Quadratic Performance of Primal-Dual Methods for Distributed Optimization

55th IEEE Conference on Decision and Control

Las Vegas, NV

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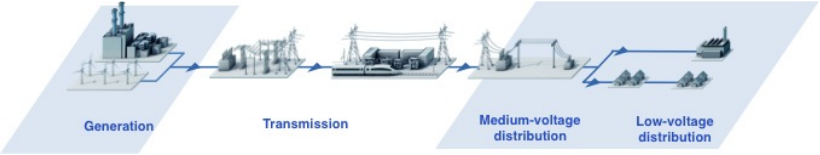
Our 20th Century Bulk Power System

A large-scale, nonlinear, hybrid, stochastic, distributed, cyber-physical . . .



Active Control → **Passive Consumption**

What kind of control is used?

Power electronics are game changers . . .

synchronous generator

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power inverters

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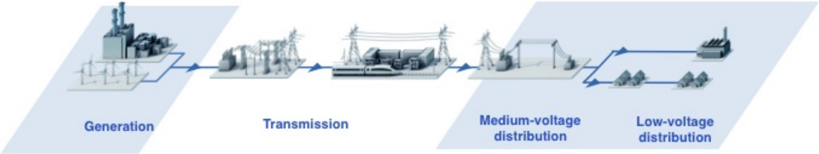
location & distributed implementation



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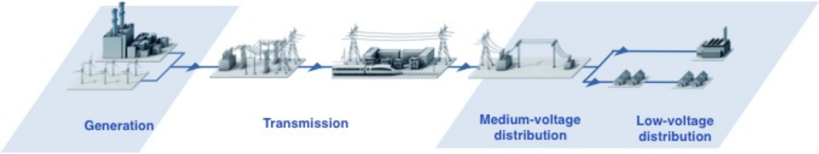

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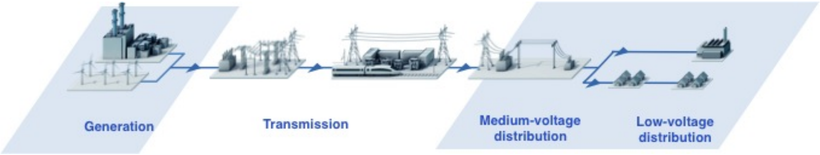

Central control authority is fading . . . how to coordinate new actuators?

Optimal Distributed Frequency Regulation

A very incomplete literature review

Connecting Automatic Generation Control and Economic Dispatch from an Optimization View

Na Li, Lijun Chen,

Achieving Real-time Economic Dispatch in Power Networks via a Saddle Point Design Approach

Xuan Zhang, Na Li and Antonis Papachristodoulou

Distributed Frequency-Preserving Optimal Load Control

Enrique Mallada* Steven H. Low*

A unifying energy-based approach to optimal frequency and market regulation in power grids

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Reverse and Forward Engineering of Frequency Control in Power N

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Popular Idea: {Grid Dynamics} ∪ {Dist. Controller}

= Distributed **Online Optimization Algorithm**

(many other approaches exist)

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A Distributed Optimization Problem

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & \frac{1}{2} x^\mathsf{T} Q x + c^\mathsf{T} x \\ \text{subject to} \quad & Sx = b\,, \end{aligned}$$

- $x \in \mathbb{R}^n$ is our decision variable, $Q = \mathrm{diag}(q_1, \ldots, q_n)$ is pos. def.
- $S \in \mathbb{R}^{r \times n}$ with $r < n$ is full rank, c, b are vectors

Each variable $x_i \in \mathbb{R}$ belongs to **agent**.

The constraint matrix S **couples** the agents together.

We want the agents to **cooperatively** solve this problem.

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The Primal-Dual / Saddle Point Algorithm

Key Idea: Write an algorithm which *seeks* the KKT points of

$$L(x,\nu) = \frac{1}{2}x^{\mathsf{T}}Qx + c^{\mathsf{T}}x + \nu^{\mathsf{T}}(Sx - b)\,.$$

Saddle Point Algorithm [Kose '56, Arrow *et al.* '58]

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1. Dynamics are **distributed** with sparsity pattern of S
2. Converges to unique primal-dual optimizer (x^*, ν^*) [Feijer '10, Cherukuri '15]
3. In control though, *stability is just the first step...*

What about **sensitivity to disturbances**?

