ECE 1659H Assignment 4 Solutions

Winter 2023
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Problem 1 (Anti-Windup Control System)

Consider the control system shown in Figure 1, where the single-input single-output plant $P : \mathcal{L}_{2e}[0,\infty) \to \mathcal{L}_{2e}[0,\infty)$ is represented by the transfer function

$$P(s) = 50\frac{(s^2+2s+1)(s+200)}{(s^2+5s+50)(s+10)^2}.$$

The system is controlled by a standard PI controller

$$C(s) = K\left(1 + \frac{1}{T_{\mathrm{i}}s}\right).$$

The objective of this control system is for y to track the reference value r.

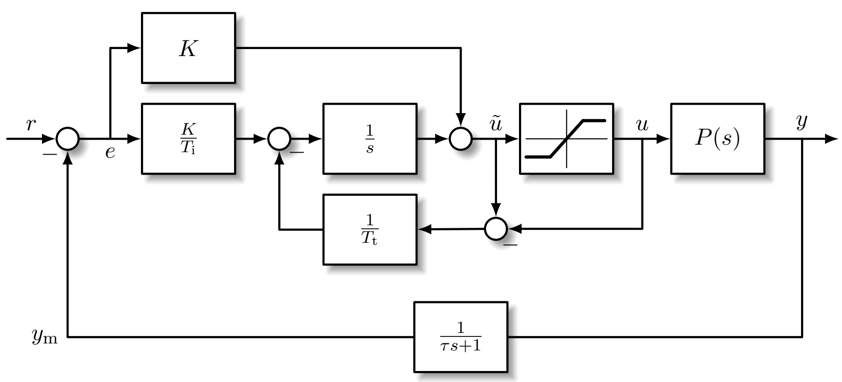


Figure 1: A feedback system with anti-windup control; note carefully the minus signs.

The system is subject to actuator saturation

$$u = \mathsf{sat}(\tilde{u}) = \begin{cases} \tilde{u} & \text{if } |\tilde{u}| \leq 1 \\ 1 & \text{if } \tilde{u} > 1 \\ -1 & \text{if } \tilde{u} < -1 \end{cases}$$

and to help compensate for this, a standard *tracking-type anti-windup controller* with time constant T_t is added in addition to the PI controller. When the output u saturates, there will be a deviation between $\tilde{u}$ and u, which is fed back to reduce the integrator state and prevent the so-called wind-up phenomena; see, for example, this link for more information on this (completely standard) control scheme. The transfer function $\frac{1}{\tau s+1}$ in the feedback path models sensor dynamics, and effectively acts as a delay the feedback loop.

The default parameters are

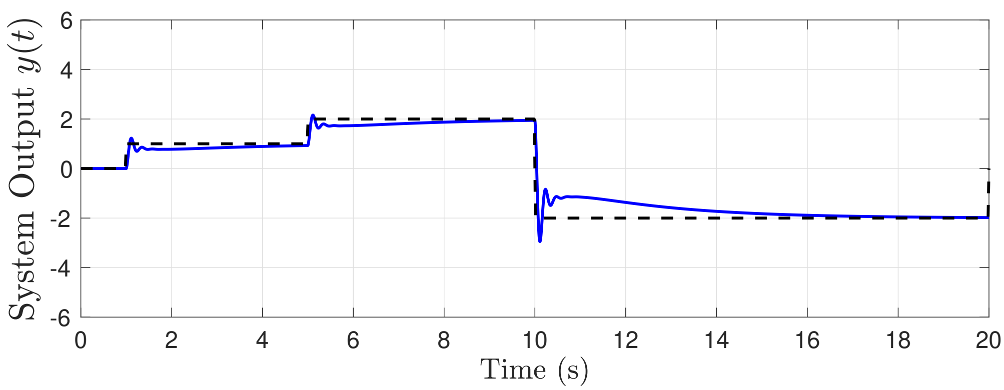
$$K = 0.07, \quad T_{\rm i} = 0.11, \quad T_{\rm t} = 0.11, \quad \tau = 0.001.$$

