# Model-Free Wide-Area Monitoring of Power Grids via Cutset Voltages

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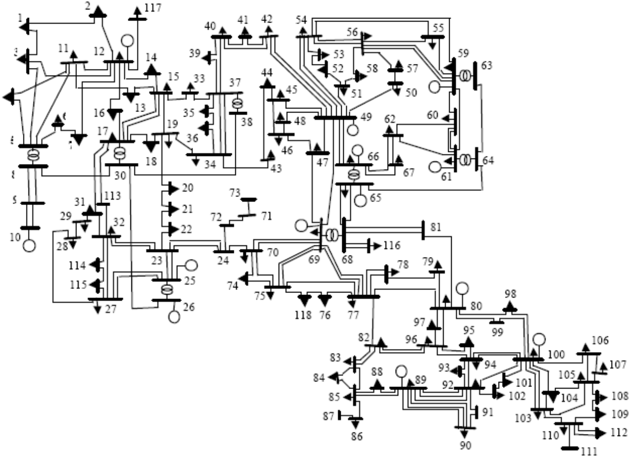

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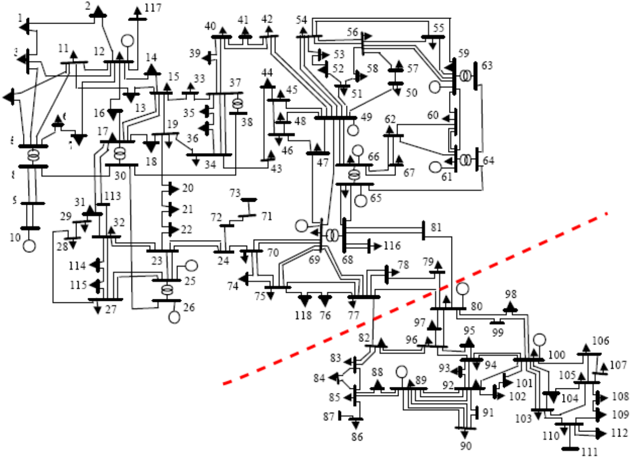

# Basic Circuit Theory

- **Graph** $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with **edge weights** $g_{ij}$, $g = \mathrm{diag}(g_{ij})$
- **Incidence matrix:** $A \in \{-1, 0, 1\}^{(\text{nodes})\times(\text{edges})}$
- **Nodal current injections** $I_i$ and **edge currents** $i_{ij}$
- **Nodal voltages** $V_i$ and **branch voltages** $v_{ij}$

**Circuit Equations**

$$I = Ai \qquad \text{(KCL)}$$
$$v = A^{\mathsf{T}} V \qquad \text{(KVL)}$$
$$i = gv \qquad \text{(Ohm)}$$

Nodal Equations $I = GV$ with **conductance matrix** $G = AgA^{\mathsf{T}}$

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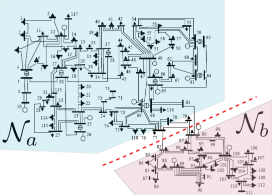
# Basic Circuit Theory for Two-Area Circuit




- Nodal partitioning: $\mathcal{N} = \mathcal{N}_a \cup \mathcal{N}_b$

$$I = \begin{bmatrix} I_a \\ I_b \end{bmatrix}, \qquad V = \begin{bmatrix} V_a \\ V_b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I_a \\ I_b \end{bmatrix} = \begin{bmatrix} G_{aa} & G_{ab} \\ G_{ba} & G_{bb} \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix}.$$

- Induced branch partitioning: $\mathcal{E} = \mathcal{E}^{aa} \cup \mathcal{E}^{ab} \cup \mathcal{E}^{bb}$

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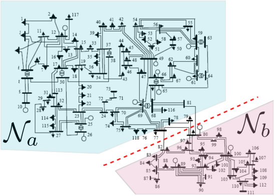


