LMIs for Joint State Estimation and Model Predictive Control

ECE1635 Final Project Presentation

Institute for Aerospace Studies
UNIVERSITY OF TORONTO

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SYSTEMS LAB

Motivation

- Safely control real-world robots

- Robotic systems: Uncertain and noisy partial state measurements
- State feedback not available → **Output feedback**

Problem Statement

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- Uncertain LTI system:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{E}\mathbf{w}_k \ ,$$
$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{F}\mathbf{v}_k \ ,$$

- Subject to state and input constraints
- Assumption: Bounded disturbances

$$\mathbf{w}_k \in \mathbb{W} = \{\mathbf{w}_k^\intercal \mathbf{w}_k \leq 1\} \, , \mathbf{v}_k \in \mathbb{V} = \{\mathbf{v}_k^\intercal \mathbf{v}_k \leq 1\}$$

- Goal: Solve the infinite horizon optimal control problem

Model Predictive Control (MPC)

- MPC approximates infinite horizon problem (state feedback, no noise):

$$V_{k\to k+N}(\mathbf{x}_k) = \min_{\mathbf{u}_{k:k+N-1|k}} \sum_{t=k}^{k+N-1} \mathbf{x}_{t|k}^{\intercal}\mathbf{Q}\mathbf{x}_{t|k} + \mathbf{u}_{t|k}^{\intercal}\mathbf{R}\mathbf{u}_{t|k}$$

$$\begin{aligned} \text{s.t.} \quad & \forall t \in \{k, \ldots, k+N-1\}, \\ & \mathbf{x}_{t+1|k} = \mathbf{A}\mathbf{x}_{t|k} + \mathbf{B}\mathbf{u}_{t|k}, \\ & \mathbf{x}_{t|k} \in \mathbb{X}, \mathbf{u}_{t|k} \in \mathbb{U}, \\ & \mathbf{x}_{k|k} = \mathbf{x}_k. \end{aligned}$$

- Algorithm:
  - Solve for current time step
  - Apply the first input
  - Repeat

Approaches to Robust Output Feedback MPC

- Min-max MPC
  - Min-max optimization problem (state feedback):

$$V_{k\to k+N}(\mathbf{x}_k) = \min_{\mathbf{u}_{k:k+N-1|k}} \max_{\mathbf{w}_{k:k+N-1|k}} \sum_{t=k}^{k+N-1} \mathbf{x}_{t|k}^{\mathsf{T}} \mathbf{Q} \mathbf{x}_{t|k} + \mathbf{u}_{t|k}^{\mathsf{T}} \mathbf{R} \mathbf{u}_{t|k}$$

$$\begin{aligned} \text{s.t.} \quad & \forall t \in \{k, \dots, k+N-1\}, \\ & \mathbf{x}_{t+1|k} = \mathbf{A}\mathbf{x}_{t|k} + \mathbf{B}\mathbf{u}_{t|k} + \mathbf{E}\mathbf{w}_{t|k}, \\ & \mathbf{x}_{t|k} \in \mathbb{X}, \mathbf{u}_{t|k} \in \mathbb{U}, \mathbf{w}_{t|k} \in \mathbb{W}, \\ & \mathbf{x}_{k|k} = \mathbf{x}_k . \end{aligned}$$

Approaches to Robust Output Feedback MPC

- Min-max MPC
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$$V_{k\rightarrow k+N}(\mathbf{x}_k) = \min_{\mathbf{u}_{k:k+N-1|k}} \max_{\mathbf{w}_{k:k+N-1|k}} \sum_{t=k}^{k+N-1} \mathbf{x}_{t|k}^{\mathsf{T}}\mathbf{Q}\mathbf{x}_{t|k} + \mathbf{u}_{t|k}^{\mathsf{T}}\mathbf{R}\mathbf{u}_{t|k}$$

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  - [Löfberg02, CoppHespanha17]: Joint estimation and control optimization
  - **[Löfberg02] formulates an SDP!**

Robust LMI

Theorem 1 (Robust LMI for Affine Uncertainty [ElGhaouiEtal98, Löfberg03]). *Let* $\mathbf{D}, \mathbf{L}, \mathbf{R}$, *and* Δ *be real matrices of appropriate size. Robust satisfaction of the uncertain matrix inequality*

$$\mathbf{D} + \mathbf{L}\Delta\mathbf{R} + \mathbf{R}^\intercal\Delta^\intercal\mathbf{L}^\intercal \succeq \mathbf{0}\,, \forall \|\Delta\|_2 \leq 1\,,$$