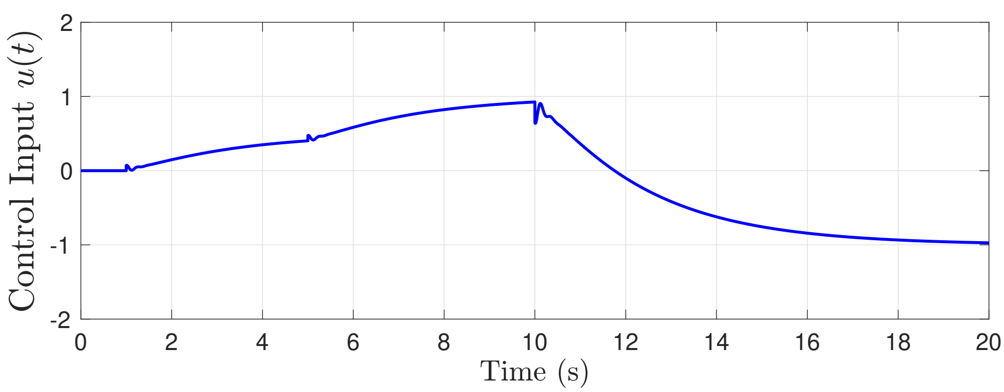
(i) To begin to get a feel for the dynamics in this system, simulate it in `MATLAB` (or even easier `Simulink`) and plot the response of the output and control signals for the following reference command:

$$r(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } 1 \leq t < 5 \\ 2 & \text{if } 5 \leq t < 10 \\ -2 & \text{if } t \geq 10. \end{cases}$$

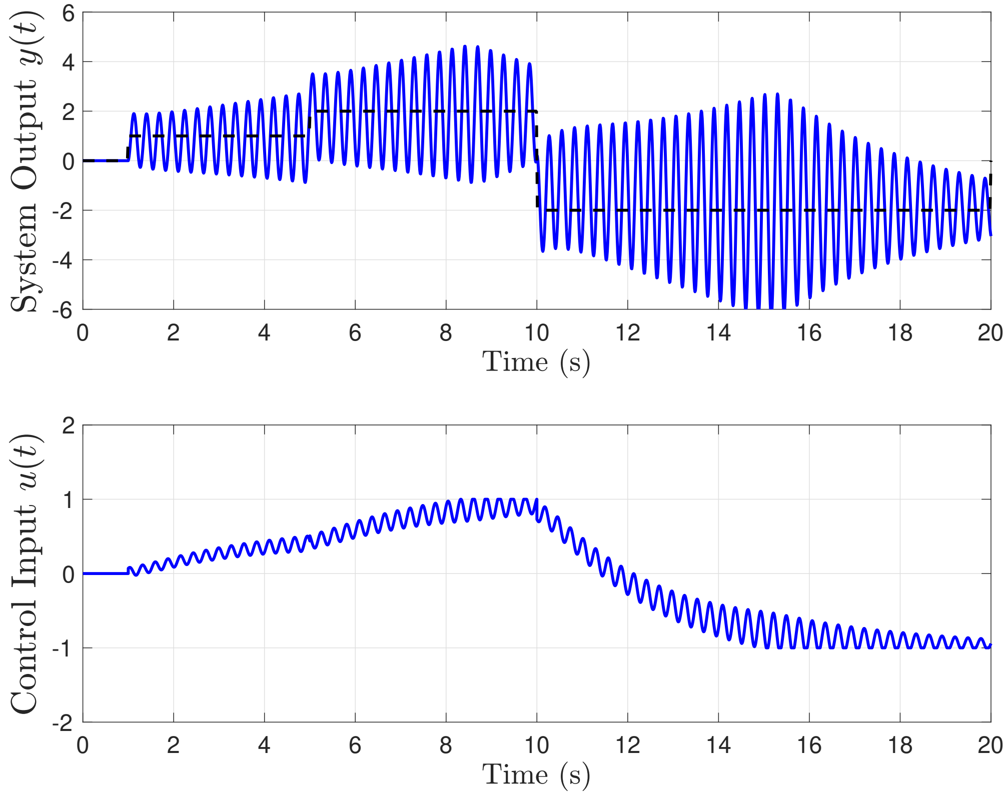
By incrementing τ and re-simulating, roughly determine the critical value of τ at which you observe sustained oscillations in the response.