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Incorporating Disturbances into Primal-Dual Dynamics

Inspired by [Bloch *et al.* '92], [Wang & Elia, '10] and [Stegink, *et al.* '15],

Primal-Dual System

$$G : \begin{cases} \tau_x \dot{x} = -Qx - c - S^\mathsf{T}\nu + d_c \\ \tau_\nu \dot{\nu} = Sx - b + d_b \\ \quad z = \dfrac{1}{\sqrt{2}} Q^{\frac{1}{2}} x \end{cases}$$

Note: $\|z\|_2^2 = \frac{1}{2} x^\mathsf{T} Q x \quad \sim \quad$ Cost Function

G defines an I/O mapping $G : (d_c, d_b) \to z$.

How much does this system amplify (or attenuate) disturbances?

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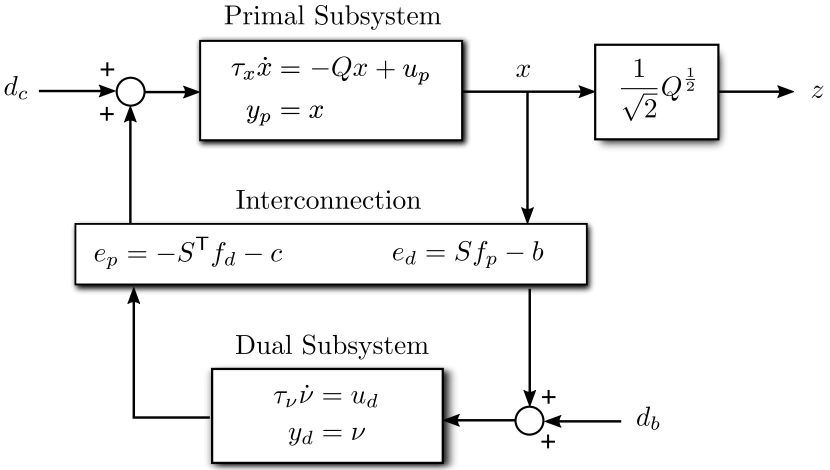
Block Diagram of Primal-Dual I/O System



Quick Review: The $\mathcal{H}_2$ System Norm

Exp. Stable Linear System

$$\begin{aligned} \dot{x} &= Ax + Bd \\ z &= Cx \end{aligned} \quad \Longleftrightarrow \quad G(s) = C(sI - A)^{-1}B$$

$$\|G\|_{\mathcal{H}_2}^2 \triangleq \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}\left[G^\mathsf{T}(-j\omega)G(j\omega)\right] \,\mathrm{d}\omega$$

Useful Interpretations:

(i) **Steady-state output variance** $\lim_{t\to\infty} \mathbb{E}[z(t)^\mathsf{T} z(t)]$ when d noise

(ii) "Average" gain over all frequencies from $d(\cdot)$ to $z(\cdot)$

If (A, C) observable

$$A^\mathsf{T} Y + YA + C^\mathsf{T} C = 0 \quad \Longrightarrow \quad \|G\|_{\mathcal{H}_2}^2 = \mathrm{Tr}(B^\mathsf{T} YB)$$

$\mathcal{H}_2$ Performance of Primal-Dual Algorithm

Primal-Dual System

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Theorem: Primal-Dual Performance

The $\mathcal{H}_2$ norm of the system $(\star)$ is

$$\|G\|_{\mathcal{H}_2}^2 = \frac{1}{4}\sum_{i=1}^{n} \frac{1}{\tau_{x,ii}} + \frac{1}{4}\sum_{k=1}^{r} \frac{1}{\tau_{\nu,kk}}\,.$$

For uniform time-constants

$$\boxed{\|G\|_{\mathcal{H}_2}^2 = \frac{1}{4\tau_x} n + \frac{1}{4\tau_\nu} r\,.}$$

$\mathcal{H}_2$ Performance of I/O Primal-Dual Algorithm

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Key Insights

1. Norm **independent** of Q, S
2. Speed/Performance Trade-off
3. Norm **grows** with # agents

Can we redesign to reduce the $\mathcal{H}_2$ norm?

