ECE 1659H Assignment 2 Solutions

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Problem 1 (Block Matrices and Davison's Integral Controller)

Consider the real symmetric block matrix

$$M = \begin{bmatrix} Q_1 + \epsilon Q_2 & \epsilon S \\ \epsilon S^{\mathsf{T}} & \epsilon R \end{bmatrix} \in \mathbb{S}^{n+p}$$

where $Q_1, R \prec 0$ and $\epsilon > 0$. Prove that there exists $\epsilon^\star > 0$ such that $M \prec 0$ for all $\epsilon \in (0, \epsilon^\star)$.

As an important application of this result, consider the LTI control system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. Assume that $p \leq m$ and that A is Hurwitz. Let $G(s) = C(sI - A)^{-1}B$ be the transfer function, with $G_0 = G(0) \in \mathbb{R}^{p\times m}$ denoting the DC gain of $G(s)$, and assume that $G(0)$ has full row rank. Consider the reference-tracking integral controller

$$\dot{\eta} = y - r, \quad u = -\epsilon K \eta \qquad K = G_0^{\mathsf{T}}(G_0 G_0^{\mathsf{T}})^{-1}$$

where $r \in \mathbb{R}^p$ is constant and $\epsilon > 0$. Use your previous result to show that there exists $\epsilon^\star > 0$ such that the closed-loop system is internally exponentially stable for all $\epsilon \in (0, \epsilon^\star)$.

Hint: Consider the change of state variables $\xi = x - \epsilon A^{-1}BK\eta$ and $\beta = \epsilon\eta$, and then write down the Lyapunov LMI for your transformed system.

Solution: Since $Q_1 \prec 0$ it follows by continuity of the mapping $\epsilon \to Q_1 + \epsilon Q_2$ that $Q_1 + \epsilon Q_2 \prec 0$ will be invertible for sufficiently small ϵ. In this case, by Schur's Lemma we have that $M \prec 0$ if and only if

$$\epsilon R - \epsilon^2 S(Q_1 + \epsilon Q_2)^{-1} S^{\mathsf{T}} \prec 0$$

which holds for $\epsilon > 0$ if and only if

$$R - \epsilon S(Q_1 + \epsilon Q_2)^{-1} S^{\mathsf{T}} \prec 0.$$

This expression is a continuous function of ϵ and is negative definite at $\epsilon = 0$. By continuity then, there exists $\epsilon^\star > 0$ such that $R - \epsilon S(Q_1 + \epsilon Q_2)^{-1} S^{\mathsf{T}} \prec 0$ for all $\epsilon \in (0, \epsilon^\star)$, which shows the result.

Now consider the control problem. The closed-loop system is described by

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A & -\epsilon BK \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} 0 \\ -I \end{bmatrix} r.$$

Consider now the change of variables

$$\begin{aligned}\xi &= x - \epsilon A^{-1}BK\eta \\ \beta &= \epsilon\eta\end{aligned} \qquad \Longleftrightarrow \qquad T = \begin{bmatrix} I & -\epsilon A^{-1}BK \\ 0 & \epsilon I \end{bmatrix}.$$

We have that

$$\begin{aligned}\begin{bmatrix}\dot{\xi} \\ \dot{\beta}\end{bmatrix} &= T\begin{bmatrix}\dot{x} \\ \dot{\eta}\end{bmatrix} = T\begin{bmatrix} A & -\epsilon BK \\ C & 0\end{bmatrix}T^{-1}\begin{bmatrix}\xi \\ \beta\end{bmatrix} \\ &= \begin{bmatrix} I & -\epsilon A^{-1}BK \\ 0 & \epsilon I\end{bmatrix}\begin{bmatrix} A & -\epsilon BK \\ C & 0\end{bmatrix}\begin{bmatrix} I & A^{-1}BK \\ 0 & \frac{1}{\epsilon}I\end{bmatrix}\begin{bmatrix}\xi \\ \beta\end{bmatrix} \\ &= \begin{bmatrix} A - \epsilon A^{-1}BKC & \epsilon A^{-1}BK\,G_0K \\ \epsilon C & -\epsilon G_0 K\end{bmatrix}\begin{bmatrix}\xi \\ \beta\end{bmatrix}\end{aligned}$$