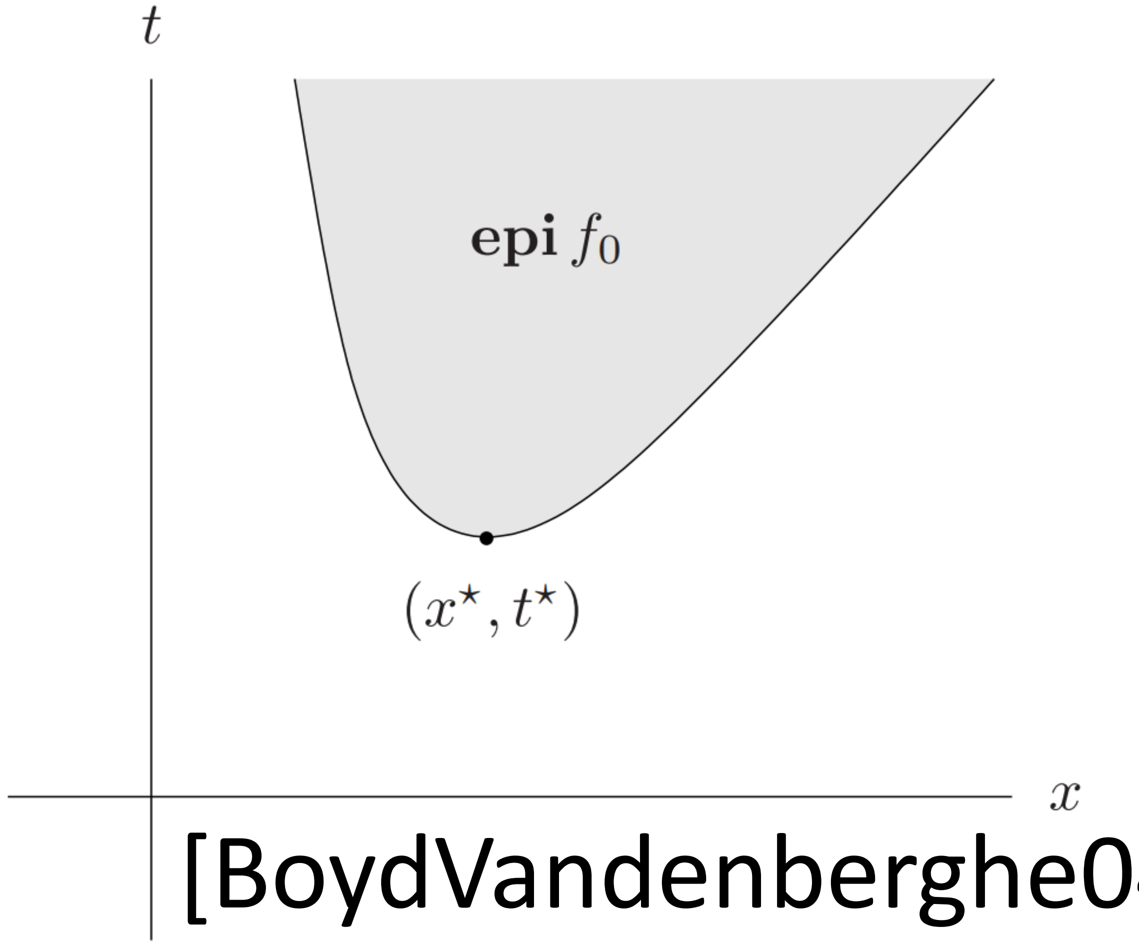
is equivalent to the matrix inequality

$$\begin{pmatrix} \mathbf{D} & \mathbf{L} \\ \mathbf{L}^\intercal & \mathbf{0} \end{pmatrix} \succeq \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}^\intercal \begin{pmatrix} \tau\mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\tau\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \tau \geq 0\,.$$

