A Theory of Solvability for Lossless Power Flow Equations

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Santa Barbara, CA

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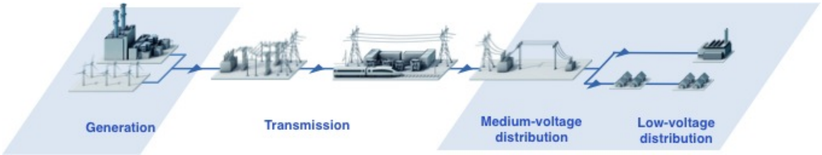
Our 20th Century Bulk Power System

A large-scale, nonlinear, hybrid, stochastic, distributed, cyber-physical . . .



Active Control → **Passive Consumption**

We are going to focus on the network.

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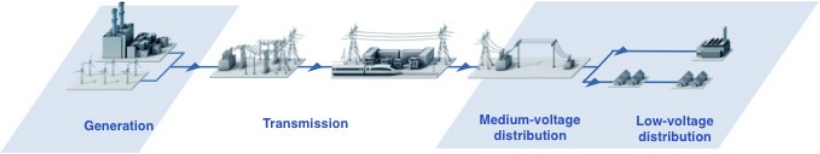

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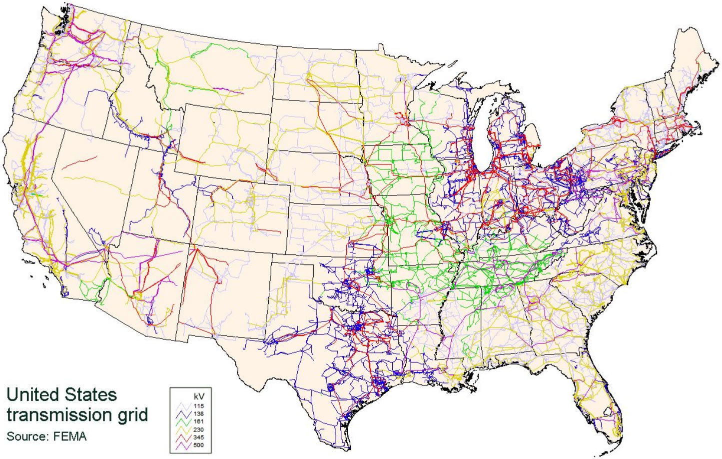
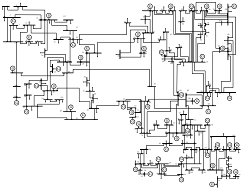
The US Power Grid

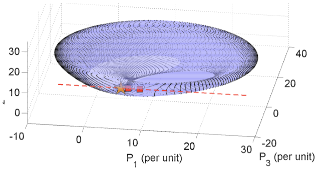


Problems in Power System Operations

Power Flow Analysis

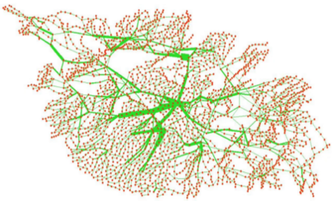


Optimal Power Flow



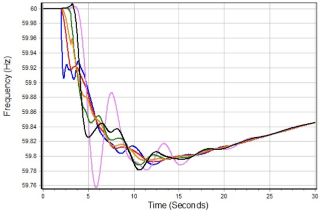
[Molzahn et al.]

Contingency Analysis



[Rezaei et al.]

Transient Stability



[Overbye et al.]

Modeling I: Physics of AC Power Flow

1. **Network Graph:** $(\mathcal{N}, \mathcal{E})$, complex weights $y_{ij} = g_{ij} + \mathbf{j} b_{ij}$
2. **Nodal Variables:** voltage $V_i e^{\mathbf{j}\theta_i}$, power $S_i = P_i + \mathbf{j} Q_i$
3. **Coupling Laws:** Kirchhoff & Ohm
4. **Admittance Matrix:** $Y = G + \mathbf{j}B =$ Laplacian w/ weights y_{ij}
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Modeling II: Bus Models

(aka boundary conditions)

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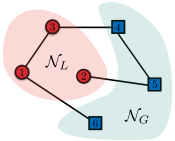

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7. **Load Model:** PQ bus constant P_i constant Q_i
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Power Flow Equations

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$2n + m$ equations in variables $\theta \in \mathbb{T}^{n+m}$ and $V_L \in \mathbb{R}^n_{>0}$.