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# The Cutset Voltage

Dobson *et. al.*

- A **scalar, aggregated** metric for stress across a cutset
- Define the **cutset current**

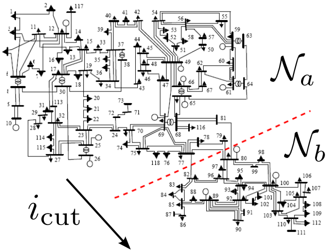
$$i_{\text{cut}} = \mathbb{1}^{\mathsf{T}} i_{ab} = \sum_{e \sim (i,j) \in \mathcal{E}^{ab}} i_e \,.$$

- and the **cutset conductance**

$$g_{\text{cut}} = \mathbb{1}^{\mathsf{T}} g_{ab} \mathbb{1} = \sum_{(i,j) \in \mathcal{E}^{ab}} g_{ij} \,,$$

- Combining, we get the **cutset voltage**

$$v_{\text{cut}} = i_{\text{cut}} / g_{\text{cut}}$$



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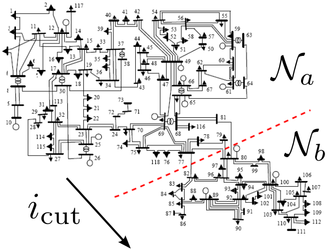

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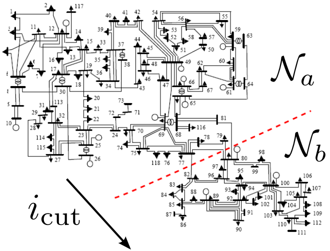

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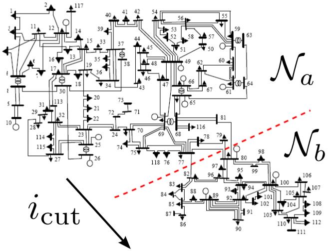
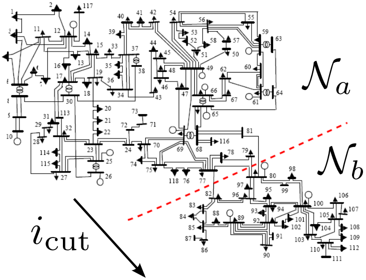


# Cutset Voltage Intuition




### Cutset Voltage

$$v_{\text{cut}} = \frac{\sum_{(i,j)\in\mathcal{E}^{ab}} i_{ij}}{\sum_{(i,j)\in\mathcal{E}^{ab}} g_{ij}}$$

- Voltage across "average" edge in cutset
- $g_{\text{cut}}$ is **parallel** combination of conductances in cut

  $\implies$ **Exact reduction** if voltages uniform on either side of cut!
- Decompose voltage vectors as

$$\begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} \mu_a \mathbb{1}_a \\ \mu_b \mathbb{1}_b \end{bmatrix} + \begin{bmatrix} \widetilde{V}_a \\ \widetilde{V}_b \end{bmatrix}, \tag{1}$$

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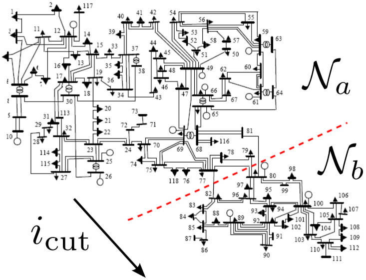


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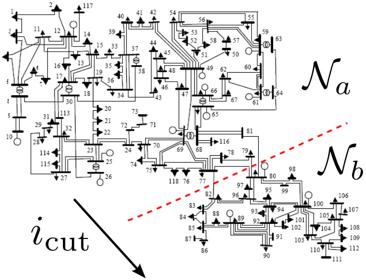


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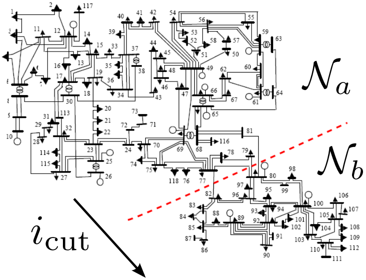


