem 7 (The Discrete Lyapunov Operator)

Given a fixed matrix $A \in \mathbb{R}^{n\times n}$, define the discrete Lyapunov operator $\mathcal{L}_\mathrm{d} : \mathbb{S}^n \to \mathbb{S}^n$ by

$$\mathcal{L}_\mathrm{d}(P) = A^\mathsf{T} P A - P$$

Show that $\mathcal{L}_\mathrm{d}$ is a linear operator. Then, considering $\mathbb{S}^n$ as a Hilbert space with inner product $\langle \cdot, \cdot \rangle_\mathsf{F}$, provide an upper bound on the induced norm $\|\mathcal{L}_\mathrm{d}\|_{\mathbb{S}^n \to \mathbb{S}^n}$.

Solution: First note that $\mathcal{L}_\mathrm{d}(0) = 0$. Given $P_1, P_2 \in \mathbb{S}^n$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ we compute that

$$\begin{aligned}
\mathcal{L}_\mathrm{d}(\alpha_1 P_1 + \alpha_2 P_2) &= A^\mathsf{T}(\alpha_1 P_1 + \alpha_2 P_2)A - (\alpha_1 P_1 + \alpha_2 P_2) \\
&= \alpha_1(A^\mathsf{T} P_1 A - P_1) + \alpha_2(A^\mathsf{T} P_2 A - P_2) \\
&= \alpha_1 \mathcal{L}_\mathrm{d}(P_1) + \alpha_2 \mathcal{L}_\mathrm{d}(P_2)
\end{aligned}$$

so $\mathcal{L}_\mathrm{d}$ is indeed linear. Since $\mathcal{L}_\mathrm{d}$ is a linear operator on the finite-dimensional space $\mathbb{S}^n$, it is bounded. To determine an upper bound on $\|\mathcal{L}\|$ we can compute that

$$\begin{aligned}
\|\mathcal{L}_\mathrm{d}(P)\|_\mathsf{F} &= \|A^\mathsf{T} P A - P\|_\mathsf{F} \\
&\leq \|A^\mathsf{T} P A\|_\mathsf{F} + \|P\|_\mathsf{F} \\
&\leq \|A^\mathsf{T}\|_\mathsf{F} \|P\|_\mathsf{F} \|A\|_\mathsf{F} + \|P\|_\mathsf{F} \\
&= (\|A\|_\mathsf{F}^2 + 1)\|P\|_\mathsf{F}.
\end{aligned}$$

In the first inequality we used the triangle inequality, in the second we used the sub-multiplicative property of $\|\cdot\|_\mathsf{F}$, and in the third we used that A and A^T have the same singular values, so their Frobenius norms are equal. Therefore, $\|\mathcal{L}_\mathrm{d}\|_{\mathsf{F}\to\mathsf{F}} \leq \|A\|_\mathsf{F}^2 + 1$.

Problem 8 (Positive Definite and Semidefinite Matrices)

In the following, assume all matrices have appropriately compatible dimensions. Some of the following statements are true, and some of them are false. For each one, either prove the statement or provide a counter-example.

(i) If $A \succ \mathbb{0}$ and $C \succ \mathbb{0}$, then $\left[\begin{smallmatrix} A & B \\ B^\mathsf{T} & C \end{smallmatrix}\right] \succ \mathbb{0}$ for any matrix B of appropriate dimensions.

(ii) If $A \succ \mathbb{0}$ and $B \succ \mathbb{0}$, then $AB \succ \mathbb{0}$.

(iii) If $A \succ \mathbb{0}$, then $A_{ii} > 0$ for all $i \in \{1, \ldots, n\}$.

(iv) If $A \succ \mathbb{0}$, then A is invertible and $A^{-1} \succ \mathbb{0}$.

(v) If $A \succ \mathbb{0}$ and T has full column rank, then $TAT^\mathsf{T} \succ \mathbb{0}$

Solution: (i): This is false; the matrix

$$\begin{bmatrix} A & B \\ B^\mathsf{T} & C \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

satisfies the assumptions, but has a zero eigenvalue.