Rewriting Min-Max Optimization Problem

- Epigraph form

$$\min_{\mathbf{x}} \quad \mathbf{f}(\mathbf{x})$$

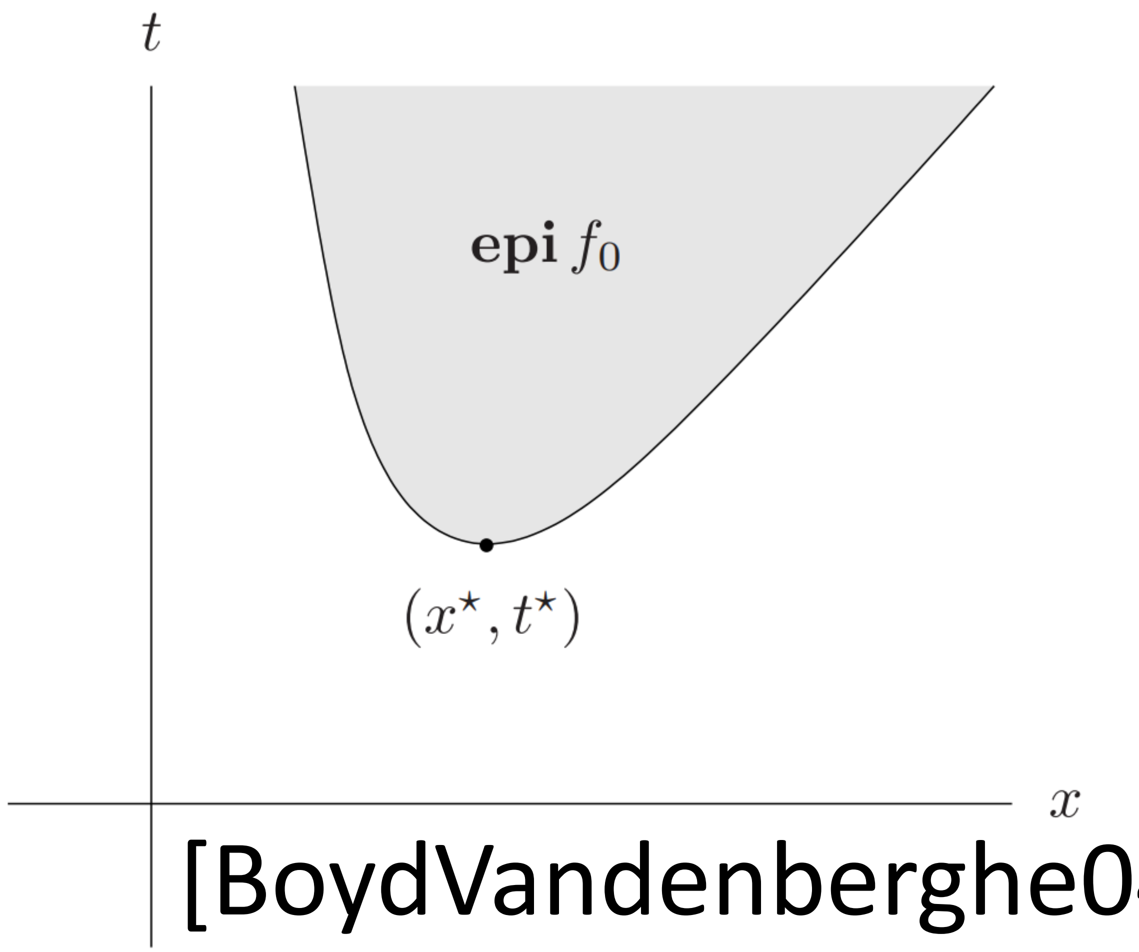


[BoydVandenberghe04]

Rewriting Min-Max Optimization Problem

- Epigraph form

$$\min_{\mathbf{x}} \quad \mathbf{f}(\mathbf{x}) \quad \Longleftrightarrow \quad \begin{aligned} \min_{\mathbf{x},t} \quad & t \\ \text{s.t.} \quad & \mathbf{f}(\mathbf{x}) \leq t \end{aligned}$$



[BoydVandenberghe04]

Rewriting Min-Max Optimization Problem

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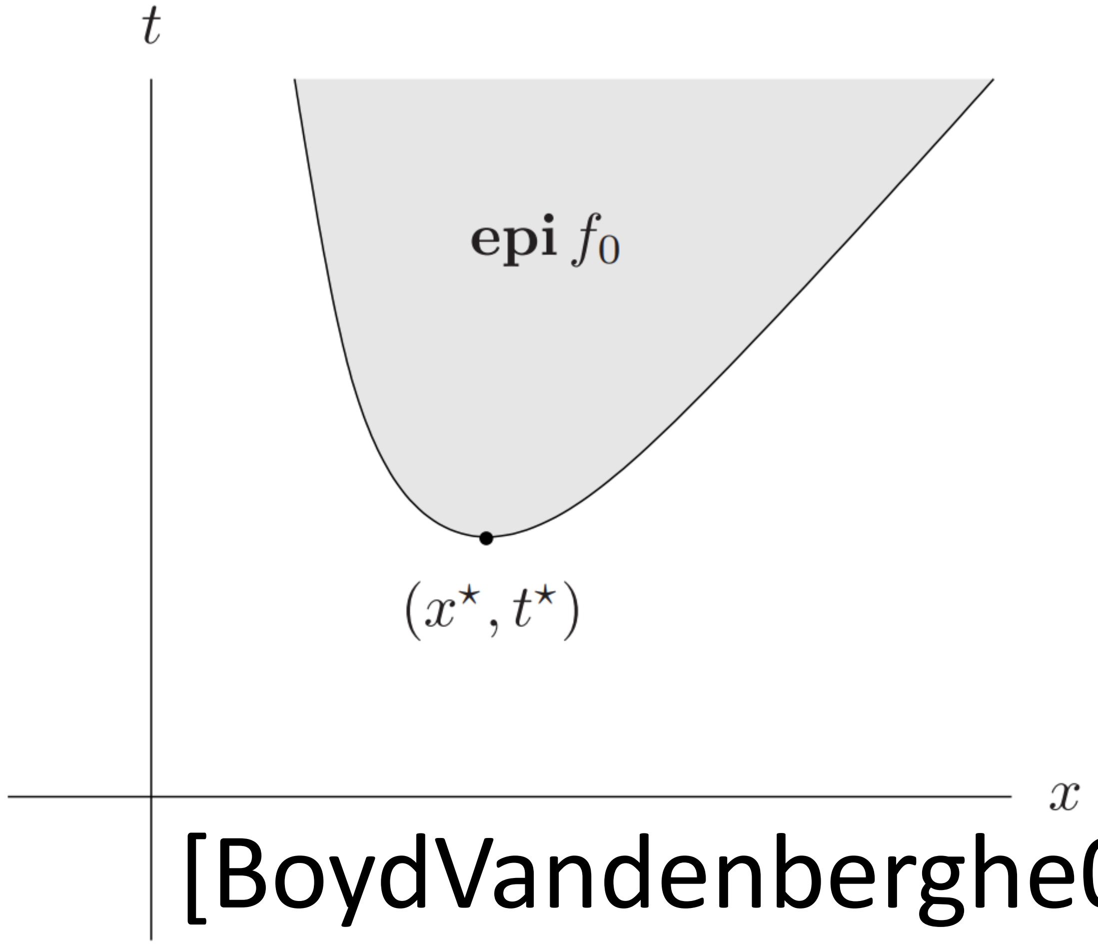
- Min-max optimization

$$\min_{\mathbf{x}} \max_{\mathbf{w}} \quad \mathbf{f}(\mathbf{x}, \mathbf{w})$$



[BoydVandenberghe04]