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# Result: Cutset Voltage and Average Voltage Levels

Define **weighted inter-area degree vectors**

$$g_a = -G_{ab}\mathbb{1}_b \qquad g_b = -G_{ba}\mathbb{1}_a$$

and the **projection matrices**

$$\Pi_a = \mathbb{I}_a - \frac{1}{|\mathcal{N}_a|}\mathbb{1}_a\mathbb{1}_a^\top, \quad \Pi_b = \mathbb{I}_b - \frac{1}{|\mathcal{N}_b|}\mathbb{1}_b\mathbb{1}_b^\top,$$

**Theorem 1**

The cutset voltage $v_{\mathrm{cut}}$ is given by

$$v_{\mathrm{cut}} = \mu_a - \mu_b + \varepsilon_{\mathrm{cut}}$$

where the *cutset error* $\varepsilon_{\mathrm{cut}}$ is

$$\varepsilon_{\mathrm{cut}} = \frac{1}{g_{\mathrm{cut}}}\begin{bmatrix} g_a^\top \Pi_a & -g_b^\top \Pi_b \end{bmatrix}\begin{bmatrix} V_a \\ V_b \end{bmatrix},$$

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The error term will vanish if:

- **Uniform Area Voltages:** $V_a = \mu_a \mathbb{1}_a$ and $V_b = \mu_b \mathbb{1}_b$
  $\implies \epsilon_{\text{cut}} = 0$
- $\Pi_a g_a = 0$ and $\Pi_b g_b = 0$ **(?)**

# Almost Equitable Partitions I

**Definition**

Let $\pi \triangleq \{\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_K\}$ be a partition of $\mathcal{G} = (\mathcal{N}, \mathcal{E})$. For any node $i \in \mathcal{N}$ and any area $q \in \{1, \ldots, K\}$, let

$$g_{\text{parallel}}(i, \mathcal{N}_q) \triangleq \sum\nolimits_{j \in \mathcal{N}_q, \{i,j\} \in \mathcal{E}} g_{ij} \,,$$

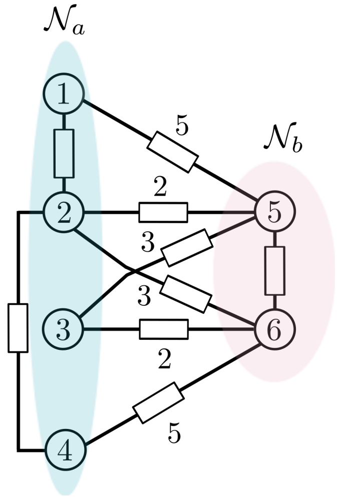
be the total parallel conductance between node $i \in \mathcal{N}$ and area $\mathcal{N}_q$. We call $\pi$ an *almost equitable partition* (AEP) of $\mathcal{G}$ if for each pair of distinct areas $p, q \in \{1, \ldots, K\}$, there exists a value $g_{pq} \in \mathbb{R}$ such that

$$g_{\text{parallel}}(i, \mathcal{N}_q) = g_{pq} \,,$$

for all nodes $i \in \mathcal{N}_p$.

- Every node in each area "sees" the same parallel conductance to the other areas.

# Almost Equitable Partitions II




- Conditions for AEP?

**Lemma**

For two-area network $\pi = \{\mathcal{N}_a, \mathcal{N}_b\}$:

$\pi$ is an AEP

$\iff$

$g_a \in \mathrm{im}(\mathbb{1}_a)$ and $g_b \in \mathrm{im}(\mathbb{1}_b)$

In this example:

$$g_a = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix} \qquad g_b = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