Solution: The response for the default parameters is shown below (I don't claim this is a particularly well-tuned response; the control signal is quite sluggish

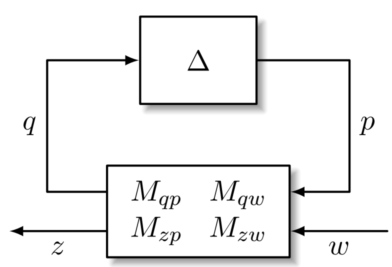




The next plot shows the response when $\tau = 0.04$; we see what effectively sustained oscillations, and the response would no longer be considered stable by any sane engineer.



(ii) Let (A, B, C) denote the system matrices for a minimal state-space realization of $P(s)$. Determine the state-space realization of the system M for which the system in Figure 1 can be written in the standard form



where Δ is the *unit deadzone* nonlinearity. Take $w = r$ as the exogenous input and the true tracking error $z = y - r$ as the performance output.

Solution: We let

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

denote a minimal state-space realization of $P(s)$. The nonlinearity is described by the relationship

$$u = \mathrm{sat}(\tilde{u}) = \tilde{u} - \mathrm{dz}(\tilde{u}).$$

We therefore let $q = \tilde{u}$ and $p = \mathrm{dz}(q) = \Delta(q)$, so the above relationship becomes

$$u = \tilde{u} - p. \tag{2}$$

The sensor model has a minimal realization

$$\begin{aligned} \dot{\xi} &= -\frac{1}{\tau}\xi + \frac{1}{\tau}y \\ y_{\mathrm{m}} &= \xi \end{aligned}$$

or, substituting the plant output

$$\begin{aligned} \dot{\xi} &= -\frac{1}{\tau}\xi + \frac{1}{\tau}Cx \\ y_{\mathrm{m}} &= \xi \end{aligned} \tag{3}$$

The PI controller with backtracking is described by

$$\begin{aligned} \tilde{u} &= K(r - y_{\mathrm{m}}) + \eta \\ \dot{\eta} &= \frac{K}{T_{\mathrm{i}}}(r - y_{\mathrm{m}}) - \frac{1}{T_{\mathrm{t}}}(\tilde{u} - u). \end{aligned}$$

As the reference is the only exogenous input here, we take $w = r$. Substituting into the PI equations, we obtain

$$\begin{aligned} \tilde{u} &= K(w - \xi) + \eta \\ \dot{\eta} &= \frac{K}{T_{\mathrm{i}}}(w - \xi) - \frac{1}{T_{\mathrm{t}}}p. \end{aligned}$$

Substituting into the plant dynamics (1), we obtain

$$\begin{aligned} \dot{x} &= Ax + B(\tilde{u} - p) \\ &= Ax + B[K(w - \xi) + \eta] - Bp \\ &= Ax - BK\xi + B\eta - Bp + BKw \end{aligned}$$

As our performance output, it is natural to select the true error $z = y - r = Cx - w$. Combining all our state-space equations, we have that the desired model M is described by

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\xi} \\ \dot{\eta} \end{bmatrix} &= \begin{bmatrix} A & -BK & B \\ \frac{1}{\tau}C & -\frac{1}{\tau} & 0 \\ 0 & -\frac{K}{T_{\mathrm{i}}} & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \\ \eta \end{bmatrix} + \begin{bmatrix} -B & BK \\ 0 & 0 \\ -\frac{1}{T_{\mathrm{t}}} & \frac{K}{T_{\mathrm{i}}} \end{bmatrix} \begin{bmatrix} p \\ w \end{bmatrix} \\ \begin{bmatrix} q \\ z \end{bmatrix} &= \begin{bmatrix} 0 & -K & 1 \\ C & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \\ \eta \end{bmatrix} + \begin{bmatrix} 0 & K \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p \\ w \end{bmatrix} \end{aligned}$$

