Input/Output Analysis of Primal-Dual Gradient Algorithms

54th Annual Allerton Conference

Monticello, IL

John W. Simpson-Porco

UNIVERSITY OF WATERLOO

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synchronous generator

Power electronics are changing the power grid . . .

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power inverters

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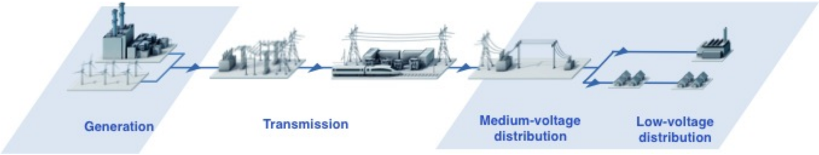
location & distributed implementation



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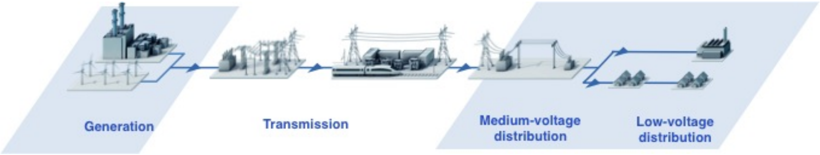

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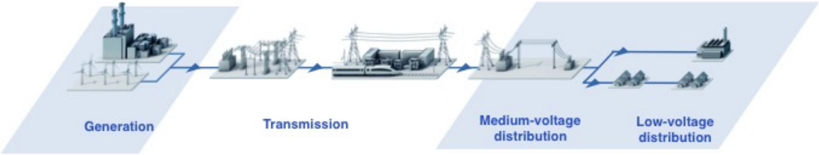
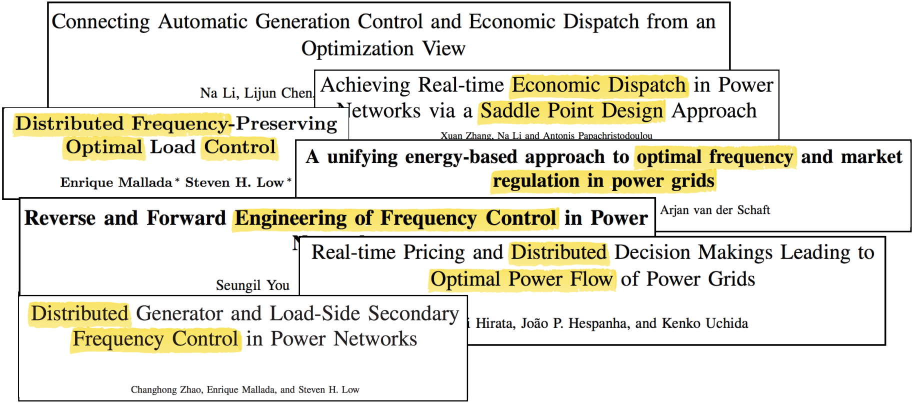


Central control authority is fading . . . how to coordinate new actuators?

Optimal Distributed Frequency Regulation

via primal-dual algorithms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Popular Idea: {Grid Dynamics} $\cup$ {Dist. Controller}

= Distributed **Online Optimization Algorithm**

(many other approaches exist, but today we'll focus on these)

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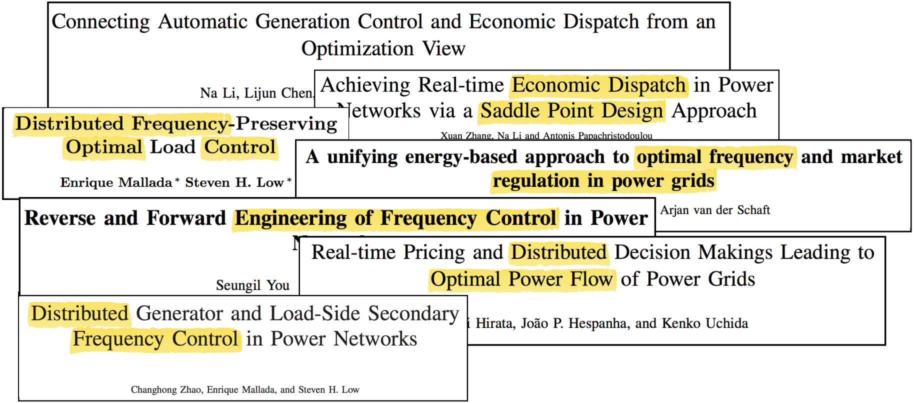