Since $G_0K = G_0G_0^{\mathsf{T}}(G_0G_0^{\mathsf{T}})^{-1} = I$ this further simplifies to

$$\begin{bmatrix}\dot{\xi} \\ \dot{\beta}\end{bmatrix} = \mathcal{A}\begin{bmatrix}\xi \\ \beta\end{bmatrix} = \begin{bmatrix} A - \epsilon A^{-1}BKC & \epsilon A^{-1}BK \\ \epsilon C & -\epsilon I\end{bmatrix}\begin{bmatrix}\xi \\ \beta\end{bmatrix}$$

Since A is Hurwitz, there exists $P \succ 0$ such that $A^{\mathsf{T}}P + PA \prec 0$. Consider the Lyapunov matrix

$$\mathcal{P} = \begin{bmatrix} P & \mathbb{0} \\ \mathbb{0} & I\end{bmatrix}$$

We compute that

$$\begin{aligned}\mathcal{P}\mathcal{A} &= \begin{bmatrix} P & \mathbb{0} \\ \mathbb{0} & I\end{bmatrix}\begin{bmatrix} A - \epsilon A^{-1}BKC & \epsilon A^{-1}BK \\ \epsilon C & -\epsilon I\end{bmatrix} \\ &= \begin{bmatrix} PA - \epsilon PA^{-1}BKC & \epsilon PA^{-1}BK \\ \epsilon C & -\epsilon I\end{bmatrix}\end{aligned}$$

and therefore

$$\mathcal{A}^{\mathsf{T}}\mathcal{P} + \mathcal{P}\mathcal{A} = \begin{bmatrix} A^{\mathsf{T}}P + PA - \epsilon(PA^{-1}BKC + C^{\mathsf{T}}K^{\mathsf{T}}B^{\mathsf{T}}A^{-\mathsf{T}}P) & \epsilon(C^{\mathsf{T}} + PA^{-1}BK) \\ \epsilon(C^{\mathsf{T}} + PA^{-1}BK)^{\mathsf{T}} & -2\epsilon I\end{bmatrix}$$

It follows from the first result that $\mathcal{A}^{\mathsf{T}}\mathcal{P} + \mathcal{P}\mathcal{A} \prec 0$, and which completes the proof.

Problem 2 (Discrete-Time Lyapunov LMI)

Consider the LTI discrete-time system autonomous system

$$x(k+1) = Ax(k)\,, \qquad x(0) = x_0 \in \mathbb{R}^n\,.$$

where $k = 0, 1, 2, \ldots$ is a discrete time index. The origin of this system is exponentially stable if there exist constants $M > 0$ and $\gamma \in (0, 1)$ such that

$$\|x(k)\|_2 \leq M\gamma^k\|x(0)\|_2.$$

for all $x(0) \in \mathbb{R}^n$. It is well-known that the origin of the system is exponentially stable if and only if all eigenvalues of $A \in \mathbb{R}^{n\times n}$ have magnitude strictly less than one; you can take this as a given in this problem. By mirroring the continuous-time proof from class, prove that the origin is exponentially stable if and only if there exists a matrix $P \succ \mathbb{0}$ such that

$$A^\mathsf{T} P A - P \prec \mathbb{0}.$$

Apply your result in `MATLAB` to check whether or not all eigenvalues of the matrix

$$A = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

have modulus less than one.

Hint: One approach is to construct a candidate Lyapunov solution via an infinite sum. To argue that this candidate is well-defined, you can use the fact that given any $A \in \mathbb{R}^{n\times n}$ and any $\epsilon > 0$, there exists a submultiplicative matrix norm $\|\cdot\|$ such that $\|A\| \le \rho(A) + \epsilon$ where $\rho(A) = \max\{|\lambda| : \lambda \in \mathrm{eig}(A)\}$ is the spectral radius of A.

Solution: We first show that all eigenvalues of A having modulus strictly less than one implies that for any $Q \succ \mathbb{0}$ there exists a solution $P \succ \mathbb{0}$ to the Lyapunov equation $A^\mathsf{T} P A - P = -Q$. We claim that the solution $P \succ \mathbb{0}$ is given explicitly by

$$P = \sum_{k=0}^{\infty} (A^k)^\mathsf{T} Q (A^k).$$

Since all eigenvalues of A have modulus strictly less than one, there exists a matrix norm $\|\cdot\|$ such that $\|A\| < 1$. We therefore compute that

$$\|P\| = \left\| \sum_{k=0}^{\infty} (A^k)^\mathsf{T} Q (A^k) \right\| \le \|Q\| \sum_{k=0}^{\infty} \|A^k\|^2 \le \|Q\| \sum_{k=0}^{\infty} \|A\|^{2k} = \|Q\| \frac{1}{1 - \|A\|}$$