Rewriting Min-Max Optimization Problem

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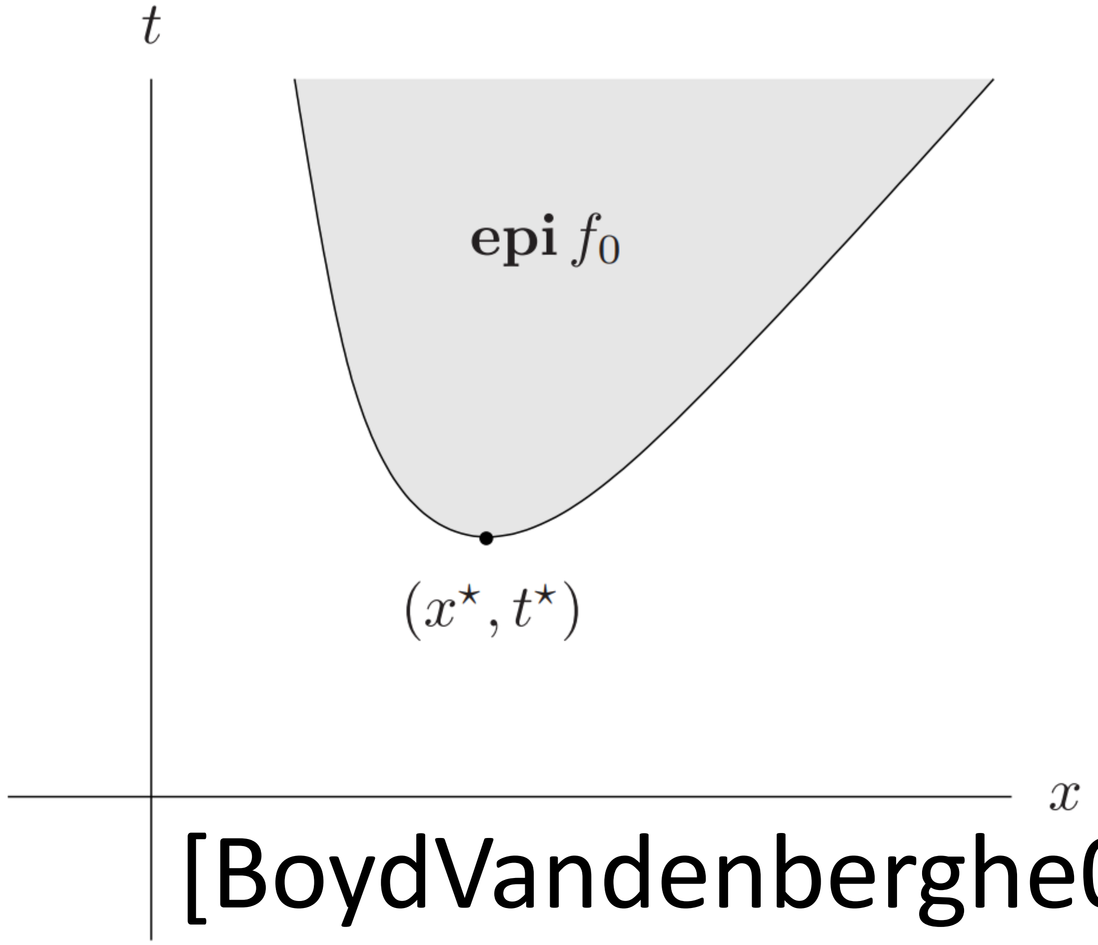


[BoydVandenberghe04]