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A Distributed Optimization Problem

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) = \sum\nolimits_{i=1}^{n} f_i(x_i) \\ \text{subject to} \quad & Sx = b\,, \end{aligned}$$

- $x \in \mathbb{R}^n$ is our decision variable, $f_i(x_i)$ is strictly convex, differentiable
- $S \in \mathbb{R}^{r \times n}$ with $r < n$ is full rank

Each variable $x_i \in \mathbb{R}$ belongs to **agent**.

The constraint matrix S **couples** the agents together.

We want the agents to **cooperatively** solve this problem.

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The Primal-Dual / Saddle Point Algorithm

Key Idea: Write an algorithm which *seeks* the KKT points of

$$L(x,\nu) = f(x) + \nu^{\mathsf{T}}(Sx - b)\,.$$

Saddle Point Algorithm [Kose '56, Arrow *et al.* '58]

$$\begin{aligned} \tau_x \dot{x} &= -\nabla_x L(x,\nu) \\ \tau_\nu \dot{\nu} &= \nabla_\nu L(x,\nu) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \tau_x \dot{x} &= -\nabla f(x) - S^{\mathsf{T}}\nu\,, \\ \tau_\nu \dot{\nu} &= Sx - b\,, \end{aligned}$$

1. Dynamics are **distributed** with sparsity of S
2. Converges to unique primal-dual optimizer (x^*, ν^*) [Feijer '10, Cherukuri '15]
3. In control though, *stability is just the first step...*

What about **sensitivity to disturbances**?

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Incorporating Disturbances into Primal-Dual Dynamics

Inspired by [Bloch *et al.* '92], [Wang & Elia, '10] and [Stegink, *et al.* '15],

Primal-Dual System

$$\Sigma_{\mathrm{pd}} : \begin{cases} \tau_x \dot{x} = -\nabla f(x) - S^{\mathsf{T}} \nu + w_p \\ \tau_\nu \dot{\nu} = Sx - b + w_d \\ y_p = x \\ y_d = \nu \end{cases}$$

Σ_{pd} defines an I/O mapping $\Sigma_{\mathrm{pd}} : (w_p, w_d) \to (x, \nu)$.

What can we say about this mapping?

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Quick Review: Passivity and L_2-Gains

Nonlinear Control System

$$\Sigma : \begin{cases} \dot{x} = F(x) + G(x)w \\ y = H(x) \end{cases}$$

Unforced Equilibrium

$$0 = F(x^*)$$
$$y^* = H(x^*)$$

Passivity

System Σ is (output-strictly) *passive* if $\exists\ V : \mathbb{R}^n \to \mathbb{R}_{>0}$, $V(x^*) = 0$, and $(\rho > 0)$ $\rho \geq 0$ such that

$$\dot{V} \leq -\rho \|y - y^*\|_2^2 + (y - y^*)^{\mathsf{T}} w\,.$$

Finite L_2-Gain

System Σ has *finite L_2-gain* $\|\Sigma_{\mathrm{pd}}\|_{L_2} \leq \gamma$ if for all $w \in L_2$

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Σ is output-strictly passive

$$\dot{V} \leq -\rho\|y - y^*\|_2^2 + (y - y^*)^{\mathsf{T}} w$$

$$\Longrightarrow$$

Σ has finite L_2-gain less than or equal to $1/\rho$:

$$\|\Sigma_{\mathrm{pd}}\|_{L_2} \leq \frac{1}{\rho}$$

Result 1: L_2-Gain of Primal-Dual System

I/O Primal-Dual System

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Theorem 1

Let (x^*, ν^*) be the optimizer. The following statements hold:

(i) If each $f_i(x_i)$ is strictly convex, then Σ_{pd} is passive at (x^*, ν^*)

(ii) If each $f_i(x_i)$ is m_i-strongly convex, then the L_2-gain $\|\Sigma_{\mathrm{pd}}\|_{L_2}$ of the map $w_p \mapsto y_p$ satisfies

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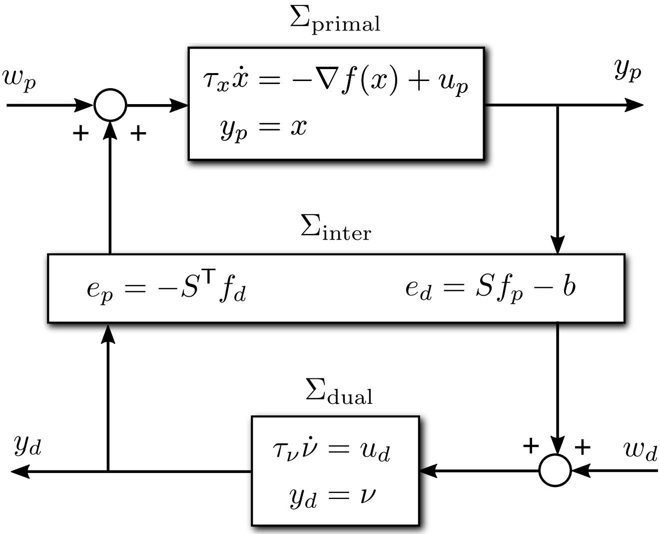
Decompose Primal-Dual Algorithm Into Subsystems



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Key Insights

1. L_2-gain independent of S, τ_x, τ_ν
2. L_2-gain independent of $\#$ agents
3. As $m_{\min} \to 0 \ldots$ ☹☹!!

Can we redesign to reduce the L_2-gain?

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Augmented Lagrangians and Primal-Dual

One interpretation: penalize constraint violation during transients

Augmented Lagrangian

$$L_K(x,\nu) = f(x) + \nu^{\mathsf{T}}(Sx - b) + \frac{K}{2}\|Sx - b\|_2^2$$

Augmented Primal-Dual System

$$\begin{aligned} \tau_x \dot{x} &= -\nabla f(x) - S^{\mathsf{T}}\nu - KS^{\mathsf{T}}(Sx - b) \\ \tau_\nu \dot{\nu} &= Sx - b \end{aligned}$$

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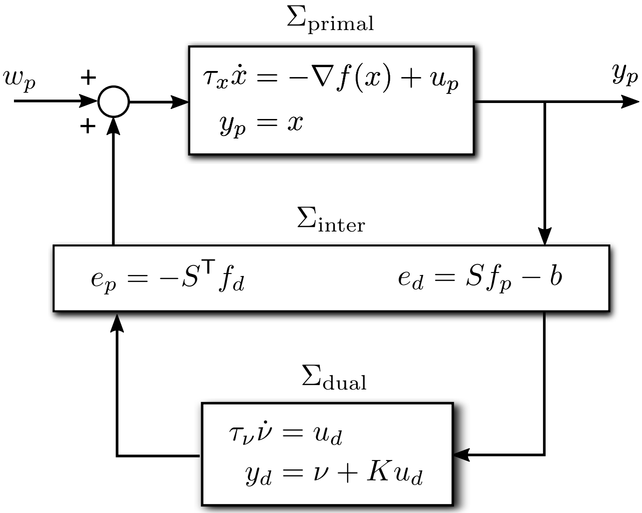

Result 2: L_2-Gain of Augmented Primal-Dual System

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$$\|\Sigma_{\mathrm{apd}}\|_{L_2} \leq \frac{1}{\lambda_{\min}(M + KS^{\mathsf{T}}S)}$$

around the optimizer, where $M = \mathrm{diag}(m_1, \dots, m_n)$.