(ii): This is false; the matrix AB need not even be symmetric. For example,

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad AB = \begin{bmatrix} -1 & 3 \\ -3 & 8 \end{bmatrix}.$$

(iii): This is true; if $A \succ \mathbb{0}$, then by definition for any $v \in \mathbb{R}^n$ we have that $v^\mathsf{T} Av > 0$. In particular then, take $\mathbb{e}_i$ as the ith unit vector of $\mathbb{R}^n$ and note that

$$\mathbb{e}_i^\mathsf{T} A \mathbb{e}_i = A_{ii} > 0.$$

(iv): This is true; if $A \succ \mathbb{0}$, then all of its eigenvalues are strictly positive, and hence non-zero, so A is invertible. Let $A = U\Lambda U^*$ denote the eigen-decomposition of A, where $\Lambda \succ 0$ is diagonal. Since U is unitary, we have that $U^{-1} = U^\star$. Therefore

$$A^{-1} = (U^*)^{-1}\Lambda^{-1}U^{-1} = U\Lambda^{-1}U^*$$

It now follows by the results on similarity (or congruence) transforms that $A^{-1} \succ 0$.

(v): This is false. For example, take

$$A = I_1, \quad T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad TAT^\mathsf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Problem 9 (Riccati Map)

Given fixed matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, define the operator $\mathcal{R} : \mathbb{S}^n \to \mathbb{S}^{n+m}$ by

$$\mathcal{R}(P) = \begin{bmatrix} A^\mathsf{T} P + PA & PB \\ B^\mathsf{T} P & -I \end{bmatrix}$$

Show that $\mathcal{R}$ is an affine mapping. What can you conclude about the image of $\mathbb{S}^n_{>0}$ under $\mathcal{R}$? What can you conclude about the set of P satisfying $\mathcal{R}(P) \prec 0$?

Solution: Let $\alpha_1, \alpha_2 \in \mathbb{R}$ be such that $\alpha_1 + \alpha_2 = 1$, and let $P_1, P_2 \in \mathbb{S}^n$. Then

$$\begin{aligned} \mathcal{R}(\alpha_1 P_1 + \alpha_2 P_2) &= \begin{bmatrix} A^\mathsf{T}(\alpha_1 P_1 + \alpha_2 P_2) + (\alpha_1 P_1 + \alpha_2 P_2)A & (\alpha_1 P_1 + \alpha_2 P_2)B \\ B^\mathsf{T}(\alpha_1 P_1 + \alpha_2 P_2) & -I \end{bmatrix} \\ &= \alpha_1 \begin{bmatrix} A^\mathsf{T} P_1 + P_1 A & P_1 B \\ B^\mathsf{T} P_1 & -I \end{bmatrix} + \alpha_2 \begin{bmatrix} A^\mathsf{T} P_2 + P_2 A & P_2 B \\ B^\mathsf{T} P_2 & -I \end{bmatrix} \\ &= \alpha_1 \mathcal{R}(P_1) + \alpha_2 \mathcal{R}(P_2) \end{aligned}$$

so $\mathcal{R}$ is indeed affine. Note that it is *not* linear, since $\mathcal{R}(0) \neq 0$. Since $\mathbb{S}^n_{>0}$ is a convex set and $\mathcal{R}$ is an affine map, we know that $\mathcal{R}(\mathbb{S}^n_{>0})$ is also a convex set. Since $\mathcal{R}(P) \prec 0$ is a strict linear matrix inequality, the set of matrices $P \in \mathbb{S}^n$ satisfying it is convex set.

Problem 10 (Hurwitz Matrices)

Consider the set $\mathcal{H}^n$ of all $n \times n$ Hurwitz matrices

(i) Provide a counter-example to the claim that $\mathcal{H}^n$ is convex.

(ii) Show that $\mathcal{H}^n \cap \mathbb{S}^n$ is convex.

(iii) For a given stabilizable pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, consider the set of stabilizing state feedback gains

$$\mathcal{K}(A, B) = \{K \in \mathbb{R}^{m\times n} \mid A + BK \in \mathcal{H}^n\}.$$

Provide a counter-example to the claim that $\mathcal{K}$ is convex. Comment on the implications of this fact for the design of stabilizing feedback gains.

Solution: (i): Consider the matrices

$$M_1 = \begin{bmatrix} -1 & 5 \\ 0 & -0.1 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} -0.1 & 0 \\ 5 & -0.1 \end{bmatrix}.$$

Note that since M_1 and M_2 are triangular, their eigenvalues are their diagonal entries, so both matrices are Hurwitz. As a particular convex combination of these two matrices, we compute that

$$\tfrac{1}{2}M_1 + \tfrac{1}{2}M_2 = \begin{bmatrix} -0.55 & 2.5 \\ 2.5 & -0.1 \end{bmatrix}$$

which has eigenvalues of -2.8351 and 2.1851, and is therefore not Hurwitz. Thus, the set of Hurwitz matrices do not form a convex set.