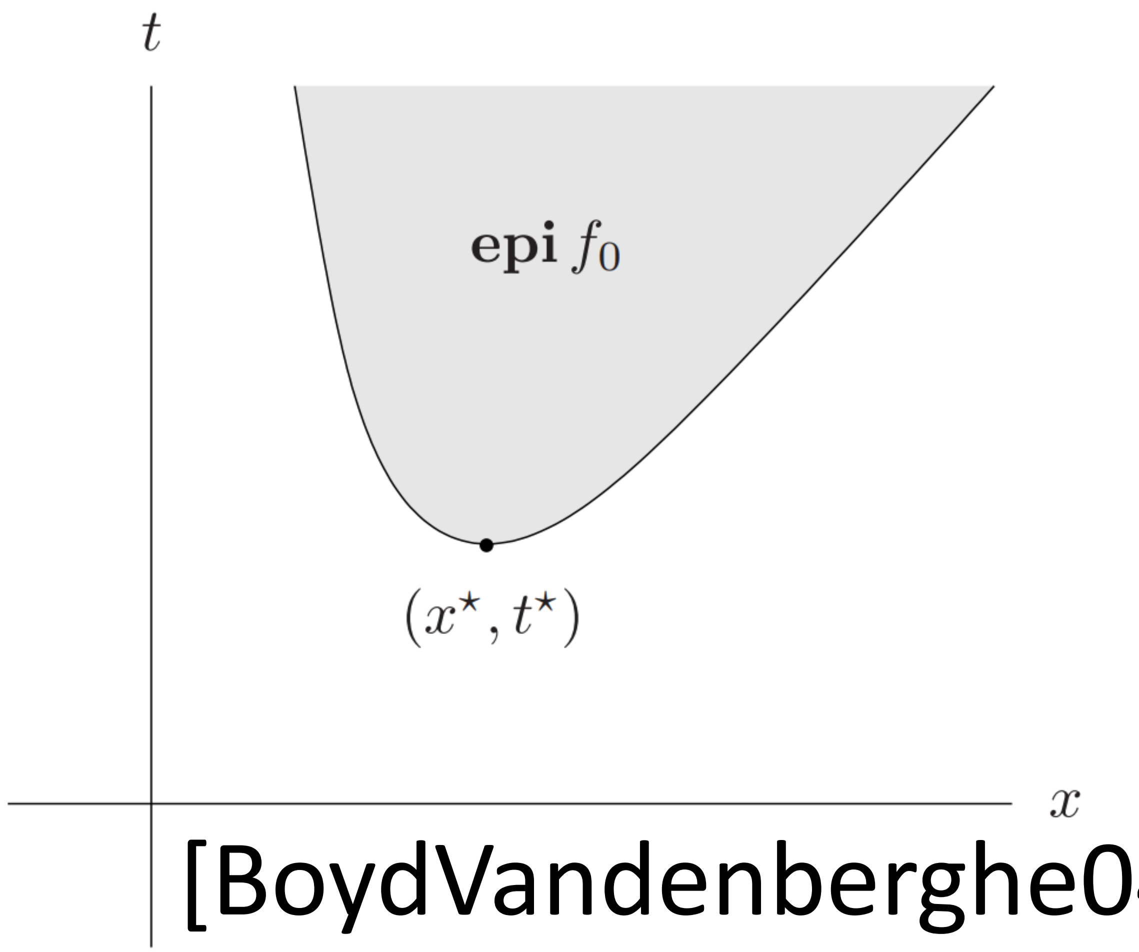
- Min-max optimization

$$\min_{\mathbf{x}} \max_{\mathbf{w}} \quad \mathbf{f}(\mathbf{x}, \mathbf{w}) \iff \begin{aligned} \min_{\mathbf{x},t} \quad & t \\ \text{s.t.} \quad & \max_{\mathbf{w}} \ \mathbf{f}(\mathbf{x}, \mathbf{w}) \leq t \end{aligned}$$

Rewriting Min-Max Optimization Problem

- Epigraph form

$$\min_{\mathbf{x}} \quad \mathbf{f}(\mathbf{x}) \iff \begin{aligned} \min_{\mathbf{x},t} \quad & t \\ \text{s.t.} \quad & \mathbf{f}(\mathbf{x}) \le t \end{aligned}$$



[BoydVandenberghe04]

- Min-max optimization

$$\min_{\mathbf{x}} \max_{\mathbf{w}} \quad \mathbf{f}(\mathbf{x}, \mathbf{w}) \iff \begin{aligned} \min_{\mathbf{x},t} \quad & t \\ \text{s.t.} \quad & \mathbf{f}(\mathbf{x}, \mathbf{w}) \le t\,, \forall \mathbf{w} \in \mathbb{W}\,. \end{aligned}$$

LMI for Min-Max MPC

- Assumption: We are given state estimate and its uncertainty
- Stacked vectors and matrices

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_{k+1|k} \\ \vdots \\ \mathbf{x}_{k+N|k} \end{pmatrix}, \ \mathbf{U} = \begin{pmatrix} \mathbf{u}_{k|k} \\ \vdots \\ \mathbf{u}_{k+N-1|k} \end{pmatrix}, \ \mathbf{W} = \begin{pmatrix} \mathbf{w}_{k|k} \\ \vdots \\ \mathbf{w}_{k+N-1|k} \end{pmatrix},$$

$$\mathbf{H} = \begin{pmatrix} \mathbf{A} \\ \mathbf{A}^2 \\ \vdots \\ \mathbf{A}^{N-1} \end{pmatrix}, \mathbf{S} = \begin{pmatrix} \mathbf{B} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{AB} & \mathbf{B} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}^{N-1}\mathbf{B} & \dots & \mathbf{AB} & \mathbf{B} \end{pmatrix}, \mathbf{G} = \begin{pmatrix} \mathbf{E} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{AE} & \mathbf{E} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}^{N-1}\mathbf{E} & \dots & \mathbf{AE} & \mathbf{E} \end{pmatrix}$$

$$\bar{\mathbf{Q}} = \mathrm{diag}(\mathbf{Q}, \dots \mathbf{Q}), \bar{\mathbf{R}} = \mathrm{diag}(\mathbf{R}, \dots \mathbf{R})$$

LMI for Min-Max MPC

- Dynamics in short-hand notation

$$\mathbf{X} = \mathbf{H}\mathbf{x}_k + \mathbf{S}\mathbf{U} + \mathbf{G}\mathbf{W}$$

LMI for Min-Max MPC

- Dynamics in short-hand notation

$$\mathbf{X} = \bar{\mathbf{X}} + \mathbf{H}\mathbf{Z}_k \mathbf{z}_k + \mathbf{G}\mathbf{W}$$

LMI for Min-Max MPC

- Dynamics in short-hand notation

$$\mathbf{X} = \bar{\mathbf{X}} + \mathbf{H}\mathbf{Z}_k \mathbf{z}_k + \mathbf{G}\mathbf{W}$$

- Epigraph form of min-max MPC

$$\min_{\mathbf{U},t} \max_{\mathbf{z}_k,\mathbf{w}_k} \quad \mathbf{X}^\intercal \bar{\mathbf{Q}} \mathbf{X} + \mathbf{U}^\intercal \bar{\mathbf{R}} \mathbf{U}$$