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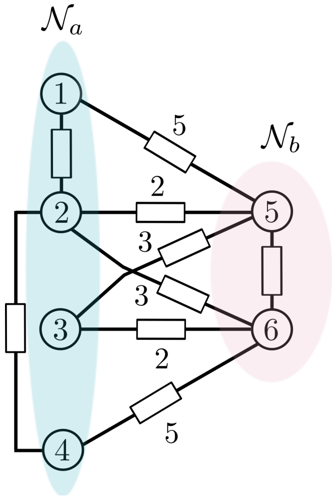


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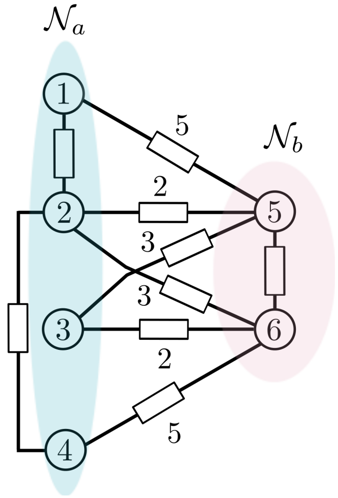


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- **Almost Equitable Partition:** $\pi = \{\mathcal{N}_a, \mathcal{N}_b\}$
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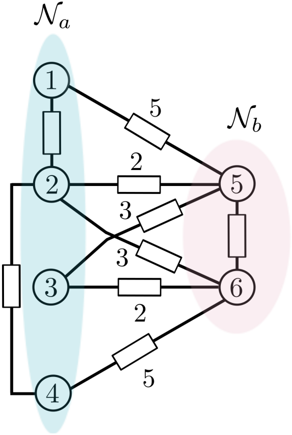
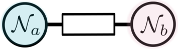
# Example



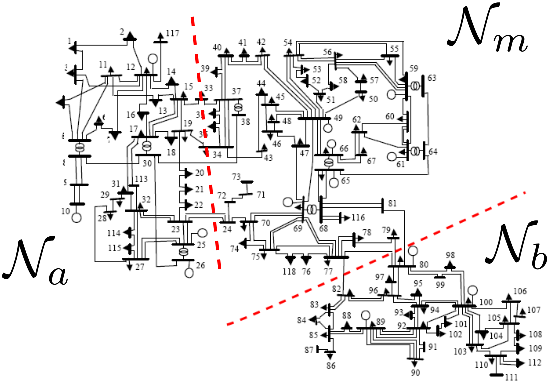

$$\mu_a = \frac{V_1 + V_2 + V_3 + V_4}{4}$$

$$\mu_b = \frac{V_5 + V_6}{2}$$

$$v_{\text{cut}} = \mu_a - \mu_b$$



# Extension to Voltage Across An Area




**Key Idea:** Kron reduction

$$\begin{bmatrix} I_a \\ I_m \\ I_b \end{bmatrix} = \begin{bmatrix} G_{aa} & G_{am} & \mathbb{0} \\ G_{ma} & G_{mm} & G_{mb} \\ \mathbb{0} & G_{bm} & G_{bb} \end{bmatrix} \begin{bmatrix} V_a \\ V_m \\ V_b \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} I_a^{\mathrm{red}} \\ I_b^{\mathrm{red}} \end{bmatrix} = \begin{bmatrix} G_{aa}^{\mathrm{red}} & G_{ab}^{\mathrm{red}} \\ G_{ba}^{\mathrm{red}} & G_{bb}^{\mathrm{red}} \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix}$$

# Voltage Across an Area

Dobson *et. al.*

We now have a **reduced graph** $\mathcal{G}_{\text{red}} = (\mathcal{N}_a \cup \mathcal{N}_b, \mathcal{E}_{\text{red}})$

- Define the **current across the area**

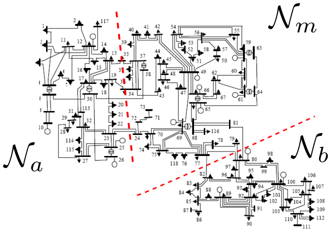
$$i_{\text{cut}}^{\text{red}} = \sum_{e \sim (i,j) \in \mathcal{E}_{\text{red}}^{ab}} i_e \,.$$