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Idea: Increase K to decrease L_2-gain?

Agreement Constraints $x_i = x_j$

$S = E^\mathsf{T}$, E incidence matrix of acyclic graph, $L = S^\mathsf{T}S$

$$\lim_{K\to\infty} \frac{1}{\lambda_{\min}(M + KL)} = \frac{1}{m_{\mathrm{avg}}}$$

Conclusions

Passivity/L_2-Framework for Primal-Dual **Input/Output Performance**:

1. L_2-gain $\leq 1/m_{\min}$, strong convexity important
2. Augmentation reduces L_2-gain ... sometimes

For quadratic costs, $\mathcal{H}_2$-norm results (**CDC, Las Vegas**)

What's next?

1. Inequality constraints
2. Other distributed optimization algorithms
3. L_2 small-gain provides solid ground for **interconnection**
4. Beyond PI controllers ...

Conclusions

Passivity/L_2-Framework for Primal-Dual **Input/Output Performance**:

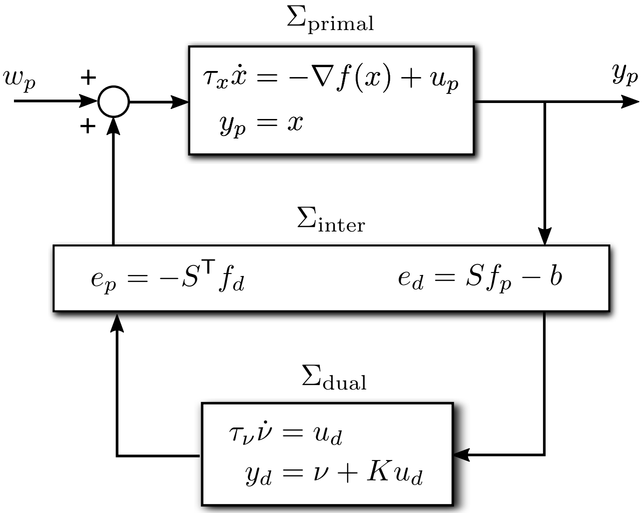
1. L_2-gain $\leq 1/m_{\min}$, strong convexity important
2. Augmentation reduces L_2-gain ... sometimes

For quadratic costs, $\mathcal{H}_2$-norm results (**CDC, Las Vegas**)

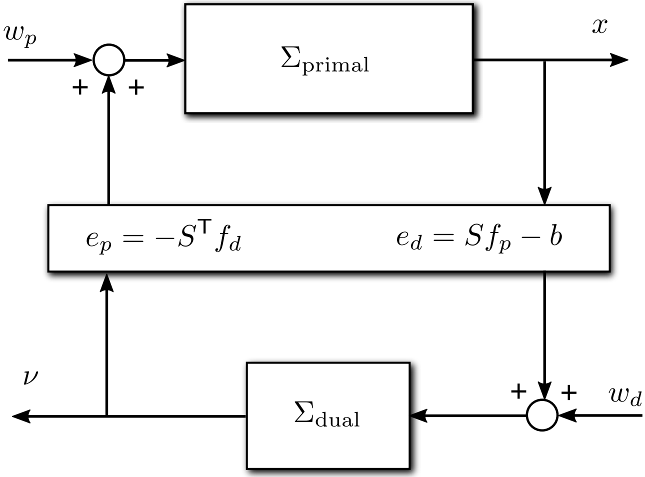
What's next?

1. Inequality constraints
2. Other distributed optimization algorithms
3. L_2 small-gain provides solid ground for **interconnection**
4. Beyond PI controllers ...

I Swept Something Under The Rug . . .