More explicitly still, we have the submatrices

$$\left[\begin{array}{c|cc} \mathcal{A} & \mathcal{B}_p & \mathcal{B}_w \\ \hline \mathcal{C}_q & \mathcal{D}_{qp} & \mathcal{D}_{qw} \\ \mathcal{C}_z & \mathcal{D}_{zp} & \mathcal{D}_{zw} \end{array}\right] = \left[\begin{array}{ccc|cc} A & -BK & B & -B & BK \\ \frac{1}{\tau}C & -\frac{1}{\tau} & 0 & 0 & 0 \\ 0 & -\frac{K}{T_{\mathrm{i}}} & 0 & -\frac{1}{T_{\mathrm{t}}} & \frac{K}{T_{\mathrm{i}}} \\ \hline 0 & -K & 1 & 0 & K \\ C & 0 & 0 & 0 & -1 \end{array}\right]$$

(iii) Perform a robust *stability* analysis of this system, and in particular, determine the largest value of τ for which you can certify that the system is robustly stable. Compare this value to your value obtained in part (i), and explain your observations.

Solution: The deadzone nonlinearity is sector-bounded in the sector $[0, 1]$, and hence for any $\sigma \geq 0$ satisfies the pointwise quadratic constraint

$$\begin{bmatrix} q(t) \\ p(t) \end{bmatrix}^\mathsf{T} \underbrace{\begin{bmatrix} 0 & \sigma \\ \sigma & -2\sigma \end{bmatrix}}_{\Pi(\sigma)} \begin{bmatrix} q(t) \\ p(t) \end{bmatrix} \geq 0.$$

We therefore apply the LMI resulting Theorem 9.1 from the notes, and seek to certify whether there exists $P \succ 0$ and $\sigma \geq 0$ satisfying

$$\begin{bmatrix} I_n & 0 \\ \mathcal{A} & \mathcal{B}_p \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ \mathcal{A} & \mathcal{B}_p \end{bmatrix} + \begin{bmatrix} \mathcal{C}_q & \mathcal{D}_{qp} \\ 0 & I_{n_p} \end{bmatrix}^\mathsf{T} \Pi(\sigma) \begin{bmatrix} \mathcal{C}_q & \mathcal{D}_{qp} \\ 0 & I_{n_p} \end{bmatrix} \prec 0.$$

The LMI is feasible at the nominal parameters, which certifies stability of the nominal system. The LMI is feasible up to a value of $\tau = 0.0103$, after which numerical difficulties and infeasibility follow quickly. This is substantially lower than the value we found of 0.04 at which sustained oscillations occur. The gap between these two values arises because the LMI is only a sufficient condition for stability of the feedback system, and is not necessary.

```
plant_ss = minreal(ss(P));
A = plant_ss.A;
B = plant_ss.B;
C = plant_ss.C;
n_plant = size(A,1);

calA = [A,-B*K,B;1/tau*C,-1/tau,0;zeros(1,n_plant),-K/Ti,0];
calB = [-B,B*K;0,0;-1/Tt,K/Ti];
    calBp = calB(:,1);
    calBw = calB(:,2);
calC = [zeros(1,n_plant),-K,1;C,0,0];
    calCq = calC(1,:);
    calCz = calC(2,:);
calD = [0,K;0,-1];
    calDqp = calD(1,1);
    calDqw = calD(1,2);
    calDzp = calD(2,1);
    calDzw = calD(2,2);

alpha = 0;
beta = 1;

n = size(calA,1);
n_p = size(calBp,2);
n_w = size(calBw,2);
n_z = size(calCz,1);
n_q = size(calCq,1);

P = sdpvar(n,n); sigma_sec = sdpvar(1,1);
small = 1e-5;

Lyap = [eye(n),zeros(n,n_p);calA,calBp]'*[zeros(n),P;P,zeros(n)]*...
                [eye(n),zeros(n,n_p);calA,calBp];
Pi = sigma_sec*[-2*alpha*beta,alpha+beta;alpha+beta,-2];
Diss = [calCq,calDqp;zeros(n_p,n),eye(n_p)]'*Pi*...
```