- and the **area conductance**

$$g_{\text{area}} = \sum_{(i,j) \in \mathcal{E}_{\text{red}}^{ab}} g_{ij}^{\text{red}} \,,$$

- Leading to the **area voltage**

$$v_{\text{area}} = i_{\text{cut}}^{\text{red}} / g_{\text{area}}$$



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$$i_{\text{cut}}^{\text{red}} = \sum_{e \sim (i,j) \in \mathcal{E}_{\text{red}}^{ab}} i_e \,.$$

- and the **area conductance**

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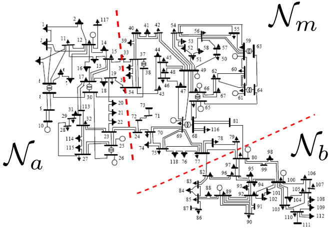
- Leading to the **area voltage**

$$v_{\text{area}} = i_{\text{cut}}^{\text{red}} / g_{\text{area}}$$



# Voltage Across an Area

Dobson *et. al.*

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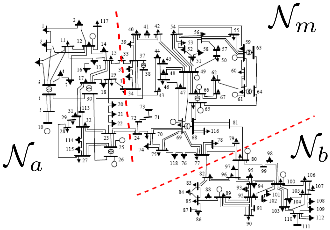

# Decomposing the Area Voltage

- Applying our cutset results, we have

**Theorem 2**

The area voltage $v_{\mathrm{cut}}$ is given by

$$v_{\mathrm{area}} = \mu_a - \mu_b + \varepsilon_{\mathrm{area}}$$

where the *area error* $\varepsilon_{\mathrm{area}}$ is

$$\varepsilon_{\mathrm{area}} = \frac{1}{g_{\mathrm{area}}} \left[ (g_a^{\mathrm{red}})^{\mathsf{T}} \Pi_a \quad -(g_b^{\mathrm{red}})^{\mathsf{T}} \Pi_b \right] \begin{bmatrix} V_a \\ V_b \end{bmatrix},$$

- We know $\varepsilon_{\mathrm{area}} = 0$ if $\pi_{\mathrm{red}} = \{\mathcal{N}_a, \mathcal{N}_b\}$ is an A.E.P. of the **reduced graph** $\mathcal{G}_{\mathrm{red}}$.

$\Longrightarrow$ Relationship to original graph $\mathcal{G}$?

# Kron Reduction & Almost Equitable Partitions

**Proposition**

$$\pi = \{\mathcal{N}_a, \mathcal{N}_m, \mathcal{N}_b\}$$
is an A.E.P. of $\mathcal{G}$

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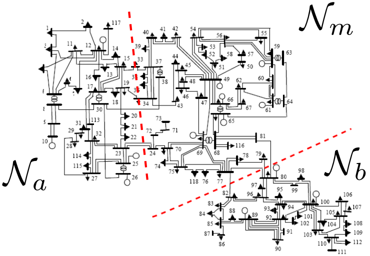


$\Longrightarrow$ A.E.P. preserved under Kron Reduction

$$v_{\text{area}} = \mu_a - \mu_b \quad \text{if} \quad \{\mathcal{N}_a, \mathcal{N}_m, \mathcal{N}_b\} \text{ is an A.E.P.}$$

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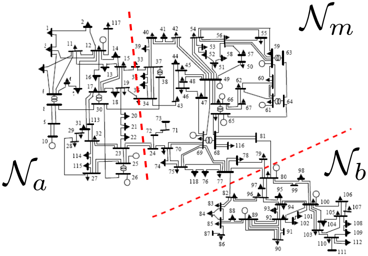


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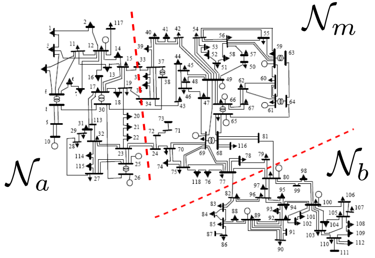