so P is well-defined. Note that P is a sum of symmetric matrices, and hence is symmetric. Moreover, we compute that

$$\begin{aligned} A^\mathsf{T} P A - P &= \sum_{k=0}^{\infty} (A^{k+1})^\mathsf{T} Q A^{k+1} - \sum_{k=0}^{\infty} (A^k)^\mathsf{T} Q A^k \\ &= \sum_{k=0}^{\infty} \left((A^{k+1})^\mathsf{T} Q A^{k+1} - (A^k)^\mathsf{T} Q A^k \right) \\ &= -Q \end{aligned}$$

so the candidate solution satisfies the equation. Next, we show that P is positive definite. Let $v \in \mathbb{R}^n$ be non-zero, and compute that

$$v^\mathsf{T} P v = \sum_{k=0}^{\infty} v^\mathsf{T} (A^k)^\mathsf{T} Q A^k v = \sum_{k=0}^{\infty} (A^k v)^\mathsf{T} Q (A_k v) = \sum_{k=0}^{\infty} w(k) Q w(k)$$

where $w(k) = A^k v$. Since $Q \succ \mathbb{0}$, all terms in the sum are nonnegative. Moreover, since $w(1) = v \neq 0$, at least one term is strictly positive, so we conclude that the sum is positive, and therefore $v^\mathsf{T} P v > 0$ for all $v \neq 0$, so $P \succ \mathbb{0}$. It now follows in particular that $A^\mathsf{T} P A - P \prec \mathbb{0}$

To complete the chain of implications we now show the Lyapunov LMI implies exponential stability. Let $P \succ \mathbb{0}$ be such that $A^\mathsf{T} PA - P \prec \mathbb{0}$. This strict LMI holds if and only if there exists a value $\rho \in (0, 1)$ such that

$$A^\mathsf{T} PA - P \preceq -\rho^2 P.$$

Now consider the function $V(x) = x^\mathsf{T} Px$. Along trajectories of the system $x(k+1) = Ax(k)$, we compute that

$$\begin{aligned} V(x(k+1)) - V(x(k)) &= x(k+1)^\mathsf{T} Px(k+1) - x(k)^\mathsf{T} Px(k) \\ &= (Ax(k))^\mathsf{T} P(Ax(k)) - x(k)^\mathsf{T} Px(k) \\ &= x(k)^\mathsf{T} \left[A^\mathsf{T} PA - P\right] x(k) \\ &\leq -\rho^2 x(k)^\mathsf{T} Px(k) \\ &= -\rho^2 V(x(k)). \end{aligned}$$

We therefore find that V satisfies

$$V(x(k+1)) \leq (1 - \rho^2)V(x(k))$$

for some $\rho \in (0, 1)$, and therefore by iteration that

$$V(x(k)) \leq (1 - \rho^2)^k V(x(0)).$$

Since $P \succ \mathbb{0}$, we have that

$$\alpha\|x\|_2^2 \leq V(x) \leq \beta\|x\|_2^2$$

for some $\alpha, \beta > 0$. Therefore,

$$\|x(k)\|_2^2 \leq \frac{\beta}{\alpha}(1 - \rho^2)^k \|x(0)\|_2^2$$

and hence

$$\|x(k)\|_2 \leq \sqrt{\frac{\beta}{\alpha}}(1 - \rho^2)^{k/2} \|x(0)\|_2$$

which shows exponential stability of the origin.

One can spot immediately in fact that the given matrix A does not have all eigenvalues with modulus less than one; it has a right-eigenvector $(1, 1, 1, 1)$ corresponding to eigenvalue $\lambda = 1$. Indeed, your solver will spit back that the problem is infeasible.

Problem 3 (Simultaneous Stabilization of LTI Systems)

Suppose you are given N linear time-invariant systems of the form

$$\dot{x}_i = A_i x_i + B_i u_i, \quad i = 1, \ldots, N,$$

where $x_1, \ldots, x_N \in \mathbb{R}^n$ are the states and $u_1, \ldots, u_N \in \mathbb{R}^m$ are the inputs. Your goal is to design a *single* state feedback gain K such that the feedback law $u_i = Kx_i$ exponentially stabilizes the ith system for all $i \in \{1, \ldots, N\}$.

(i) Formulate this design problem using linear matrix inequalities. If instead you believe you cannot formulate this stabilization problem as an LMI problem, explain why.

