A Hill-Moylan Lemma for Equilibrium Independent Dissipativity

American Control Conference

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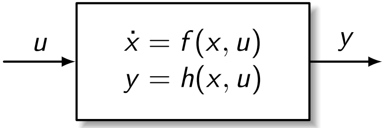
Dissipative Dynamical Systems: State-Space Models



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$$0 = h(0, 0)$$

1. Use C^1 **storage function** $V_0 : \mathcal{X} \to \mathbb{R}_{\geq 0}$ to measure "energy"
2. Use function $w : \mathcal{U} \times \mathcal{Y} \to \mathbb{R}$ to measure "dissipation"

Dissipativity [Willems '72]

System is dissipative w.r.t. supply rate $w(\cdot, \cdot)$ if there exists a positive definite storage function with $V_0(0) = 0$ s.t.

$$\mathcal{L}_f V_0(x) = \nabla V_0(x)^\mathsf{T} f(x, u) \leq w(u, y)$$

for all $t \geq 0$ and all input signals $u : \mathbb{R}_{\geq 0} \to \mathcal{U}$

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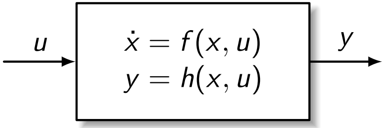

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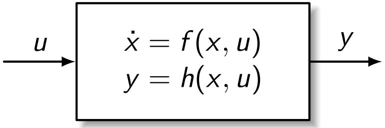

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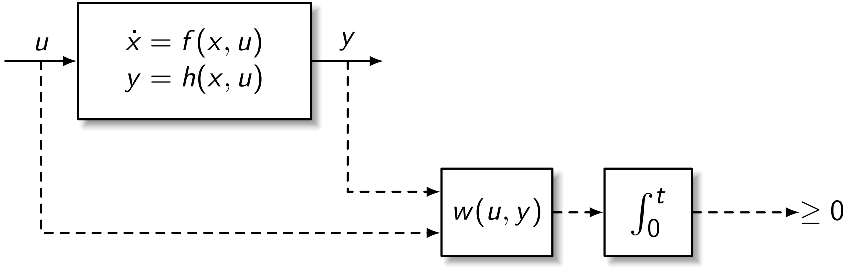
Dissipative Dynamical Systems: State-Space Models

(Lyapunov Theory with Inputs and Outputs)



- many interesting cases in restriction to **quadratic supply rates**

$$w(u, y) = \begin{bmatrix} y \\ u \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}$$

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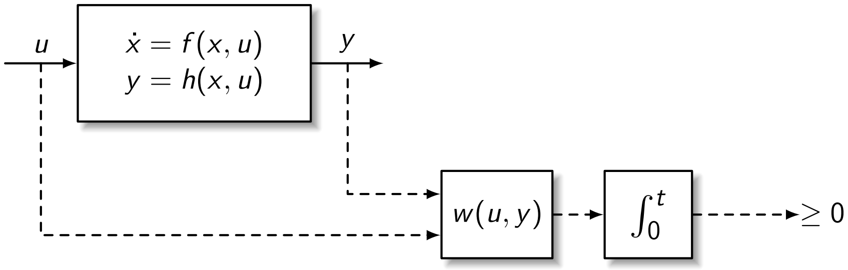

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Applications of Dissipativity Theory

1. System analysis
   - Passivity / small-gain / conic sector theorems
   - Absolute stability
   - Diagonal stability / large-scale systems
   - Network analysis
2. Design methodologies
   - Nonlinear $\mathcal{H}_\infty$ control [van der Schaft *et al.*]
   - Backstepping [Kokotovic *et al.*]
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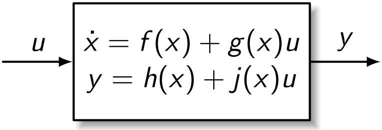
Algebraic Characterization for Control-Affine Systems



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Lemma [Hill-Moylan '76]

System is dissipative w.r.t. supply rate $w(\cdot, \cdot)$ with C^1 storage function $V_0 : \mathcal{X} \to \mathbb{R}_{\geq 0}$ iff $\exists k \in \mathbb{Z}_{>0}$, $W : \mathcal{X} \to \mathbb{R}^{k \times m}$ and $l : \mathcal{X} \to \mathbb{R}^k$ s.t.