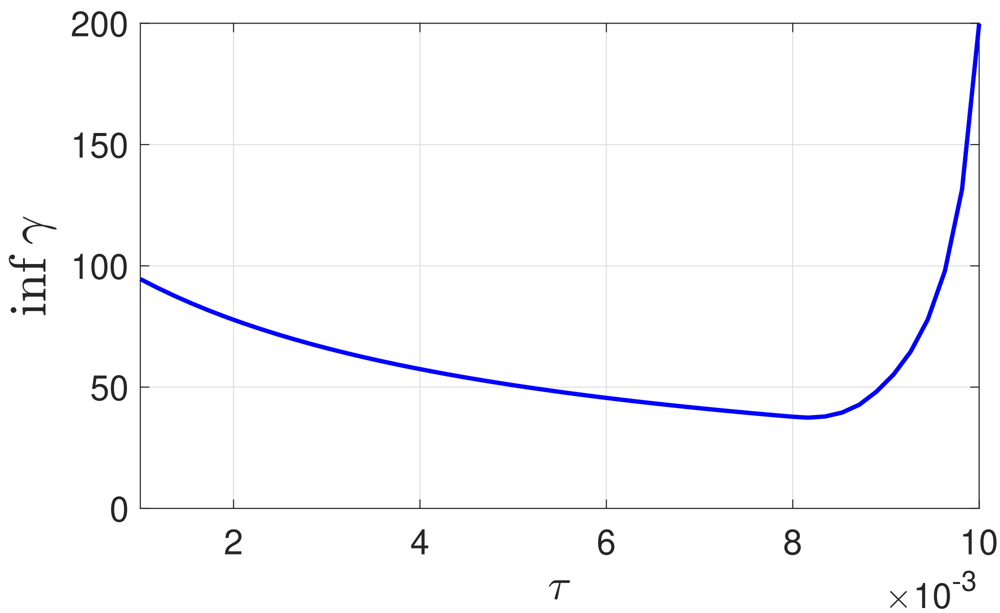
```
                                [calCq,calDqp;zeros(n_p,n),eye(n_p)];

Constraints = [P ≥ small*eye(n), sigma_sec ≥ 0, Lyap + Diss ≤ ...
    -small*eye(n+n_p)];
Cost = 0;
options = sdpsettings('solver','sdpt3','verbose',1);
sol = optimize(Constraints,Cost,options);

sol.info
```

(iv) Perform a robust $\mathcal{L}_2$ performance analysis of this system. In particular, produce a plot of the least upper $\mathcal{L}_2$-gain bound you can certify from w to z, as a function of $\tau \in [0.001, 0.01]$. Interpret your result.

Solution: We now follow Theorem 9.2 and seek to satisfy the LMI described there. Since the performance LMI is sufficient for feasibility of the stability LMI, we know that the performance LMI will be feasible up to at most the value $\tau = 0.0103$. The plot obtained is shown below.