(ii) Are you conditions sufficient, or necessary and sufficient? If they are merely sufficient, indicate clearly at which step conservatism has been introduced into the design procedure.

(iii) Create a non-trivial[1] example to which you can apply your method, and do so using `MATLAB` or your preferred solver. Provide solver output indicating that your method was successful.

Solution: (i): The closed-loop systems are described by

$$\dot{x}_i = (A_i + B_i K)x_i, \quad i = 1, \dots, N.$$

Stability of all these systems is equivalent to the existence of matrices $P_i \succ 0$ such that

$$P_i(A_i + B_i K) + (A_i + B_i K)^\mathsf{T} P_i \prec 0$$

in which case $V_i(x_i) = x_i^\mathsf{T} P_i x_i$ defines a quadratic Lyapunov function for the ith system. Let us assume the existence of a *common Lyapunov function*, i.e., there exists a matrix $P \succ 0$ such that $P_i = P$ for all $i \in \{1, \dots, N\}$. It follows that the existence of $P \succ 0$ satisfying

$$P(A_i + B_i K) + (A_i + B_i K)^\mathsf{T} P \prec 0$$

is sufficient for stability. Performing a congruence transformation with $X = P^{-1}$, this is equivalent to

$$(A_i + B_i K)X + X(A_i + B_i K)^\mathsf{T} \prec 0$$

Defining $Z = KX$ one finally obtains the linear matrix inequality: find $X \succ 0$ such that

$$A_i X + X A_i^\mathsf{T} + B_i Z + Z^\mathsf{T} B_i^\mathsf{T} \prec 0,$$

and the controller K can be recovered as $K = ZX^{-1}$.

(ii): The conditions are only sufficient; conservatism has been introduced by requiring that the systems be stable with a common Lyapunov function $V(\xi) = \xi^\mathsf{T} X^{-1} \xi$.

(iii): Any example is fine. The following code generated a feasible example for me after a few runs

```
clc
clear all
close all

%%
n = 3; m = 2; N = 4;
A = rand(n,n,N);
B = rand(n,m,N);

A1 = A(:,:,1); B1 = B(:,:,1);
A2 = A(:,:,2); B2 = B(:,:,2);
A3 = A(:,:,3); B3 = B(:,:,3);
A4 = A(:,:,4); B4 = B(:,:,4);

%% Define SDP Problem
X = sdpvar(n,n); Z = sdpvar(m,n);
small = 1e-5;
Constraints = [X ≥ small*eye(n),   A1*X+X*A1'+B1*Z+Z'*B1' ≤ -small*eye(n), ...
```

[1] Note that if all A_i are Hurwitz, then the problem is solved by $K = 0$, so this case is not particularly interesting.

```
                                             A2*X+X*A2'+B2*Z+Z'*B2' ≤ -small*eye(n), ...
                                             A3*X+X*A3'+B3*Z+Z'*B3' ≤ -small*eye(n), ...
                                             A4*X+X*A4'+B4*Z+Z'*B4' ≤ -small*eye(n)];
Cost = 0;

%% Solve
options = sdpsettings('solver','sdpt3','verbose',1);
sol = optimize(Constraints,Cost,options);

value(X) %print value

%% Check

K = value(Z)*inv(value(X));
A_cl = blkdiag(A1+B1*K,A2+B2*K,A3+B3*K,A4+B4*K);
```

Problem 4 (Passive Systems and PI Control)

An LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ z &= Cx \end{aligned}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^p$, and output $z \in \mathbb{R}^p$ is said to be *input-strictly passive* if it is dissipative with positive-definite storage function $V(x) = x^\mathsf{T} P x$ with $P \succ 0$ and supply rate $s(u, z) = z^\mathsf{T} u - \nu u^\mathsf{T} u$ where $\nu > 0$. Assume that A is Hurwitz, that (C, A) is observable, and that $CA^{-1}B$ is invertible.

Consider now such a system with a proportional-integral controller

$$\dot{\eta} = z - r, \qquad u = -K_\mathrm{i}\eta - K_\mathrm{p}(z - r)$$

where r is a constant reference signal and $K_\mathrm{p}, K_\mathrm{i} \succ 0$.

(i) Show that for any $r \in \mathbb{R}^p$, the closed-loop system possess a unique equilibrium point $(\bar{x}, \bar{\eta})$ satisfying $\bar{z} = C\bar{x} = r$.

(ii) Show that $(\bar{x}, \bar{\eta})$ is exponentially stable.