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Augmented Lagrangians and Primal-Dual

One interpretation: penalize constraint violation during transients

Augmented Lagrangian

$$L_\rho(x,\nu) = \frac{1}{2}x^\mathsf{T} Qx + c^\mathsf{T} x + \nu^\mathsf{T}(Sx - b) + \frac{\rho}{2}\|Sx - b\|_2^2$$

Augmented Primal-Dual System

$$\tau_x \dot{x} = -Qx - S^\mathsf{T}\nu - c - \rho S^\mathsf{T}(Sx - b)$$

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Uniformity Assumption: $Q_{ii} = q\,,\ \tau_{x,ii} = \tau_x\,,\ \tau_{\nu,ii} = \tau_\nu\,.$

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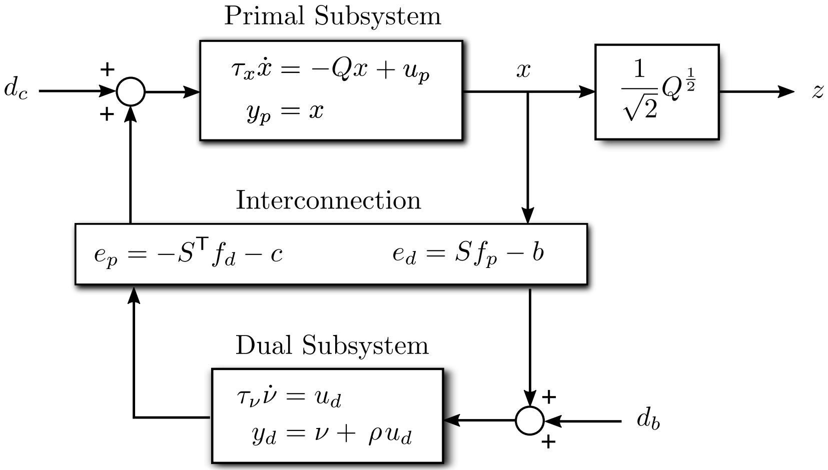

$\mathcal{H}_2$ Performance of I/O Augmented Primal-Dual

Theorem: Aug. Primal-Dual Performance

The $\mathcal{H}_2$ norm of the augmented P-D system is

$$\|G_\rho\|_{\mathcal{H}_2}^2 = \frac{1}{4\tau_x}(n-r) + \frac{1}{4}\left(\frac{1}{\tau_x} + \frac{1}{\tau_\nu}\right)\sum_{k=1}^{r} \frac{q}{q + \rho\,\sigma_k(S)}$$

where $\sigma_k(S) =$ singular values. Moreover

$$\boxed{\|G_\rho\|_{\mathcal{H}_2} \leq \|G\|_{\mathcal{H}_2}} \quad \text{and} \quad \boxed{\lim_{\rho\to\infty} \|G_\rho\|_{\mathcal{H}_2}^2 = \frac{1}{4\tau_x}(n-r)}$$

Recall:

$$\|G\|_{\mathcal{H}_2}^2 = \frac{1}{4\tau_x}n + \frac{1}{4\tau_\nu}r$$

Key Insights

1. **Not** independent of Q, S
2. More constraints $=$ good
3. More gain $=$ good

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Key Insights

1. **Not** independent of Q, S
2. More constraints $=$ good
3. More gain $=$ good

$\mathcal{H}_2$ Performance of I/O Augmented Primal-Dual

Theorem: Aug. Primal-Dual Performance

The $\mathcal{H}_2$ norm of the augmented P-D system is

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Connection to Frequency Control