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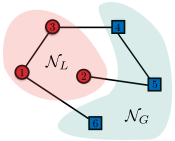

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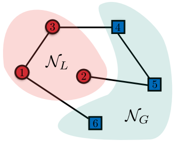

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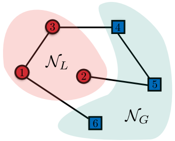

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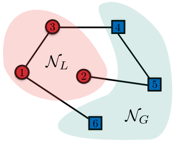

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Motivation I: Numerical Methods for Power Flow

Power flow always solved with variant of Newton iteration

$$x = \begin{pmatrix}\theta & V_L\end{pmatrix}^{\mathsf{T}}, \qquad x^{k+1} = x^k - J(x^k)^{-1} f(x^k)\,.$$

- If **convergent**, may converge to "wrong" solution
- If **non-convergent**, several possibilities:

(a) No power flow solution exists

(b) Numerical instability (conditioning)

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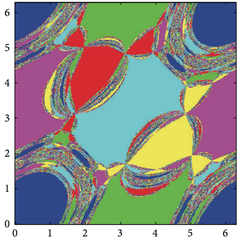

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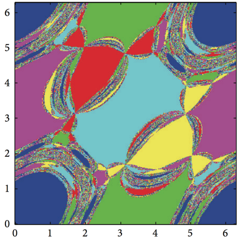

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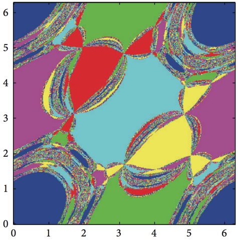

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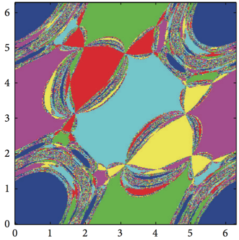

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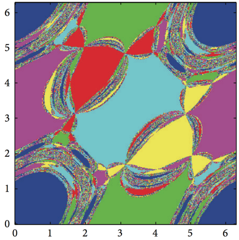

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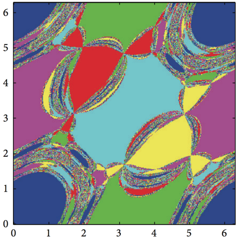

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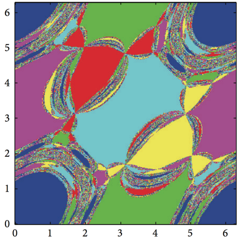

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To differentiate, need theory of power flow solvability

Motivation II: Multimachine Transient Stability

Constrained Swing Dynamics

$$\text{Gen}: \begin{cases} \dot{\theta}_i = \omega_i \\ M_i\dot{\omega}_i = -D_i\omega_i + P_i - \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \end{cases}$$

$$\text{Load}: \begin{cases} D_i\dot{\theta}_i = P_i - \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \\ Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j) \end{cases}$$

Challenge: Characterize equilibria, stability, basin of attraction

Approaches: Energy functions, nearest unstable eq. point, S.O.S., ...

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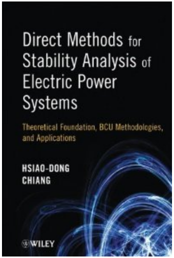

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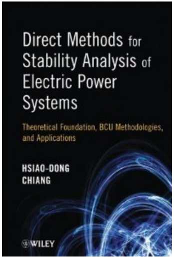

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$$\{\text{Equilibria}\} = \{\text{Power Flow Solutions}\}$$

Motivation III: Optimal Power Flow

Idea: Optimally match supply and demand (with constraints)

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& V_i^{\min} \leq V_i \leq V_i^{\max} && i \in \mathcal{N}_L \,, \\
& S_i^{\min} \leq |P_i + \mathbf{j} Q_i| \leq S_i^{\max} && i \in \mathcal{N}_G \,, \\
& s_{ij}^{\min} \leq |p_{i \to j} + \mathbf{j} q_{i \to j}| \leq s_{ij}^{\max} && (i,j) \in \mathcal{E} \,,
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\end{aligned}$$

- non-convex, solved every 5-15 min. via linearization, ($$$)

Motivation III: Optimal Power Flow

Idea: Optimally match supply and demand (with constraints)

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\underset{\theta, V_L, P_G}{\text{minimize}} \quad & \sum\nolimits_{i \in \mathcal{N}_G} f_i(P_i) & \\
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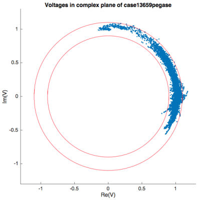
Intuition on Power Flow Solutions

1. 'Normally', exists unique **high-voltage** soln:
   - voltage magnitude $V_i \simeq 1$
   - phase diff $|\theta_i - \theta_j| \ll 1$



[Josz et al.]