LMI for Min-Max MPC

- Dynamics in short-hand notation

$$\mathbf{X} = \bar{\mathbf{X}} + \mathbf{H}\mathbf{Z}_k \mathbf{z}_k + \mathbf{G}\mathbf{W}$$

- Epigraph form of min-max MPC

$$\min_{\mathbf{U},t} \quad t$$

$$\text{s.t.} \quad \mathbf{X}^\intercal \bar{\mathbf{Q}} \mathbf{X} + \mathbf{U}^\intercal \bar{\mathbf{R}} \mathbf{U} \leq t \,, \forall \mathbf{z}_k \in \mathbb{Z}, \forall \mathbf{w}_k \in \mathbb{W} \,,$$

LMI for Min-Max MPC

- Dynamics in short-hand notation

$$\mathbf{X} = \bar{\mathbf{X}} + \mathbf{H}\mathbf{Z}_k \mathbf{z}_k + \mathbf{G}\mathbf{W}$$

- Epigraph form of min-max MPC

$$\min_{\mathbf{U},t} \quad t$$
$$\text{s.t.} \quad \mathbf{X}^\intercal \bar{\mathbf{Q}} \mathbf{X} + \mathbf{U}^\intercal \bar{\mathbf{R}} \mathbf{U} \leq t \,, \forall \mathbf{z}_k \in \mathbb{Z}, \forall \mathbf{w}_k \in \mathbb{W} \,,$$

- Schur complement of constraint

$$t - \begin{pmatrix} \mathbf{X} \\ \mathbf{U} \end{pmatrix}^\intercal \begin{pmatrix} \bar{\mathbf{Q}} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{R}} \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{U} \end{pmatrix} \geq 0 \iff \begin{pmatrix} t & (\bar{\mathbf{X}} + \mathbf{H}\mathbf{Z}_k \mathbf{z}_k + \mathbf{G}\mathbf{W})^\intercal & \mathbf{U}^\intercal \\ (\star)^\intercal & \bar{\mathbf{Q}}^{-1} & \mathbf{0} \\ (\star)^\intercal & (\star)^\intercal & \bar{\mathbf{R}}^{-1} \end{pmatrix} \succeq \mathbf{0}$$

LMI for Min-Max MPC

- Extract state estimate uncertainty

$$\begin{pmatrix} t & (\bar{\mathbf{X}} + \mathbf{G}\mathbf{W})^\intercal & \mathbf{U}^\intercal \\ (\star)^\intercal & \bar{\mathbf{Q}}^{-1} & \mathbf{0} \\ (\star)^\intercal & (\star)^\intercal & \bar{\mathbf{R}}^{-1} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{H}\mathbf{Z}_k \\ \mathbf{0} \end{pmatrix} \mathbf{z}_k \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \mathbf{z}_k^\intercal \begin{pmatrix} \mathbf{0} & \mathbf{Z}_k\mathbf{H} & \mathbf{0} \end{pmatrix} \succeq 0$$

LMI for Min-Max MPC

- Extract state estimate uncertainty

$$\begin{pmatrix} t & (\bar{\mathbf{X}}+\mathbf{G}\mathbf{W})^\intercal & \mathbf{U}^\intercal \\ (\star)^\intercal & \bar{\mathbf{Q}}^{-1} & \mathbf{0} \\ (\star)^\intercal & (\star)^\intercal & \bar{\mathbf{R}}^{-1} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \mathbf{H}\mathbf{Z}_k \\ \mathbf{0} \end{pmatrix} \mathbf{z}_k \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \mathbf{z}_k^\intercal \begin{pmatrix} \mathbf{0} & \mathbf{Z}_k\mathbf{H} & \mathbf{0} \end{pmatrix} \succeq 0$$

- Apply Theorem 1 and simplify

$$\begin{pmatrix} t-\tau_x & (\bar{\mathbf{X}}+\mathbf{G}\mathbf{W})^\intercal & \mathbf{U}^\intercal & \mathbf{0} \\ (\star)^\intercal & \bar{\mathbf{Q}}^{-1} & \mathbf{0} & \mathbf{H}\mathbf{Z}_k \\ (\star)^\intercal & (\star)^\intercal & \bar{\mathbf{R}}^{-1} & \mathbf{0} \\ (\star)^\intercal & (\star)^\intercal & (\star)^\intercal & \tau_x \mathbf{I} \end{pmatrix} \succeq \mathbf{0}\,, \quad \tau_x \geq 0$$

- Repeat for remaining uncertainties!

LMI for Min-Max MPC

- SDP for min-max MPC

$$\min_{\mathbf{U},t,\tau_x,\Omega} \quad t$$

$$\text{s.t.} \quad \begin{pmatrix} t-\tau_x-\mathrm{trace}(\Omega) & \bar{\mathbf{X}}^\intercal & \mathbf{U}^\intercal & \mathbf{0} & \mathbf{0} \\ (\star)^\intercal & \bar{\mathbf{Q}}^{-1} & \mathbf{0} & \mathbf{H}\mathbf{Z}_k & \mathbf{G} \\ (\star)^\intercal & (\star)^\intercal & \bar{\mathbf{R}}^{-1} & \mathbf{0} & \mathbf{0} \\ (\star)^\intercal & (\star)^\intercal & (\star)^\intercal & \tau_x\mathbf{I} & \mathbf{0} \\ (\star)^\intercal & (\star)^\intercal & (\star)^\intercal & (\star)^\intercal & \Omega \end{pmatrix} \succeq \mathbf{0}\,,$$