We observe that the best certifiable value of γ is blowing up as τ approaches the point where the LMI becomes infeasible. The interpretation of this of course is that the system is no longer certifiably stable at this point, so the induced $\mathcal{L}_2$-performance becomes arbitrarily bad.

```
tau_list = linspace(0.001,0.01,50)';

gamma = zeros(length(tau_list),1);
for k=1:length(tau_list)

    tau = tau_list(k);

    calA = [A,-B*K,B;1/tau*C,-1/tau,0;zeros(1,n_plant),-K/Ti,0];

    KYP = [eye(n),zeros(n,n_p),zeros(n,n_w);
```

```
            calA, calBp, calBw;
            calCq, calDqp, calDqw;
            zeros(n_p,n), eye(n_p), zeros(n_p,n_w);
            calCz,calDzp, calDzw;
            zeros(n_w,n), zeros(n_w,n_p), eye(n_w)];

    P = sdpvar(n,n); sigma_sec = sdpvar(1,1); gamsq = sdpvar(1,1);
    small = 1e-5;
    Pi = sigma_sec*[-2*alpha*beta,alpha+beta;alpha+beta,-2];
    Pi_p = blkdiag(eye(n_z),-gamsq*eye(n_w));
    Middle = blkdiag([zeros(n),P;P,zeros(n)],Pi,Pi_p);

    Constraints = [P ≥ small*eye(n), sigma_sec ≥ 0, gamsq ≥ 0, ...
        KYP'*Middle*KYP ≤ -small*eye(n+n_p+n_w)];
    Cost = gamsq;
    options = sdpsettings('solver','sdpt3','verbose',1);
    sol = optimize(Constraints,Cost,options);

    gamma(k) = sqrt(value(Cost));
    myInfo(k) = sol.problem;

end

figure('Position', [300, 200, 500, 295]);
plot(tau_list,gamma,'b','linewidth',2);
set(gca, 'FontSize', 16);
ylabel('$\inf \gamma$','interpreter','latex','FontSize',24);
xlabel('$\tau$','interpreter','latex','FontSize',20);
xlim([0.001,0.01])
box on
grid on
```

Problem 2 (Analysis of The (Scalar) Gradient Method)

A continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ is called *strongly convex* with parameter $m > 0$ if

$$(\nabla f(x) - \nabla f(y))(x - y) \geq m(x - y)^2$$

for all $x, y \in \mathbb{R}$, and is called *strongly smooth* with parameter $L > 0$ if

$$(\nabla f(x) - \nabla f(y))^2 \leq L^2(x - y)^2$$

for all $x, y \in \mathbb{R}$. If both these inequalities hold, then necessarily one has that $L \geq m$. Any such function essentially looks like a "bowl", and has a unique minimum value $f^\star$ which is achieved by a unique minimizer $x^\star \in \mathbb{R}$ satisfying $\nabla f(x^\star) = 0$. The classical gradient method is the discrete-time dynamical system

$$x_{k+1} = x_k - \alpha \nabla f(x_k), \qquad k \in \{0, 1, 2, \ldots\},$$

which attempts to iteratively compute the minimizer by moving in the direction of steepest descent. The quantity $\alpha > 0$ is the step size of the method.

(i) Show that the error $e_k = x_k - x^\star$ obeys the nonlinear difference equation

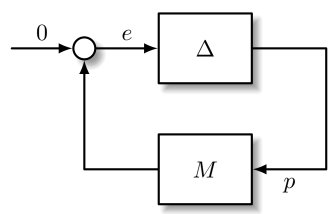
$$e_{k+1} = e_k - \alpha\Phi(e_k)$$

where $\Phi(e) = \nabla f(e + x^\star)$.

Solution: We compute directly that

$$\begin{aligned} e_{k+1} = x_{k+1} - x^\star &= x_k - x^\star - \alpha\nabla f(x_k) \\ &= e_k - \alpha\nabla f(e_k + x^\star) \\ &= e_k - \alpha\Phi(e_k). \end{aligned}$$

(ii) Express this nonlinear system in the form of the feedback interconnection shown in the diagram below where M is a ***stable*** discrete-time LTI system and Δ is a static memoryless nonlinearity



satisfying $|\Delta(e)| \leq \gamma|e|$ for some $\gamma \geq 0$ that you will determine. You may place any restrictions on α that you wish.

Solution: There are many possible choices for how to define the operator Δ here, which will lead to results of varying conservatism when the small gain theorem is applied in the subsequent step; we show one approach. We always have that $L \geq m$, and if $L = m$ the function ∇f must be affine. In this case, the gradient dynamics are in fact a scalar linear system and the analysis is rather trivial, so we assume without loss of generality that $L > m$. Define

$$\Delta(e) = \frac{2}{L-m}\left(\Phi(e) - \frac{m+L}{2}e\right) \tag{4}$$

and set $p = \Delta(e)$. Rearranging this formula, we have that

$$\Phi(e) = \frac{(L-m)}{2}\Delta(e) + \frac{m+L}{2}e.$$

With this, the e-dynamics can be written as

$$M: \quad \begin{aligned} e_{k+1} &= e_k - \alpha\left(\frac{L-m}{2}\Delta(e) + \frac{m+L}{2}e\right) \\ &= \left(1 - \alpha\tfrac{m+L}{2}\right)e_k - \alpha\tfrac{L-m}{2}p_k \end{aligned}$$