2. **Lightly loaded systems:** many **low-voltage** solutions

3. **Heavily loaded systems:** Few solutions or **infeasible**
   - saddle node bifurcations
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   - non-convex feasible set in (P, Q)-space

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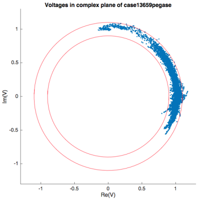

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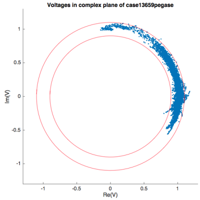

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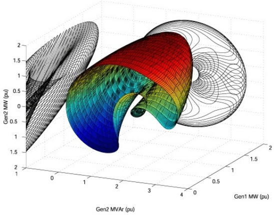

[Hiskens & Davy]

Mysteries of Power Flow I

Given: network topology, impedances, generation & loads

Q: $\exists$ "stable high-voltage" solution? unique? properties?

Partial answers from **45+ years** of literature:

- Jacobian singularity [Weedy '67]
- Multiple dynamically stable solutions [Korsak '72]
- Existence conditions [Wu & Kumagai '80, '82]
- Active power flow singularity [Araposthatis, Sastry & Varaiya, '81]
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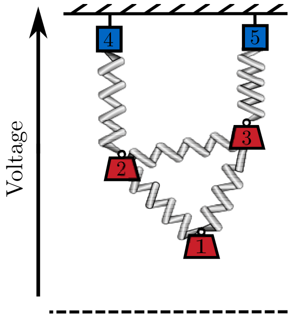
Main insight: stiffness vs. loading

1. Stiff network + light loading ⇒ feasible
2. Weak network + heavy loading ⇒ infeasible

Q: How to quantify network stiffness vs. loading?



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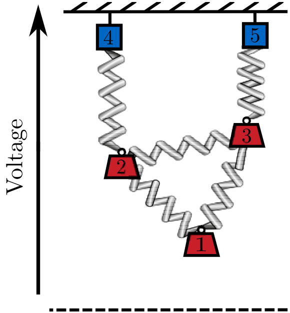

Mysteries of Power Flow II

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Solution of Two-Bus System I

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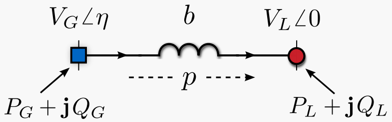

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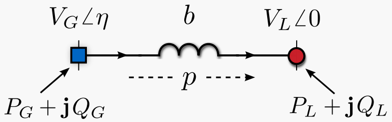

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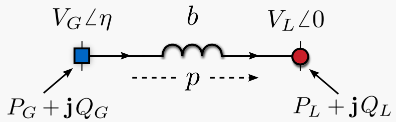

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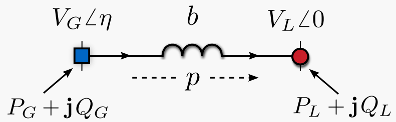

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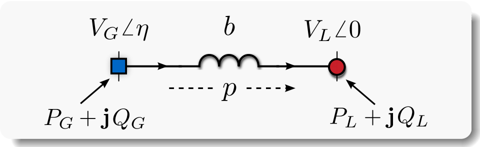

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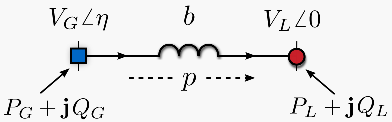

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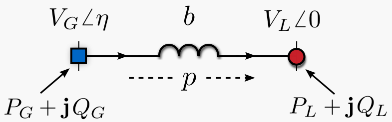

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   $v_+ \in [\frac{1}{2}, 1)$
2. **Low-voltage** solution
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Angle: $\sin(\eta_\mp) = \Gamma / v_\pm$

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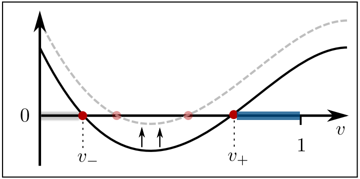

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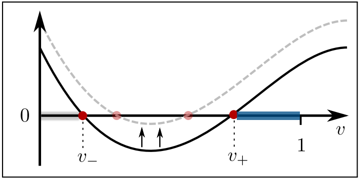