Optimal Frequency Regulation Problem

$$\begin{aligned} \underset{p \in \mathbb{R}^n}{\text{minimize}} \quad & \sum\nolimits_{i=1}^{n} \frac{1}{2} k_i p_i^2 \\ \text{subject to} \; & \sum\nolimits_{i=1}^{n} (P_i^* + p_i) = 0 \end{aligned}$$

After dual decomposition: $P_i^* = c_i =$ uncontrolled load

$\Longrightarrow$ Quantify $\mathcal{H}_2$-norm from **load disturbance** d_c to **generation cost**

1. For Primal-Dual : $\|G\|_{\mathcal{H}_2} \propto \sqrt{n}$
   $\Longrightarrow$ Large systems are more sensitive
2. For Augmented Primal-Dual: $\|G_\rho\|^2_{\mathcal{H}_2} \to$ const. as $\rho \to \infty$
   $\Longrightarrow$ Sensitivity removed by high-gain (Laplacian!!) feedback

For details, see the paper.

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For L_2-gain or $\mathcal{H}_\infty$ results, see **Allerton 2016** paper
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What's next?

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2. Other distributed optimization algorithms
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Acknowledgements

Bala K. Poolla

Nima Monshizadeh

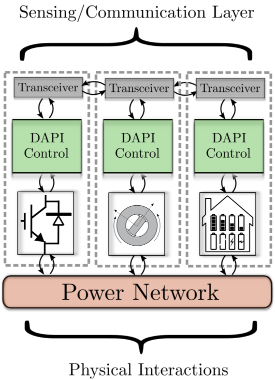
Florian Dörfler

ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

UNIVERSITY OF WATERLOO

UNIVERSITY OF CAMBRIDGE

Question Time



https://ece.uwaterloo.ca/~jwsimpso/
jwsimpson@uwaterloo.ca

Easy Proof of $\mathcal{H}_2$ Result

1. **Simplify**: Assume block-diagonal solution to avoid coupling

2. **Lyapunov Equation:**

$$\begin{bmatrix} Y_{11} & \mathbb{0} \\ \mathbb{0} & Y_{22} \end{bmatrix} \begin{bmatrix} -\tau_x^{-1} Q & -\tau_x^{-1} S^\top \\ \tau_\nu^{-1} S & \mathbb{0} \end{bmatrix} + \begin{bmatrix} \star & \star \\ \star & \star \end{bmatrix}^\top = -\begin{bmatrix} \frac{1}{2} Q & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{bmatrix}$$

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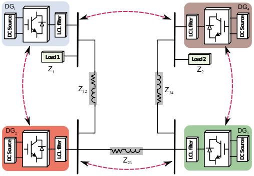
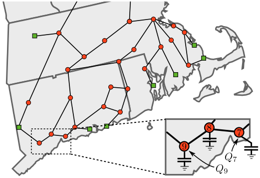
Smart Grid Project Samples

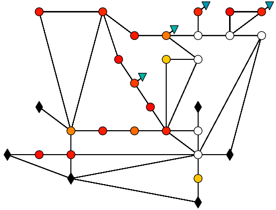
Distributed Inverter Control



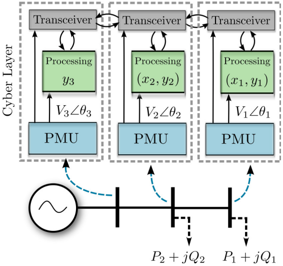
Voltage Collapse (Nat. Comms.)



Optimal Distrib. Volt/Var (CDC)



