A Theory of Solvability for Lossless Power Flow Equations

John W. Simpson-Porco

UNIVERSITY OF WATERLOO

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This talk is based on these papers

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A Theory of Solvability for Lossless Power Flow Equations—Part I: Fixed-Point Power Flow

John W. Simpson-Porco, *Member, IEEE*

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A Theory of Solvability for Lossless Power Flow Equations—Part II: Conditions for Radial Networks

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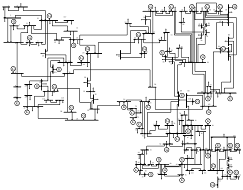
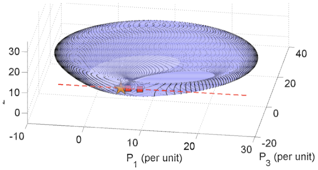
Problems in power system operations

Power Flow Analysis

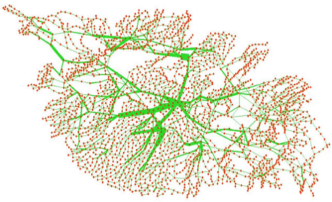


Optimal Power Flow



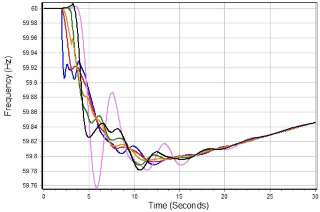
[Molzahn et al.]

Contingency Analysis



[Rezaei et al.]

Transient Stability



[Overbye et al.]

Modeling AC power flow

- active power: $P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)$
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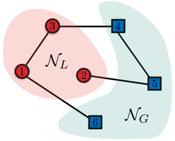

6. n **Loads** (●) **and** m **Generators** (■) $\mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G$
7. **Load Model:** PQ bus constant P_i constant Q_i
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$2n + m$ equations in variables $\theta \in \mathbb{T}^{n+m}$ and $V_L \in \mathbb{R}^n_{>0}$.

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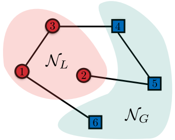

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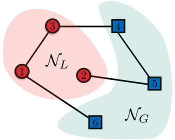

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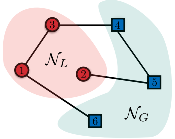

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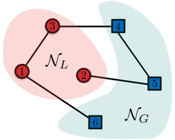

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1. **Because it is interesting to do so**
2. **Numerical methods**
   - understand convergence, divergence, and initialization issues

- **State vector:** $x = (\theta, V_L)$
- **Newton iteration:**

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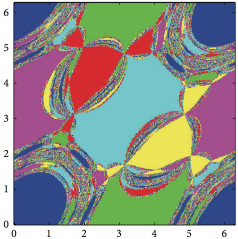

[Deng et al.]

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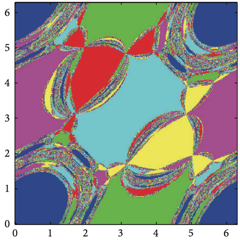

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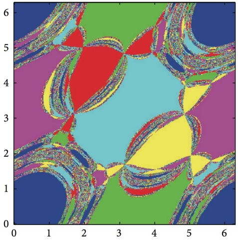

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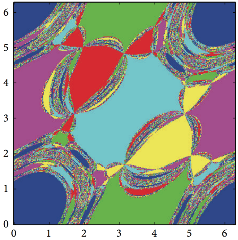

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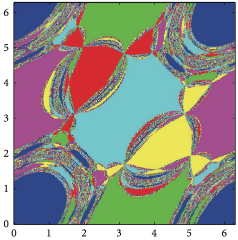

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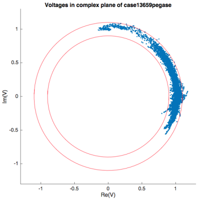
1. 'Normally', exists unique **high-voltage** soln:
    - voltage magnitude $V_i \simeq 1$
    - phase diff $|\theta_i - \theta_j| \ll 1$



[Josz et al.]

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    - saddle node bifurcations
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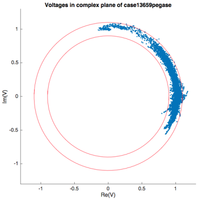

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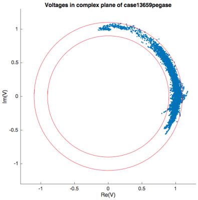

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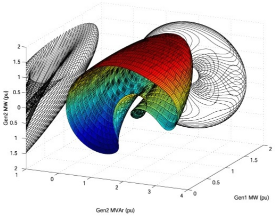

[Hiskens & Davy]

Mysteries of power flow

Given data: network topology, impedances, generation & loads

Q: ∃ "stable high-voltage" solution? unique? properties?