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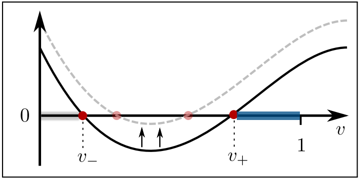

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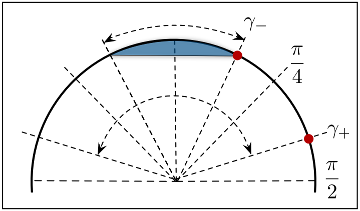

Solution of Two-Bus System IV

- Squaring and adding equations **does not generalize** to networks.
- Is there any hope then?

$$\Gamma = v\sin(\eta)$$
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- Use $\cos(\eta) = \sqrt{1-\sin^2(\eta)} \implies \Delta = -4v^2 + 4v\sqrt{1-(\Gamma/v)^2}$
- Rearrange to get *fixed-point equation*

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$$v = f(v) := -\frac{1}{4}\frac{\Delta}{v} + \sqrt{1 - \left(\frac{\Gamma}{v}\right)^2}$$

This generalizes!

Solution of Two-Bus System IV

- Squaring and adding equations **does not generalize** to networks.
- Is there any hope then?

$$\begin{aligned} \Gamma &= v \sin(\eta) \\ \Delta &= -4v^2 + 4v\cos(\eta) \end{aligned}$$

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Network Notation I: Branches Between Bus Types

Power Flow Equations

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$

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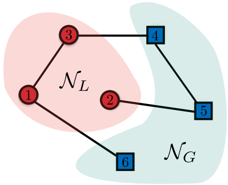

- Bus partitioning $\mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G$ induces **branch partitioning**

$$\mathcal{E} = \mathcal{E}^{\ell\ell} \cup \mathcal{E}^{g\ell} \cup \mathcal{E}^{gg}\,, \quad A = \left(\begin{array}{c} A_L \\ \hline A_G \end{array}\right) = \left(\begin{array}{c|c|c} A_L^{\ell\ell} & A_L^{g\ell} & \mathbb{0} \\ \hline \mathbb{0} & A_G^{g\ell} & A_G^{gg} \end{array}\right).$$

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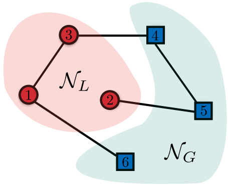

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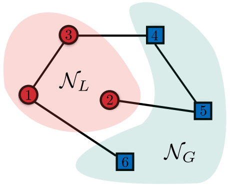

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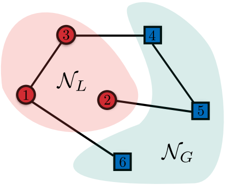

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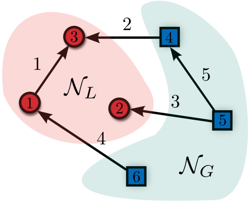

$$A = \overbrace{}^{\mathcal{E}^{\ell\ell}}\overbrace{}^{\mathcal{E}^{g\ell}}\overbrace{}^{\mathcal{E}^{gg}} \left(\begin{array}{c|ccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \mathcal{N}_L \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \mathcal{N}_G \end{array}$$

Network Notation II: Open-Circuit Voltages

Power Flow Equations

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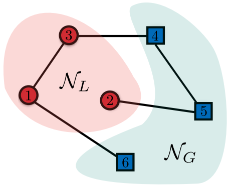
- **Generators $\mathcal{N}_G$:** V_i fixed
- **Loads $\mathcal{N}_L$:** V_i free



Partitioned Variables

$$V = \begin{pmatrix} V_L \\ \hline V_G \end{pmatrix}, \qquad B = \left(\begin{array}{c|c} B_{LL} & B_{LG} \\ \hline B_{GL} & B_{GG} \end{array}\right)$$

Open-circuit voltages

$$V_L^* \triangleq \underbrace{-B_{LL}^{-1} B_{LG}}_{\text{Generators}\to\text{Loads}} \cdot V_G$$

Scaled voltages

$$v_i \triangleq V_i / V_i^*$$

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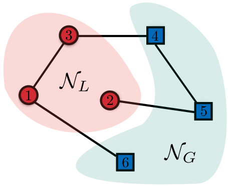

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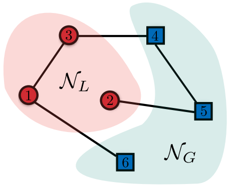