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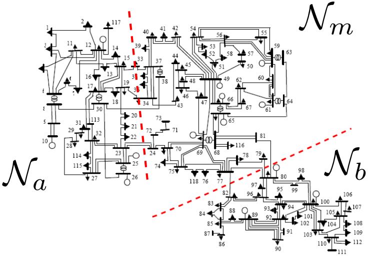


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# Conclusions

Theoretical development for **cutset and area voltage**:

1. $v_{\mathrm{cut}}$ = (diff. in mean voltage levels) + remainder $\varepsilon_{\mathrm{cut}}$
2. Remainder vanishes for almost equitable partitions
3. Also holds for voltage across area
4. Kron Reduction preserves A.E.P.

**Open Problems:**

1. Optimal cutset selection, "spanning cutsets"
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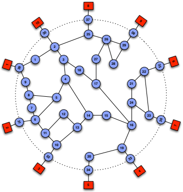
https://ece.uwaterloo.ca/~jwsimpso/
jwsimpson@uwaterloo.ca

# appendix

# The Optimal Frequency Regulation Problem

(Simplified, Linearized, and Network-Reduced)

**Grid**: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, B)$



### Swing Dynamics

$$\dot{\theta}_i = \omega_i$$

$$M_i \dot{\omega}_i = -D_i \omega_i + P_i^* - P_{\mathrm{e},i}(\theta) + p_i$$

### (Linearized) Power Flow

$$P_{\mathrm{e},i}(\theta) = \sum\nolimits_{j=1}^{n} B_{ij}(\theta_i - \theta_j)\,,$$

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$$\underset{p \in \mathbb{R}^n}{\text{minimize}} \quad \sum\nolimits_{i=1}^{n} \frac{1}{2} k_i p_i^2$$

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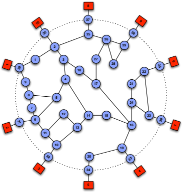

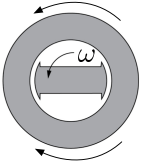


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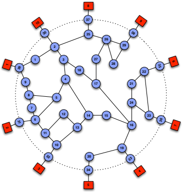

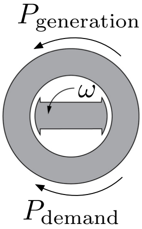


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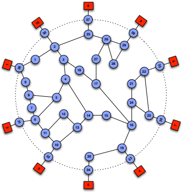

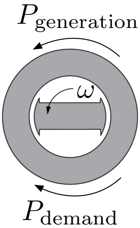


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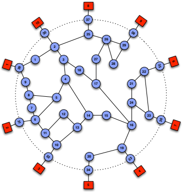

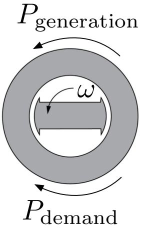


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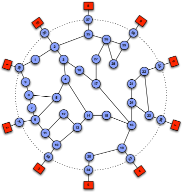

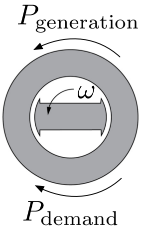


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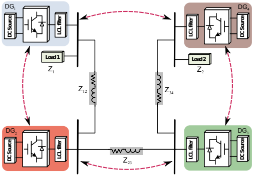
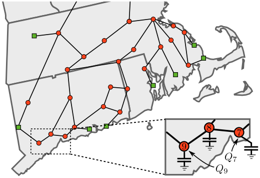
# Smart Grid Project Samples

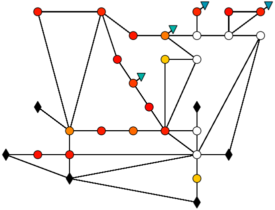
## Distributed Inverter Control




## Voltage Collapse (Nat. Comms.)




## Optimal Distrib. Volt/Var (CDC)



## Wide-Area Monitoring (TSG)