Partial answers from **45+ years** of literature:

- Jacobian singularity [Weedy '67]
- Multiple dynamically stable solutions [Korsak '72]
- Existence conditions [Wu & Kumagai '80, '82]
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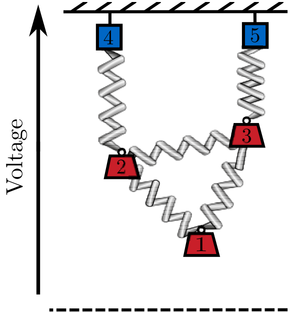
Main insight: stiffness vs. loading

1. Stiff network + light loading $\Rightarrow$ feasible
2. Weak network + heavy loading $\Rightarrow$ infeasible

Q: How to quantify network stiffness vs. loading?



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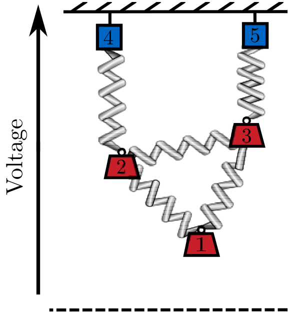

Solution of Two-Bus System I

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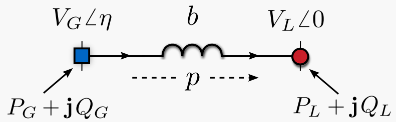

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2. **Eliminate η**

$$p^2 + (Q_L - bV_L^2)^2 = b^2 V_G^2 V_L^2 ,$$

3. **Change Variables**

$$v := \frac{V_L}{V_G} \qquad \Gamma := \frac{p}{bV_G^2} \qquad \Delta := \frac{Q_L}{-\frac{1}{4}bV_G^2}$$

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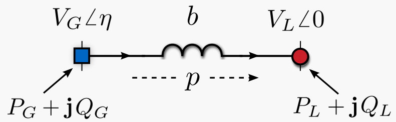

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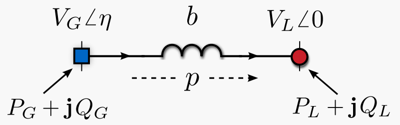

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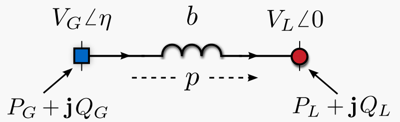

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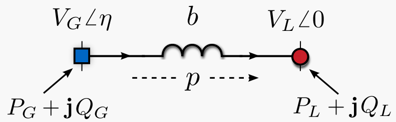

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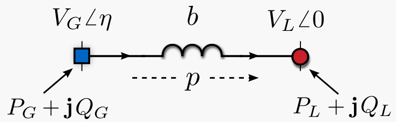

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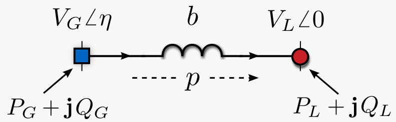
$$Q_L = bV_L^2 - bV_L V_G \cos(\eta)$$



3. **Change Variables**

$$v := \frac{V_L}{V_G} \qquad \Gamma := \frac{p}{bV_G^2} \qquad \Delta := \frac{Q_L}{-\frac{1}{4}bV_G^2}$$

4. **Solve Quadratic in v^2**

$$v_\pm = \sqrt{\frac{1}{2}\left(1 - \frac{\Delta}{2} \pm \sqrt{1 - (4\Gamma^2 + \Delta)}\right)}$$

5. **Nec. & Suff. Condition**

$$4\Gamma^2 + \Delta < 1$$

Solution of Two-Bus System III

$$\Gamma = v \sin(\eta)$$
$$\Delta = -4v^2 + 4v\cos(\eta)$$

$$v := \frac{V_L}{V_G} \quad \Gamma := \frac{p}{bV_G^2} \quad \Delta := \frac{Q_L}{-\frac{1}{4}bV_G^2}$$
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1. **High-voltage** solution
   $v_+ \in [\frac{1}{2}, 1)$
2. **Low-voltage** solution
   $v_- \in [0, \frac{1}{\sqrt{2}})$

Angle: $\sin(\eta_\mp) = \Gamma / v_\pm$

1. **Small-angle** solution
   $\eta_- \in [0, \pi/4)$
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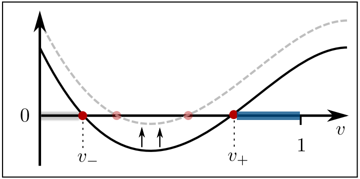