Wide-Area Monitoring (TSG)



appendix

An incomplete literature review of a busy field

ntwk with unknown disturbances ∪ integral control ∪ distributed averaging

- all-to-all source frequency & injection averaging [Q. Shafiee, J. Vasquez, & J. Guerrero, '13] & [H. Liang, B. Choi, W. Zhuang, & X. Shen, '13] & [M. Andreasson, D. V. Dimarogonas, K. H. Johansson, & H. Sandberg, '12]
- optimality w.r.t. economic dispatch [E. Mallada & S. Low, '13] & [M. Andreasson, D. V. Dimarogonas, K. H. Johansson, & H. Sandberg, '13] & [X. Zhang and A. Papachristodoulou, '13] & [N. Li, L. Chen, C. Zhao & S. Low '13]
- ratio consensus & dispatch [S.T. Cady, A. Garcıa-Domınguez, & C.N. Hadjicostis, '13]
- load balancing in Port-Hamiltonian networks [J. Wei & A. Van der Schaft, '13]
- passivity-based network cooperation and flow optimization [M. Bürger, D. Zelazo, & F. Allgöwer, '13, M. Bürger & C. de Persis '13, He Bai & S.Y. Shafi '13]
- distributed PI avg optimization [G. Droge, H. Kawashima, & M. Egerstedt, '13]
- PI avg consensus [R. Freeman, P. Yang, & K. Lynch '06] & [M. Zhu & S. Martinez '10]
- decentralized "practical" integral control [N. Ainsworth & S. Grijalva, '13]

Distributed Averaging PI (DAPI) Frequency Control

$$\omega_i = \omega^* - m_i P_i(\theta) - \Omega_i$$

$$k_i \dot{\Omega}_i = (\omega_i - \omega^*) - \sum_{j \subseteq \text{inverters}} a_{ij} \cdot (\Omega_i - \Omega_j)$$

1. no tuning, no model dependence
2. weak comm. requirements
3. maintains load sharing
   (share burden of sec. control)

Simple & Intuitive

Theorem: Stability of DAPI
[JWSP, FD, & FB, '13]

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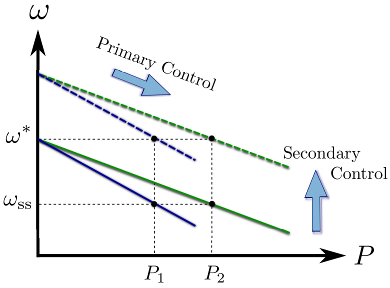

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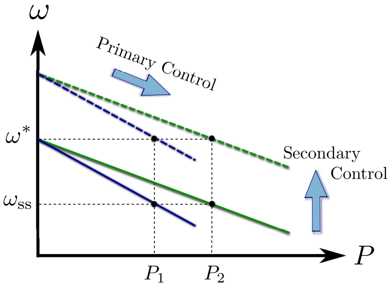

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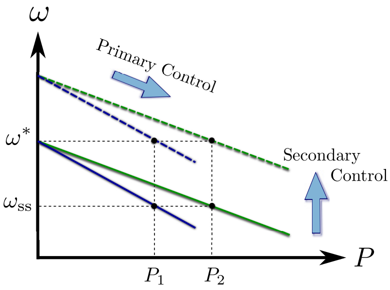

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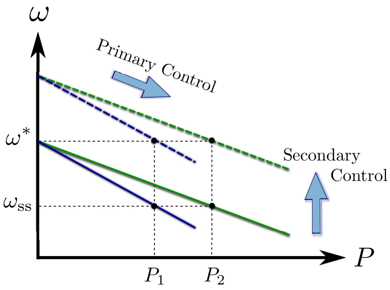

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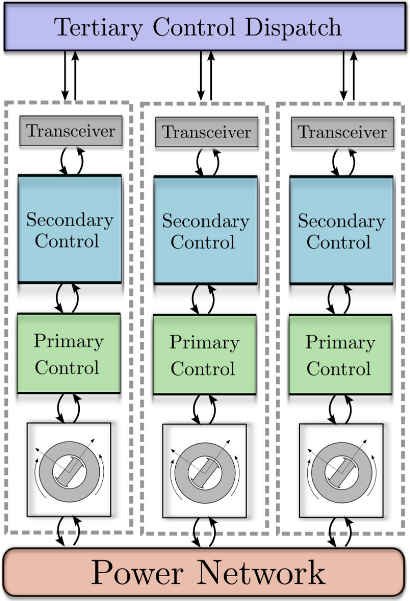
From Hierarchical Control to Distributed Control

flat hierarchy, no time-scale separations, & model-free



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