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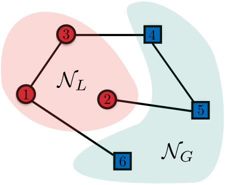

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$$V = \begin{pmatrix} V_L \\ \hline V_G \end{pmatrix}, \quad B = \left(\begin{array}{c|c} B_{LL} & B_{LG} \\ \hline B_{GL} & B_{GG} \end{array}\right), \quad V_L^* = -B_{LL}^{-1} B_{LG} V_G$$

- Need to **non-dimensionalize** power flow equations
- **Stiffness matrices** quantify **grid strength** in **units of power**

1. **Nodal** stiffness matrix $\quad \mathsf{S} \triangleq \frac{1}{4}[V_L^*] \cdot B_{LL} \cdot [V_L^*]$
2. **Branch** stiffness matrix $\quad \mathsf{D} \triangleq [V_i^* V_j^* B_{ij}]_{(i,j)\in\mathcal{E}}$
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Active Power Flow Reformulation

Notation:

$$h_e(v) = \begin{cases} v_i v_j & \text{if } e = (i,j) \in \mathcal{E}^{\ell\ell} \\ v_j & \text{if } e = (i,j) \in \mathcal{E}^{g\ell} \\ 1 & \text{if } e = (i,j) \in \mathcal{E}^{gg} \end{cases}$$

Active Power:

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)$$

$$P = \underbrace{A}_{\text{Incidence}} \underbrace{\mathsf{D}}_{\text{Branch Stiff.}} \underbrace{[h(v)]}_{\text{Voltages}} \underbrace{\mathbf{sin}(A^{\mathsf{T}}\theta)}_{\sin(\theta_i - \theta_j)}$$

- Let columns of C be a basis for $\ker(A)$, let $p_c \in \mathbb{R}^c$

Semi-Explicit Solution

$$\mathbf{sin}(A^{\mathsf{T}}\theta) = \psi(v, p_c) \triangleq [h(v)]^{-1} \left(A^{\mathsf{T}} \mathsf{L}^{\dagger} P + \mathsf{D}^{-1} C p_c \right)$$

$$\mathbb{0} = C^{\mathsf{T}} \mathbf{arcsin}(\psi)$$

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Reactive Power Reformulation

Skipping some details ...

$$Q_L = -4[v]\mathsf{S}(v - \mathbb{1}_n) + |A|_L \mathsf{D}\,[h(v)](\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^\mathsf{T}\theta))\,.$$

- Rearrange for v

$$v = f(v,\theta) = \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\,(\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^\mathsf{T}\theta))\,,$$

- Now plug in $\cos(z) = \sqrt{1-\sin^2(z)}$!

Reactive Power Reformulation

Skipping some details . . .

$$Q_L = -4[v] \underbrace{S}_{\text{Nodal stiff.}} (v - \mathbb{1}_n) + |A|_L \mathrm{D}\,[h(v)](\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^\mathsf{T}\theta))\,.$$

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- Rearrange for v

$$\begin{aligned} v = f(v,\theta) = \mathbb{1}_n &- \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n \\ &+ \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\,(\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^\mathsf{T}\theta))\,, \end{aligned}$$

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$$Q_L = -4[v]\mathsf{S}(v - \mathbb{1}_n) + |A|_L \mathsf{D}\,[h(v)](\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^\mathsf{T}\theta))\,.$$

- Rearrange for v

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- Now plug in $\cos(z) = \sqrt{1 - \sin^2(z)}$!

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Main Modeling Result

Fixed-Point Power Flow: Meshed Networks

(θ, V_L) is a power flow solution iff (v, p_c) solves the FPPF

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New Approximate Power Flow Solution

- The model says $v = f(v, p_c)$, and $\mathbf{sin}(A^\mathsf{T}\theta) = \psi(v, p_c)$.
- By construction, when $P = Q_L = \mathbb{0}$, a solution is

$$v = \mathbb{1}_n, \quad p_c = \mathbb{0}_c\,, \quad A^\mathsf{T}\theta = \mathbb{0}_{|\mathcal{E}|}\,.$$

- **Taylor expand** FPPF model around this solution

$$\begin{aligned} A^\mathsf{T}\theta_{\text{approx}} &= A^\mathsf{T} L^\dagger P \\ v_{\text{approx}} &\simeq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}Q_L + \frac{1}{8}\mathsf{S}^{-1}|A|_L \mathsf{D}[A^\mathsf{T}\mathsf{L}^\dagger P]A^\mathsf{T}\mathsf{L}^\dagger P \\ p_{c,\text{approx}} &= \mathbb{0} \end{aligned}$$