- Add state estimation LMI

$$\min_{\mathbf{U},\mathbf{Z}_k,\hat{\mathbf{x}}_k,t,\tau_x,\Omega} \quad t$$

$$\text{s.t.} \quad \begin{pmatrix} t-\tau_x-\text{trace}(\Omega) & \bar{\mathbf{X}}^\intercal & \mathbf{U}^\intercal & \mathbf{0} & \mathbf{0} \\ (\star)^\intercal & \bar{\mathbf{Q}}^{-1} & \mathbf{0} & \mathbf{H}\mathbf{Z}_k & \mathbf{G} \\ (\star)^\intercal & (\star)^\intercal & \bar{\mathbf{R}}^{-1} & \mathbf{0} & \mathbf{0} \\ (\star)^\intercal & (\star)^\intercal & (\star)^\intercal & \tau_x\mathbf{I} & \mathbf{0} \\ (\star)^\intercal & (\star)^\intercal & (\star)^\intercal & (\star)^\intercal & \Omega \end{pmatrix} \succeq \mathbf{0},$$

$$\begin{pmatrix} \Gamma & \mathbf{T}_{\mathbf{e}_k}^\intercal \\ \mathbf{T}_{\mathbf{e}_k} & \mathbf{Z}_k\mathbf{Z}_0 + \mathbf{Z}_0^\intercal\mathbf{Z}_k - (\mathbf{Z}_0^\intercal)^2 \end{pmatrix} \succeq \mathbf{0}$$

Simulation

- Stabilization task on double integrator

$$\mathbf{x}_{k+1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x}_k + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbf{u}_k$$

$$\mathbf{y}_k = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x}_k + 0.05\mathbf{v}_k \,,$$

$$\mathbb{X} = \{ \begin{pmatrix} x_1 & x_2 \end{pmatrix}^\mathsf{T} \in \mathbb{R}^2 : x_2 \leq 1.3 \} \,,$$

$$\mathbb{U} = \{ u \in \mathbb{R} : u \in [-10, 10] \} \,,$$

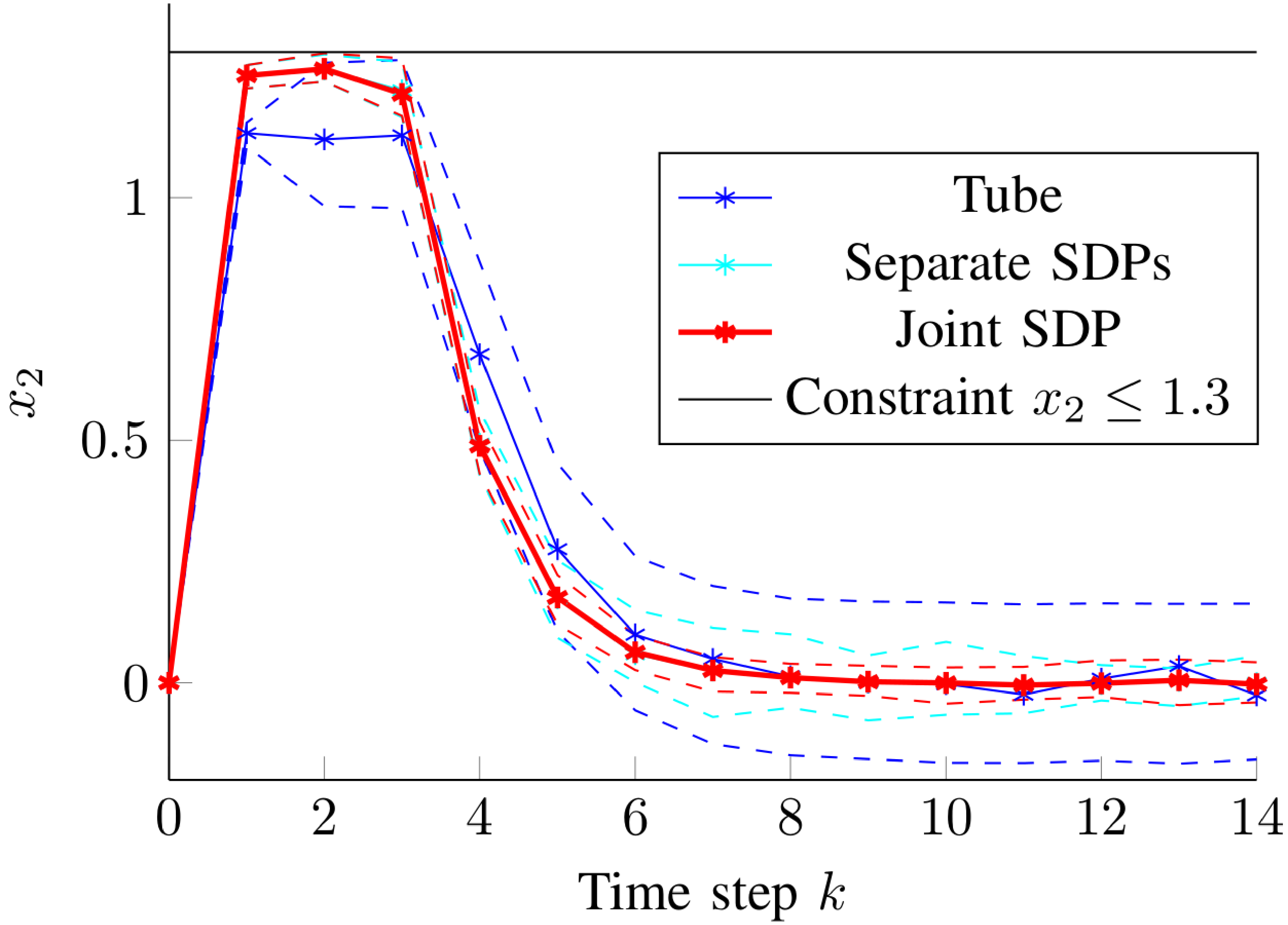
$$\mathbb{V} = \{ v \in \mathbb{R} : |v| \leq 1 \}$$