(ii): Symmetry implies that the eigenvalues of any such matrix $A \in \mathcal{H}^n \cap \mathbb{S}^n$ must be real, and hence the set of symmetric Hurwitz matrices is nothing but the set of negative definite matrices. In other words, A is symmetric and Hurwitz if and only if $A \prec 0$, and we already know that this is a convex set.

(iii): Consider the fully actuated system $A = 0_{2\times 2}$ and $B = I_2$, and using the matrices from part (i), let $K_1 = M_1$ and $K_2 = M_2$. By the same argument as in part (i), the set $\mathcal{K}$ is not convex. The naive implication of this would seem to be that state-feedback controller design is a non-convex problem and therefore inherently difficult. This is, of course, nonsense; we know that state feedback design is actually quite a straightforward problem. The proper implication to draw is that *directly optimizing over the set of stabilizing state-feedback controller gains* is an inherently difficult problem.

Problem 11 (Other SDPs)

Look up an example of a semidefinite program which arises *outside* of systems and control, and write a brief summary (no more than 1/2 page).

Solution: Any appropriate example is acceptable.

Problem 12 (A Small-Gain Theorem)

Let V be a normed vector space with $\mathrm{Id} : \mathsf{V} \to \mathsf{V}$ denoting the identity operator on V, i.e., $\mathrm{Id}(v) = v$ for all $v \in \mathsf{V}$. Let $M : \mathsf{V} \to \mathsf{V}$ be a bounded operator satisfying $\|M\|_{\mathsf{V}\to\mathsf{V}} \leq \gamma$ for some $\gamma \in [0, 1)$. Show that if $(\mathrm{Id} - M) : \mathsf{V} \to \mathsf{V}$ is invertible, then $(\mathrm{Id} - M)^{-1}$ is a bounded operator and

$$\|(\mathrm{Id} - M)^{-1}\|_{\mathsf{V}\to\mathsf{V}} \leq \frac{1}{1-\gamma}.$$

Hint: For $u, v \in \mathsf{V}$, the reverse triangle inequality says that $\|u - v\|_{\mathsf{V}} \geq \|u\|_{\mathsf{V}} - \|v\|_{\mathsf{V}}$.

Solution: Let $x \in \mathsf{V}$. Using the reverse triangle inequality and gain bound on M, we can compute that

$$\begin{aligned}
\|(\mathrm{Id} - M)(v)\|_{\mathsf{V}} &= \|v - M(v)\|_{\mathsf{V}} \\
&\geq \|v\|_{\mathsf{V}} - \|M(v)\|_{\mathsf{V}} \\
&\geq \|v\|_{\mathsf{V}} - \|M\|_{\mathsf{V}\to\mathsf{V}}\|v\|_{\mathsf{V}} \\
&\geq \|v\|_{\mathsf{V}} - \gamma\|v\|_{\mathsf{V}} \\
&= (1-\gamma)\|v\|_{\mathsf{V}}
\end{aligned}$$

where we note that $1 - \gamma > 0$. By assumption, for an arbitrary $w \in \mathsf{V}$ the equation $v = M(v) + w$ — or equivalently $(\mathrm{Id} - M)(v) = w$ — has a unique solution given by $v = (\mathrm{Id} - M)^{-1}(w)$. The previous inequality is therefore precisely that

$$\|w\|_{\mathsf{V}} \geq (1-\gamma)\|(\mathrm{Id} - M)^{-1}(w)\|_{\mathsf{V}}$$

or

$$\|(\mathrm{Id} - M)^{-1}(w)\|_{\mathsf{V}} \leq \frac{1}{1-\gamma}\|w\|_{\mathsf{V}}$$

for all $w \in \mathsf{V}$. Therefore,

$$\sup_{w \in \mathsf{V}\setminus\{0\}} \frac{\|(\mathrm{Id} - M)^{-1}(w)\|_{\mathsf{V}}}{\|w\|_{\mathsf{V}}} = \|(\mathrm{Id} - M)^{-1}\|_{\mathsf{V}\to\mathsf{V}} \leq \frac{1}{1-\gamma}.$$