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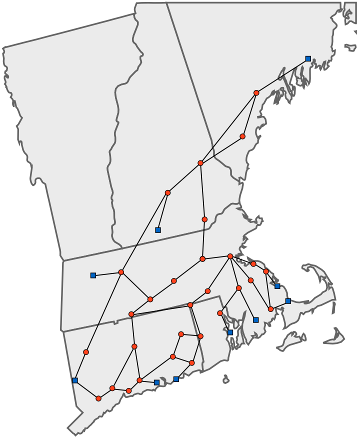

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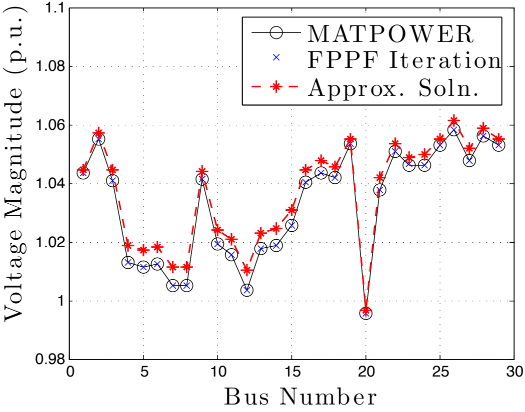

Numerical Results

$$\delta_{\max} = \|v - v_{\text{approx}}\|_\infty , \quad \delta_{\text{avg}} = \frac{1}{n}\|v - v_{\text{approx}}\|_1$$

	Base Load			High Load	
Test Case	FPPF Iters.	$\delta_{\max}$ (p.u.)	δ_{avg} (p.u.)	FPPF Iters.	$\delta_{\max}$ (p.u.)
New England 39	4	0.006	0.004	8	0.086
57 bus system	5	0.011	0.003	8	0.118
RTS '96 (3 area)	4	0.003	0.001	8	0.084
118 bus system	3	0.001	0.000	7	0.054
300 bus system	6	0.022	0.004	8	0.059
PEGASE 1,354	5	0.011	0.001	8	0.070
Polish 2,383 wp	4	0.003	0.000	8	0.078
PEGASE 2,869	5	0.015	0.002	8	0.098
PEGASE 9,241	6	0.063	0.003	9	0.133

FPPF Simplifies for Acyclic Networks

Fixed-Point Power Flow: Radial Networks

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On what invariant set is f a **contraction**?

Solvability Results for Different Acyclic Topologies

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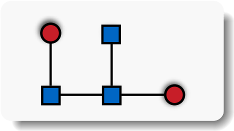
PQ buses have one PV bus neighbor



Sufficient + Necessary
Existence + Uniqueness

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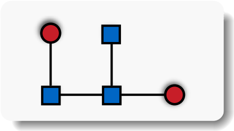

Sufficient + Necessary
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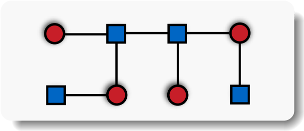
PQ buses have many PV bus neighbors



Sufficient + Tight
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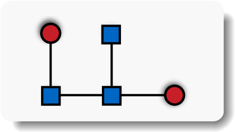

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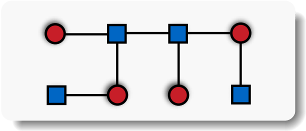

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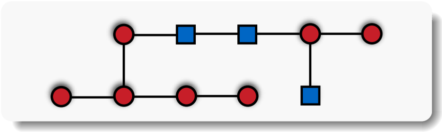
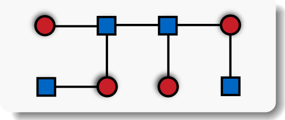
General interconnections



Sufficient
Existence

Solvability Results for Type II Networks



Theorem: Conditions for Power Flow Solvability

Define the dimensionless loading margins

$$\Delta_i \triangleq Q_i/\mathsf{S}_{ii}\,, \qquad \Gamma_i \triangleq \max_j |p_{ji}|/\mathsf{D}_{ji}\,, \qquad i \in \mathcal{N}_L$$

$$\Gamma_{ij} \triangleq |p_{ij}|/\mathsf{D}_{ij}\,, \qquad (i,j) \in \mathcal{E}^{gg}\,.$$