- Comparing:

1. Joint SDP [Löfberg02]
2. Separate SDPs [Löfberg01]
3. Tube-based MPC [LorenzettiPavone20]

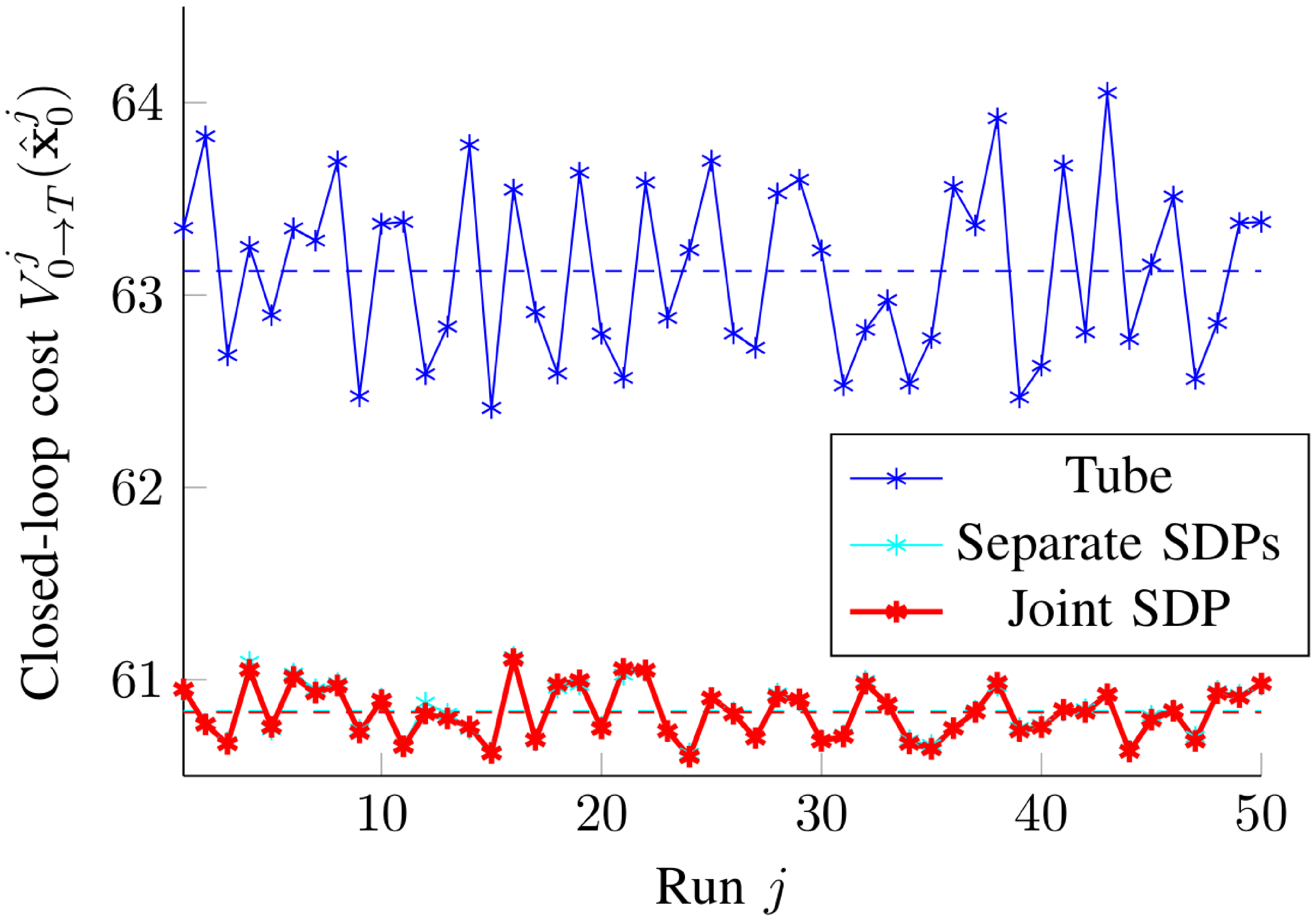
Comparison Results

- Conservativeness



Comparison Results

- Performance



Comparison Results

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- Computational time per run

Controller	Elapsed time [s] Average	Elapsed time [s] Standard deviation
Tube	**18.40**	**2.67**
Separate SDPs	20.43	5.34
Joint SDPs	24.55	2.79

Conclusions

- Reviewed the joint state estimation and MPC approach in [Löfberg02]
- Derived the LMIs step-by-step
- Future work:
  - Further investigate performance
  - Add terminal state set and terminal cost to guarantee stability

Thank you!

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