$$\begin{aligned} \nabla V_0(x)^{\mathsf{T}} f(x) &= h(x)^{\mathsf{T}} Q h(x) - l(x)^{\mathsf{T}} l(x) \\ \frac{1}{2} \nabla V_0(x)^{\mathsf{T}} g(x) &= h(x)^{\mathsf{T}} (Q j(x) + S) - l(x)^{\mathsf{T}} W(x) \\ W(x)^{\mathsf{T}} W(x) &= R + j(x)^{\mathsf{T}} S + S^{\mathsf{T}} j(x) + j(x)^{\mathsf{T}} Q j(x) \,. \end{aligned}$$

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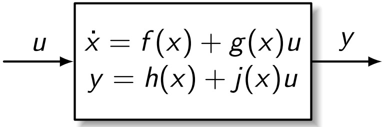

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Forced Equilibria and Dissipativity

- Dissipation inequality is w.r.t. $(u, x, y) = (0, 0, 0)$ equilibrium

$$\frac{\mathrm{d}}{\mathrm{d}t} V_0(x(t)) \leq w(u - 0, y - 0)$$

- Often however interested in **forced equilibria** $(\bar{u}, \bar{x})$

$$\mathcal{E} := \{\bar{x} \in \mathcal{X} \ : \ \exists \bar{u} \in \mathcal{U} \text{ s.t. } f(\bar{x}) + g(\bar{x})\bar{u} = 0\}$$

- Why care? Changing operating points, uncertain interconnections

Storage function $V_0(x)$ need not be useful for establishing dissipativity at **other** operating points

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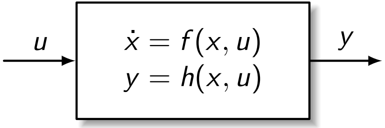
Equilibrium-Independent Dissipativity



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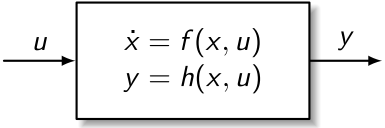

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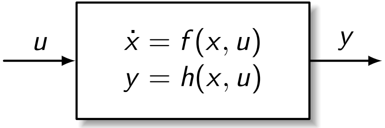

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Hill-Moylan Characterization of Equilibrium-Indep. Diss

What is the analogous Hill-Moylan-type algebraic characterization of equilibrium-independent dissipative systems?

- We will restrict attention to
  - control-affine systems with **constant** input/throughput matrices
  - quadratic supply rates

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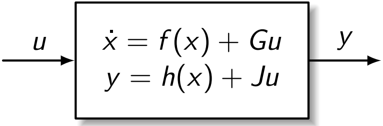

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Storage Functions via Bregman Divergence

[Jayawardhana et al.]

- For $V : \mathbb{R}^n \to \mathbb{R}$ convex, define **Bregman divergence** of V at $\bar{x}$:

$$V_{\bar{x}}(x) = V(x) - \underbrace{\left[V(\bar{x}) + \nabla V(\bar{x})^{\mathsf{T}}(x - \bar{x})\right]}_{\text{Linearization at } \bar{x}}$$

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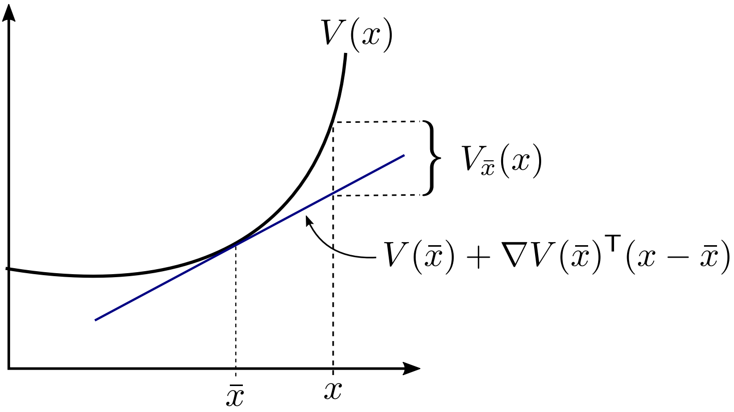

Hill-Moylan Characterization of Equilibrium-Indep. Diss.