subject to the interconnection $p_k = \Delta(e_k)$. Note that with $\alpha_1^\star \triangleq \frac{4}{m+L}$, if $\alpha \in (0, \alpha_1^\star)$, the A matrix of the LTI system is Schur stable, so we make this assumption. Next, note that

$$\begin{aligned} |\Delta(e)|^2 &= \frac{4}{(L-m)^2}\left(\Phi(e) - \frac{m+L}{2}e\right)^2 \\ &= \frac{4}{(L-m)^2}\left(\Phi(e)^2 + \frac{(m+L)^2}{4}e^2 - (m+L)\Phi(e)e\right). \end{aligned}$$

To obtain the best possible upper bound on this quantity requires a bit of care. First, from strong convexity we have that

$$\begin{aligned}\Phi(e)e &= \nabla f(e+x^\star)e = (\nabla f(x) - \nabla f(x^\star))(x-x^\star)\\ &\geq m(x-x^\star)^2 = me^2.\end{aligned}$$

Similarly, due to L-smoothness we have

$$\Phi(e)e \leq |\Phi(e)e| = |\Phi(e)||e| \leq Le^2,$$

so we conclude that

$$\begin{aligned}(Le - \Phi(e))e \geq 0\\ (me - \Phi(e))e \leq 0.\end{aligned}$$

These two inequalities are satisfied if and only if

$$(Le - \Phi(e))(me - \Phi(e)) \leq 0$$

from which it follows that

$$\Phi(e)e \geq \frac{mL}{m+L}e^2 + \frac{1}{m+L}\Phi(e)^2.$$

Inserting this into our inequality for $|\Delta(e)|$, we find that

$$\begin{aligned}|\Delta(e)|^2 &\leq \frac{4}{(L-m)^2}\left(\Phi(e)^2 + \frac{(m+L)^2}{4}e^2 - mLe^2 - \Phi(e)^2\right)\\ &= \frac{4}{(L-m)^2}\left(\frac{(m+L)^2}{4}e^2 - \frac{4mL}{4}e^2\right)\\ &= e^2\end{aligned}$$

and therefore $|\Delta(e)| \leq |e|$.

(iii) Use the small gain theorem and your LMI from Assignment 3 Problem 3 to derive sufficient conditions on α to ensure stability.

Solution: By the small-gain theorem, the closed-loop system will be stable if the product of the ℓ_2 gains of M and Δ is strictly less than one. We therefore want the ℓ_2 gain of M to be strictly less than 1. The LMI of Assignment 3 said that the induced ℓ_2-gain of M would be less than γ if there exists $P \succ 0$ such that

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^\mathsf{T} \begin{bmatrix} -P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} - \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} -I & 0 \\ 0 & \gamma^2 I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \prec 0$$

Specializing this to the current scenario, we are asking for the existence of a constant $p > 0$ such that

$$\begin{bmatrix} 1 & 0 \\ \left(1-\alpha\frac{m+L}{2}\right) & -\alpha\frac{L-m}{2} \end{bmatrix}^\mathsf{T} \begin{bmatrix} -p & 0 \\ 0 & p \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \left(1-\alpha\frac{m+L}{2}\right) & -\alpha\frac{L-m}{2} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^\mathsf{T} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \prec 0$$

or more simply

$$\begin{bmatrix} p[1-\left(1-\alpha\frac{m+L}{2}\right)^2]-1 & \alpha\frac{L-m}{2}p\left(1-\alpha\frac{m+L}{2}\right) \\ \alpha\frac{L-m}{2}p\left(1-\alpha\frac{m+L}{2}\right) & 1-\alpha^2\frac{(L-m)^2}{4}p \end{bmatrix} \succ 0.$$