Solution: (i): After eliminating, closed-loop equilibrium points are determined by the equations

$$\begin{aligned} 0 &= C\bar{x} - r \\ 0 &= A\bar{x} - BK_\mathrm{i}\bar{\eta}. \end{aligned}$$

Since A is Hurwitz, it is invertible, and we can further reduce this to

$$CA^{-1}BK_\mathrm{i}\bar{\eta} = r$$

Since $K_\mathrm{i} \succ 0$, we conclude that this equation is solvable for each $r \in \mathbb{R}^p$ if and only if the square matrix $CA^{-1}B$ is invertible, which is true by assumption. Therefore, the unique equilibrium is given by

$$\bar{\eta} = K_\mathrm{i}^{-1}(CA^{-1}B)^{-1}r, \qquad \bar{x} = A^{-1}BK_\mathrm{i}\bar{\eta}.$$

with associated control signal $\bar{u} = -K_{\mathrm{i}}\bar{\eta}$ and plant output $\bar{z} = C\bar{x}$.

(ii): Given the equilibrium from part (i), we introduce the deviation variables

$$\begin{aligned}\tilde{x} &= x - \bar{x}\\ \tilde{\eta} &= \eta - \bar{\eta}\\ \tilde{u} &= u - \bar{u}\\ \tilde{z} &= z - \bar{z},\end{aligned}$$

which leads to the dynamics

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} + B\tilde{u}\\ \tilde{z} &= C\tilde{x}\end{aligned} \tag{1}$$

and the controller

$$\begin{aligned}\dot{\tilde{\eta}} &= \tilde{z}\\ \tilde{u} &= -K_{\mathrm{i}}\tilde{\eta} - K_{\mathrm{p}}\tilde{z}.\end{aligned} \tag{2}$$

By assumption (1) is dissipative with quadratic supply rate matrix $\Pi_1 = \begin{bmatrix} 0 & \frac{1}{2}I \\ \frac{1}{2}I & -\nu I \end{bmatrix}$ and positive definite storage function $V(\tilde{x}) = \tilde{x}^{\mathsf{T}}P\tilde{x}$.

Consider the storage function candidate $W(\tilde{\eta}) = \tilde{\eta}^{\mathsf{T}}K_{\mathrm{i}}\tilde{\eta}$. We compute that

$$\begin{aligned}\dot{W}(\eta(t)) &= 2\eta^{\mathsf{T}}K_{\mathrm{i}}\dot{\eta}\\ &= 2\eta^{\mathsf{T}}K_{\mathrm{i}}\tilde{z}\\ &= 2(-K_{\mathrm{i}}^{-1}(\tilde{u} + K_{\mathrm{p}}\tilde{z}))^{\mathsf{T}}K_{\mathrm{i}}\tilde{z}\\ &= -(\tilde{u} + K_{\mathrm{p}}\tilde{z})^{\mathsf{T}}\tilde{z}\\ &= -\tilde{z}^{\mathsf{T}}K_{\mathrm{p}}\tilde{z} - \tilde{u}^{\mathsf{T}}\tilde{z}\\ &= \begin{bmatrix}\tilde{u}\\ \tilde{z}\end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & -\frac{1}{2}I \\ -\frac{1}{2}I & -K_{\mathrm{p}} \end{bmatrix} \begin{bmatrix}\tilde{u}\\ \tilde{z}\end{bmatrix}\\ &\triangleq \begin{bmatrix}\tilde{u}\\ \tilde{z}\end{bmatrix}^{\mathsf{T}} \Pi_2 \begin{bmatrix}\tilde{u}\\ \tilde{z}\end{bmatrix}.\end{aligned}$$

so we now have a dissipation inequality for (2). Following our discussion of interconnections of dissipative systems from the notes, we compute that

$$\Pi_1 + \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \Pi_2 \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}I \\ \frac{1}{2}I & -\nu I \end{bmatrix} + \begin{bmatrix} -K_{\mathrm{p}} & -\frac{1}{2}I \\ -\frac{1}{2}I & 0 \end{bmatrix} = \begin{bmatrix} -K_{\mathrm{p}} & 0 \\ 0 & -\nu I \end{bmatrix} \prec 0.$$

By assumption (C, A) is observable. Moreover, since $K_{\mathrm{i}} \succ 0$, one may apply any desired observability test to check that (2) is also observable. We conclude that the origin of the closed-loop system (1)–(2) in deviation variables is exponentially stable, which shows the desired result. This result is a special case of a more general result that states, roughly speaking, that the negative feedback interconnection of two input-strictly passive systems is stable.