Main Result

Let $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ be C^1 and convex and for $\bar{x} \in \mathcal{E}$ let

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- Result simplifies when W **has full row rank**

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$$\text{HM '76:} \quad \begin{cases} \frac{1}{2}\nabla V(x)^{\mathsf{T}} G = h(x)^{\mathsf{T}}(QJ + S) - l(x)^{\mathsf{T}} W \\ W^{\mathsf{T}} W = R + J^{\mathsf{T}} S + S^{\mathsf{T}} J + J^{\mathsf{T}} QJ \end{cases}$$

Special Case of Main Result: Equilibrium-Indep. Passivity

1. For **passive** systems without feedthrough

$$[\nabla V(x) - \nabla V(\bar{x})]^{\mathsf{T}} [f(x) - f(\bar{x})] \leq -\|\ell(x, \bar{x})\|_2^2 \qquad (\star)$$
$$G^{\mathsf{T}} \nabla V(x) = h(x) \qquad (\star\star)$$

2. If $V(x) = x^{\mathsf{T}} P x$, then $(\star)$ implies **Krasovskii**-type condition

$$\left(\frac{\partial f}{\partial x}(x)\right)^{\mathsf{T}} P + P \left(\frac{\partial f}{\partial x}(x)\right) \preceq 0\,, \qquad x \in \mathcal{X}$$

   for incremental stability.

3. Generalization of SISO Popov-type Lyapunov result in [Arcak, Meissen, Packard '16] in the paper

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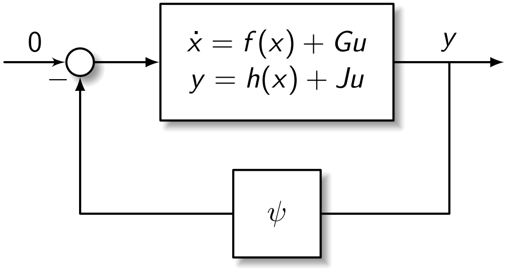
Application: Equilibrium-Independent Absolute Stability



- Static map $\psi : \mathbb{R}^m \to \mathbb{R}^m$ is $[K_1, K_2]$ **slope-restricted**

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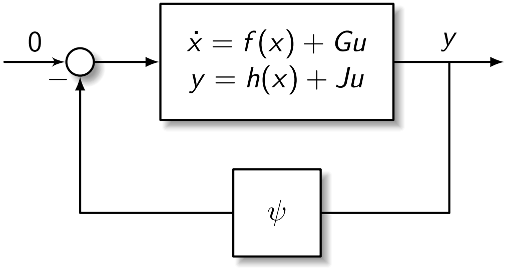
Application: Equilibrium-Independent Absolute Stability



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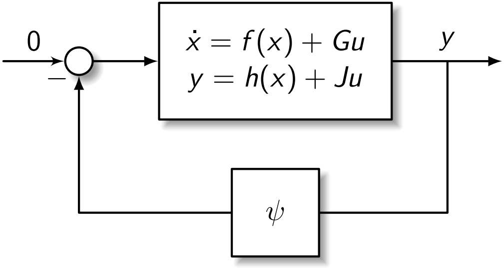

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Equilibrium-Independent Circle Criterion

Suppose that the *loop-transformed system*

$$\Sigma' : \begin{cases} \dot{x} = f(x) - GK_1 h(x) + Gu_\ell \\ y_\ell = (K_2 - K_1)h(x) + u_\ell \end{cases}$$

satisfies the main result w.r.t. supply rate

$$w(u_\ell, y_\ell) = -\varepsilon \|y_\ell - \bar{y}_\ell\|_2^2 + (y_\ell - \bar{y}_\ell)^\mathsf{T}(u_\ell - \bar{u}_\ell)$$

for some $\varepsilon > 0$ and with $V(x)$ strongly convex. Then the closed-loop possesses a unique and globally asymptotically stable equilibrium point.

- Existence/uniqueness of equilibrium is *inferred* through **maximal monotonicity** of subsystem input-output relations

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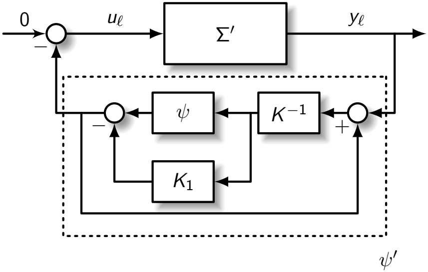
Proof Sketch

1. Establish 1-1 correspondence between equilibria of original and **loop transformed** system



2. ψ' is maximally monotone, static I/O relation of Σ' is **strongly** maximally montotone $\implies$ Existence/uniqueness of equilibrium $\bar{x}$
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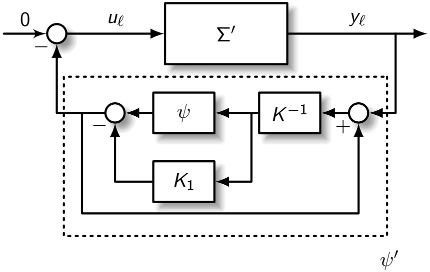