A More General Picture



UNIVERSITY OF
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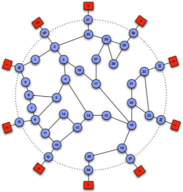
appendix

The Optimal Frequency Regulation Problem

(Simplified, Linearized, and Network-Reduced)

Grid: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, B)$



Swing Dynamics

$$\dot{\theta}_i = \omega_i$$

$$M_i \dot{\omega}_i = -D_i \omega_i + P_i^* - P_{\mathrm{e},i}(\theta) + p_i$$

(Linearized) Power Flow

$$P_{\mathrm{e},i}(\theta) = \sum\nolimits_{j=1}^{n} B_{ij}(\theta_i - \theta_j)\,,$$

Problem: How to pick controls p_i for (i) $\omega_{\mathrm{ss}} = 0$ (ii) optimality?

OFR Problem

$$\begin{aligned} \underset{p \in \mathbb{R}^n}{\text{minimize}} \quad & \sum\nolimits_{i=1}^{n} \frac{1}{2} k_i p_i^2 \\ \text{subject to} \quad & \sum\nolimits_{i=1}^{n} (P_i^* + p_i) = 0 \\ & \underline{p}_i \leq p_i \leq \overline{p}_i \end{aligned}$$

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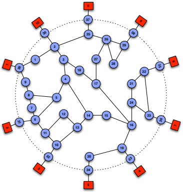

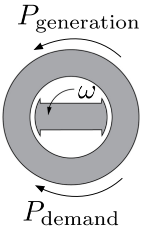

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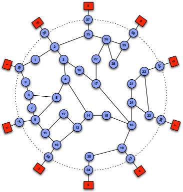

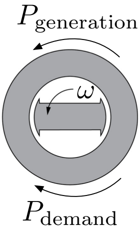

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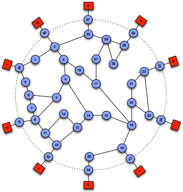

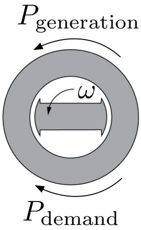

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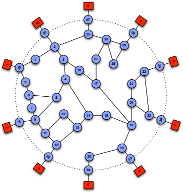

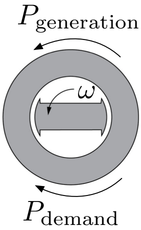

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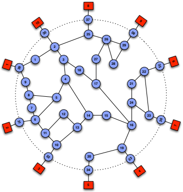

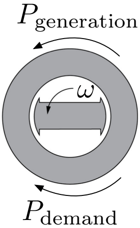

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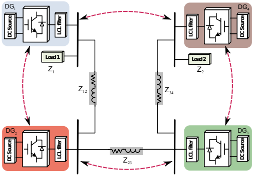
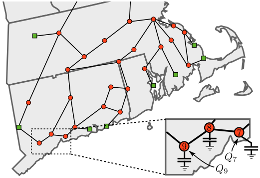
Smart Grid Project Samples

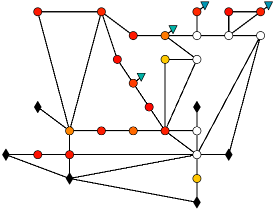
Distributed Inverter Control



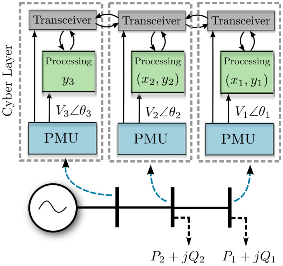
Voltage Collapse (Nat. Comms.)



Optimal Distrib. Volt/Var (CDC)



Wide-Area Monitoring (TSG)



Some Intuition

To first order

$$f(x) = f(x^*) + \nabla f(x^*)^\mathsf{T}(x - x^*)\,.$$

Therefore

$$\|f(x) - f(x^*)\|_{L_2} \leq \frac{m_{\max}}{m_{\min}} \cdot \text{(size of disturbance)}$$