Problem 5 (Numerical Problem)

Consider the LTI system

$$\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx + Dw \end{aligned}$$

where

$$A = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 4 & -3 \\ 1 & -3 & -1 & -3 \\ 0 & 4 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Write a program to compute the smallest value $\gamma > 0$ such that the system is input-strictly dissipative with respect to the supply rate $s(w, z) = -\|z\|_2^2 + \gamma^2 \|w\|_2^2$. Your answer is your code and your numerical value.

Solution: The smallest such γ is 1.3791; here is some sample code

```
clc
clear all
close all

%%
n = 4;
m = 2;
p = 2;
A = [-1,0,0,1;0,-1,4,-3;1,-3,-1,-3;0,4,2,-1];
B = [0,1;0,0;-1,0;0,0];
C = [-1,0,1,0; 0,1,0,1];
D = [0,1;0,0];

%% Define SDP Problem
P = sdpvar(n,n); gamsq = sdpvar(1,1);
small = 1e-5;

Lyap = [eye(n),zeros(n,m);A,B]'*[zeros(n),P;P,zeros(n)]*[eye(n),zeros(n,m);A,B];
Pi = [-eye(p),zeros(p,m);zeros(m,p),gamsq*eye(m)];
Diss = [C,D;zeros(m,n),eye(m)]'*Pi*[C,D;zeros(m,n),eye(m)];

Constraints = [P ≥ small*eye(n),   Lyap - Diss ≤ small*eye(n+m)];
Cost = gamsq;

%% Solve
options = sdpsettings('solver','sdpt3','verbose',1);
sol = optimize(Constraints,Cost,options);

gamma = sqrt(value(gamsq)) %print value
```

Problem 6 (A Small-Gain Theorem)

Following the discussion of interconnected systems on slides 5-112/5-113, suppose that the systems are dissipative with respect to the $\mathscr{L}_2$ supply rates

$$s_1(w_1, z_1) = -\|z_1\|_2^2 + \gamma_1^2\|w_1\|_2^2$$
$$s_2(w_2, z_2) = -\|z_2\|_2^2 + \gamma_2^2\|w_2\|_2^2$$

respectively, where $\gamma_1, \gamma_2 \geq 0$. Show that the LMI on page 5-113 is feasible if and only if $\gamma_1\gamma_2 < 1$, and hence conclude exponential stability of the origin under this small-loop-gain condition.

Solution: The LMI on 5-112/5-113 reduces to the following: find $\alpha_1, \alpha_2 > 0$ such that

$$\alpha_1 \begin{bmatrix} -I_p & 0 \\ 0 & \gamma_1^2 I_m \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} -I_m & 0 \\ 0 & \gamma_2^2 I_p \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \prec 0$$

or more simply

$$\alpha_1 \begin{bmatrix} -I_p & 0 \\ 0 & \gamma_1^2 I_m \end{bmatrix} + \alpha_2 \begin{bmatrix} \gamma_2^2 I_p & 0 \\ 0 & -I_m \end{bmatrix} \prec 0.$$

We may define $\alpha = \alpha_2/\alpha_1$ and divide the above by $\alpha_1 > 0$ to obtain the equivalent LMI: find $\alpha > 0$ such that

$$\begin{bmatrix} -I_p & 0 \\ 0 & \gamma_1^2 I_m \end{bmatrix} + \alpha \begin{bmatrix} \gamma_2^2 I_p & 0 \\ 0 & -I_m \end{bmatrix} = \begin{bmatrix} -(1-\alpha\gamma_2^2)I_p & 0 \\ 0 & -(\alpha - \gamma_1^2)I_m \end{bmatrix} \prec 0.$$

We therefore find that we need to find $\alpha > 0$ satisfying

$$1 - \alpha\gamma_2^2 > 0, \qquad \alpha - \gamma_1^2 > 0.$$

or equivalently

$$\gamma_1^2 < \alpha < \frac{1}{\gamma_2^2}$$

or equivalently

$$\gamma_1^2\gamma_2^2 < \alpha < 1.$$

There obviously exists $\alpha > 0$ satisfying this final inequality if and only if

$$\gamma_1\gamma_2 < 1.$$

The interpretation of this condition is that γ_1 and γ_2 are bounds on the (induced $\mathcal{L}_2$) gain of each system; the condition $\gamma_1\gamma_2 < 1$ requires the gain around the loop to be less than 1, so that signals tend to decay to zero as they traverse the feedback loop multiple times.