We can simplify our lives here by selecting p to equalize the (1,1) and (2,2) elements of this matrix. It is not completely obvious that this is a good decision at this point, as it could lead to conservatism in the selection of α, or even to an inequality that is infeasible for all α. The rationale however is that the (2,2) element is a decreasing function of α, while the (1,1) element is an increasing function of α, so it seems reasonable to try to equalize the positivity between the (1,1) and (2,2) blocks. Thus, if we require that

$$p\left[1-\left(1-\alpha\frac{m+L}{2}\right)^2\right]-1 = 1-\alpha^2\frac{(L-m)^2}{4}p$$

then we find that the required value of p is

$$p = \frac{2}{\alpha}\frac{1}{L+m-\alpha mL}$$

which is well-defined only under the condition that

$$\alpha \in (0, \alpha_2^\star), \qquad \alpha_2^\star \triangleq \frac{L+m}{mL}.$$

With this particular choice of p, the inequality becomes

$$\begin{bmatrix} 1-\alpha\frac{(L-m)^2}{2}\frac{1}{L+m-\alpha mL} & \frac{L-m}{L+m-\alpha mL}\left(1-\alpha\frac{m+L}{2}\right) \\ \frac{L-m}{L+m-\alpha mL}\left(1-\alpha\frac{m+L}{2}\right) & 1-\alpha\frac{(L-m)^2}{2}\frac{1}{L+m-\alpha mL} \end{bmatrix} \succ 0.$$

Since $\alpha \in (0, \alpha_2^\star)$ we can multiply through by $L+m-\alpha mL > 0$ and to obtain

$$\begin{bmatrix} L+m-\alpha mL-\alpha\frac{(L-m)^2}{2} & (L-m)\left(1-\alpha\frac{m+L}{2}\right) \\ (L-m)\left(1-\alpha\frac{m+L}{2}\right) & L+m-\alpha mL-\alpha\frac{(L-m)^2}{2} \end{bmatrix} \succ 0.$$

which after simplification becomes

$$\begin{bmatrix} L+m-\frac{\alpha}{2}(L^2+m^2) & (L-m)\left(1-\alpha\frac{m+L}{2}\right) \\ (L-m)\left(1-\alpha\frac{m+L}{2}\right) & L+m-\frac{\alpha}{2}(L^2+m^2) \end{bmatrix} \succ 0.$$

Assuming now that

$$\alpha \in (0, \alpha_3^\star), \qquad \alpha_3^\star \triangleq (0, \tfrac{2(L+m)}{L^2+m^2})$$

the (1,1) block is positive definite, and the overall matrix is positive definite if and only if the determinant is positive. Some algebra shows that this simplifies to the inequality

$$(\alpha L-2)(\alpha m-2) > 0$$

which, since $L > m$, holds if and only if $\alpha \in (0, 2/L)$. Thus, we have established that if

$$\alpha \in (0, \min\{\alpha_1^\star, \alpha_2^\star, \alpha_3^\star, 2/L\})$$

then the equilibrium $x^\star$ is globally exponentially stable. Note that

$$\alpha_1^\star = \frac{4}{m+L} = \frac{2}{L} + \frac{2(L-m)}{L(m+L)} > \frac{2}{L}$$

and

$$\alpha_2^\star = \frac{L+m}{mL} = \frac{2}{L} + \frac{L-m}{mL} > \frac{2}{L}$$

and

$$\alpha_3^\star = \frac{2L+2m}{L^2+m^2} = \frac{2}{L} + \frac{2m(L-m)}{L^3+m^2L} > \frac{2}{L}$$

so the inequality $\alpha \in (0, 2/L)$ implies all the other inequalities that we used. Thus, $\alpha \in (0, 2/L)$ is a sufficient condition for stability of gradient descent, and in fact this is the standard maximum permissible step size, which can be found in any optimization book containing an analysis of the gradient method with fixed step size.

n.b.: The solution pathway I took here was obviously not painless; it is likely there are other choices one could have made which may have simplified the calculations. If you have a simpler approach, feel free to share it on Piazza!

n.b.: The same analysis works with almost no changes in the vector case, and the same condition results.