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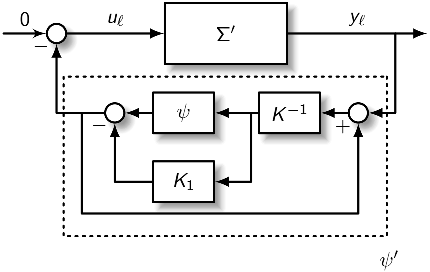

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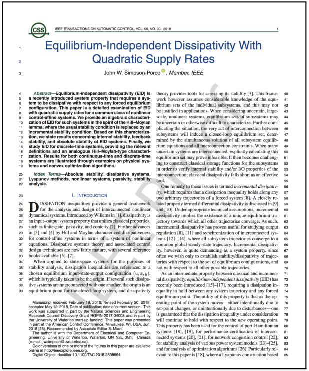
Conclusions

Hill-Moylan-type result for **equilibrium-independent dissipativity**:

1. result is an "incremental" variant of classic result
2. provides framework for equilibrium-independent stability studies

Extended version to appear in TAC

1. examples
2. unabridged proofs
3. monotonicity of static I/O relations
4. feedback stability theorems, existence of closed-loop equilibria
5. analogous discrete-time results
6. application to gradient method


IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 00, NO. 00, 2018

Equilibrium-Independent Dissipativity With Quadratic Supply Rates

John W. Simpson-Porco, Member, IEEE

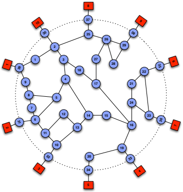
https://ece.uwaterloo.ca/~jwsimpso/
jwsimpson@uwaterloo.ca

appendix

The Optimal Frequency Regulation Problem

(Simplified, Linearized, and Network-Reduced)

Grid: $\mathcal{G} = (\mathcal{V}, \mathcal{E}, B)$



Swing Dynamics

$$\dot{\theta}_i = \omega_i$$

$$M_i \dot{\omega}_i = -D_i \omega_i + P_i^* - P_{\mathrm{e},i}(\theta) + p_i$$

(Linearized) Power Flow

$$P_{\mathrm{e},i}(\theta) = \sum_{j=1}^{n} B_{ij}(\theta_i - \theta_j)\,,$$

Problem: How to pick controls p_i for (i) $\omega_{\mathrm{ss}} = 0$ (ii) optimality?

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$$\underset{p \in \mathbb{R}^n}{\text{minimize}} \quad \sum_{i=1}^{n} \frac{1}{2} k_i p_i^2$$

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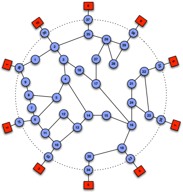

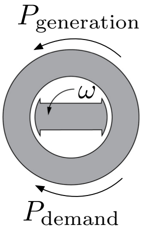

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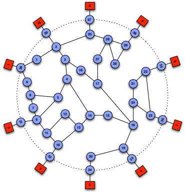

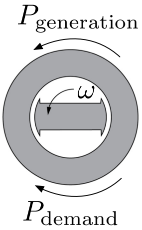

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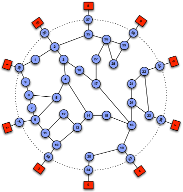

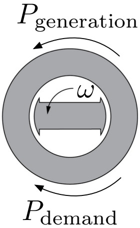

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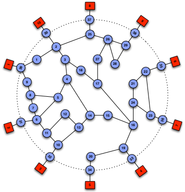

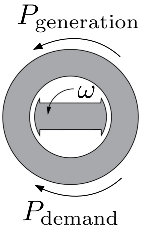

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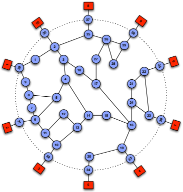

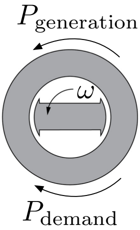

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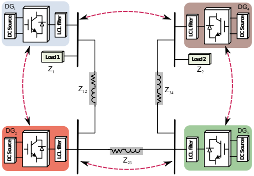
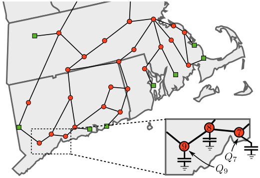
Smart Grid Project Samples

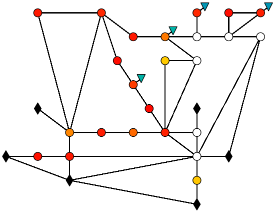
Distributed Inverter Control



Voltage Collapse (Nat. Comms.)



Optimal Distrib. Volt/Var (CDC)



Wide-Area Monitoring (TSG)