If it holds that

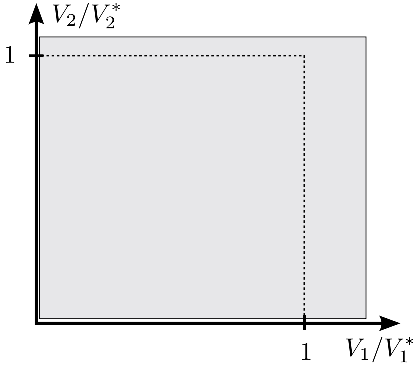
$$\max_{i \in \mathcal{N}_L} \; \Delta_i + 4\Gamma_i^2 < 1\,, \qquad \text{and} \qquad \max_{(i,j) \in \mathcal{E}^{gg}} \Gamma_{ij} < 1\,,$$

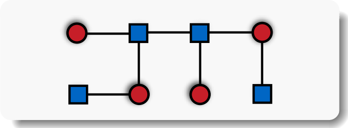
then $\exists!$ solution (V_L, θ) satisfying

$$\frac{1}{2} \leq v_i^+ \leq V_i/V_i^* \leq 1\,, \qquad \max_j |\theta_i - \theta_j| \leq \gamma_i < \frac{\pi}{4}\,, \qquad i \in \mathcal{N}_L$$

$$|\theta_i - \theta_j| \leq \gamma_{ij} < \frac{\pi}{2}\,, \qquad (i,j) \in \mathcal{E}^{gg}\,.$$

Partitioning of Voltage Space





$$\max_{i\in\mathcal{N}_L} \Delta_i + 4\Gamma_i^2 < 1$$

$$\max_{(i,j)\in\mathcal{E}^{gg}} \Gamma_{ij} < 1\,,$$

$$v_{i,\pm} \triangleq \sqrt{\frac{1}{2}\left(1 - \frac{\Delta_i}{2} \pm \sqrt{1-(\Delta_i + 4\Gamma_i^2)}\right)}$$

$$v_{i,+}^2 - v_{i,-}^2 = \sqrt{1-(\Delta_i + 4\Gamma_i^2)}\,.$$

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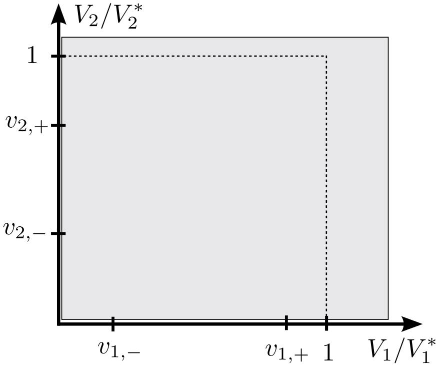

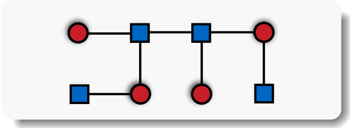

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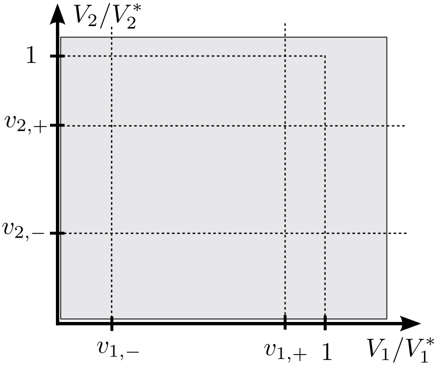

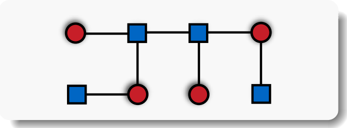

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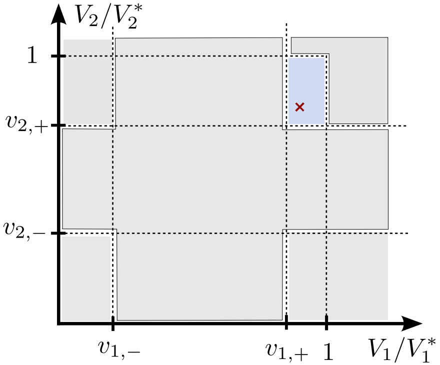

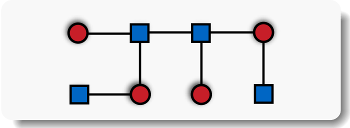

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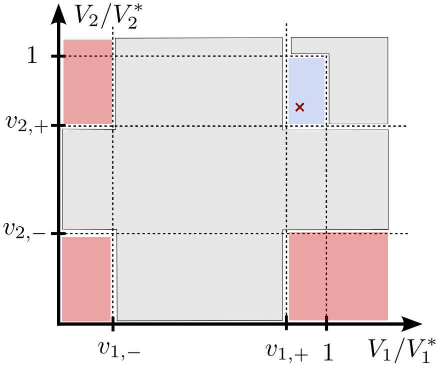

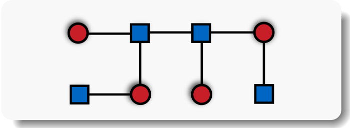

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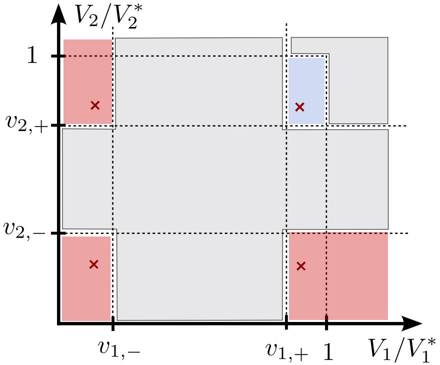

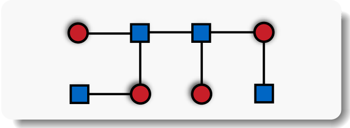

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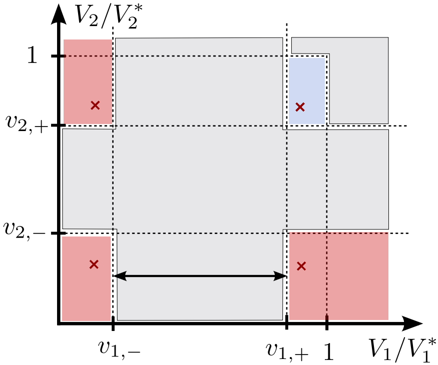

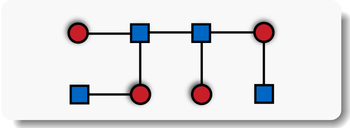

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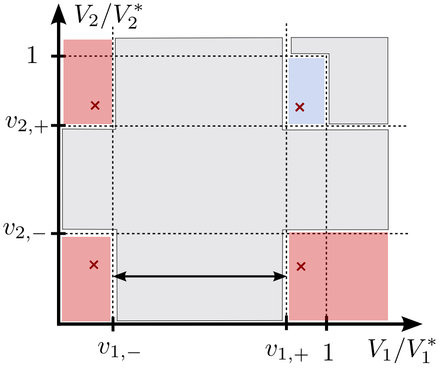

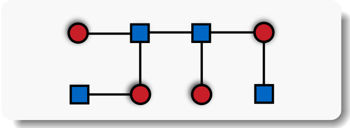

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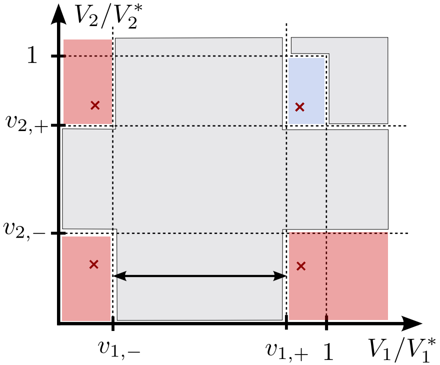

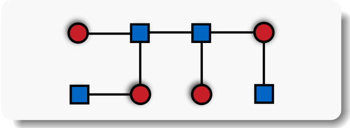

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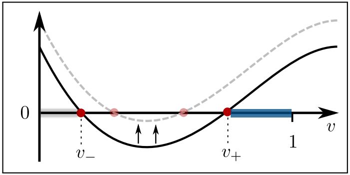

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Conclusions

Framework for studying **Lossless Power Flow**:

1. Fixed-Point Power Flow
2. Approximate solution

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John W. Simpson-Porco, *Member, IEEE*

New **conditions for power flow solvability**:

3. Contractive iteration
4. Existence/uniqueness
5. Generalizes known results

What's next?

1. Analysis for **meshed networks**
2. Extension for lossy networks
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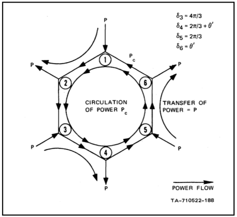

Questions

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appendix