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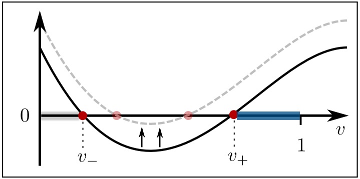

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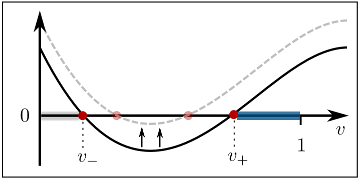

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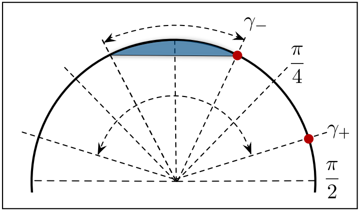

Solution of Two-Bus System IV

- Squaring and adding equations **does not generalize** to networks.
- Is there any hope then?

$$\Gamma = v\sin(\eta)$$
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- Use $\cos(\eta) = \sqrt{1-\sin^2(\eta)} \implies \Delta = -4v^2 + 4v\sqrt{1-(\Gamma/v)^2}$
- Rearrange to get *fixed-point equation*

$$v = f(v) := -\frac{1}{4}\frac{\Delta}{v} + \sqrt{1-\left(\frac{\Gamma}{v}\right)^2}$$

This generalizes!

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Network Notation I: Branches Between Bus Types

Power Flow Equations

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$

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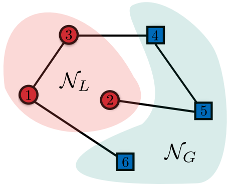

- Bus partitioning $\mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G$ induces **branch partitioning**

$$\mathcal{E} = \mathcal{E}^{\ell\ell} \cup \mathcal{E}^{g\ell} \cup \mathcal{E}^{gg}\,, \quad A = \left(\begin{array}{c} A_L \\ \hline A_G \end{array}\right) = \left(\begin{array}{c|c|c} A_L^{\ell\ell} & A_L^{g\ell} & \mathbb{0} \\ \hline \mathbb{0} & A_G^{g\ell} & A_G^{gg} \end{array}\right).$$

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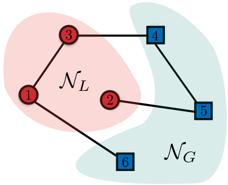

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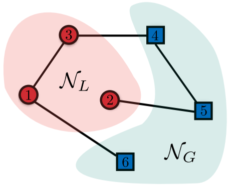

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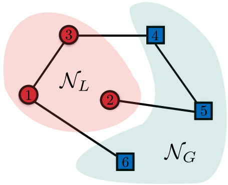

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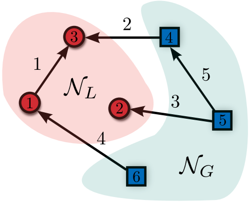

$$A = \begin{array}{c} \overbrace{}^{\mathcal{E}^{\ell\ell}} \quad \overbrace{}^{\mathcal{E}^{g\ell}} \quad \overbrace{}^{\mathcal{E}^{gg}} \\ \left(\begin{array}{c|ccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} \mathcal{N}_L \\ \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} \mathcal{N}_G \end{array}$$

Network Notation II: Open-Circuit Voltages

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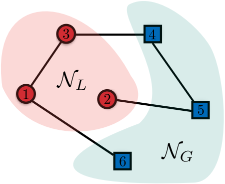
- **Loads $\mathcal{N}_L$:** V_i free



Partitioned Variables

$$V = \begin{pmatrix} V_L \\ \hline V_G \end{pmatrix}, \qquad B = \left(\begin{array}{c|c} B_{LL} & B_{LG} \\ \hline B_{GL} & B_{GG} \end{array}\right)$$

Open-circuit voltages

$$V_L^* \triangleq \underbrace{-B_{LL}^{-1} B_{LG}}_{\text{Generators}\to\text{Loads}} \cdot V_G$$

Scaled voltages

$$v_i \triangleq V_i / V_i^*$$

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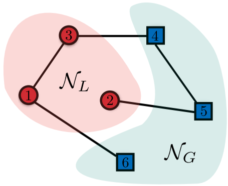

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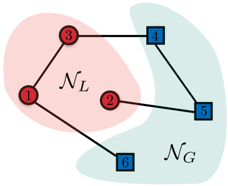

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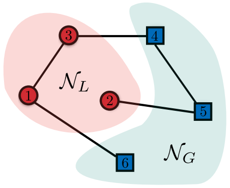

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- Need to **non-dimensionalize** power flow equations
- **Stiffness matrices** quantify **grid strength** in **units of power**

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Main Modeling Result

Fixed-Point Power Flow: Meshed Networks

(θ, V_L) is a power flow solution iff (v, p_c) solves the FPPF

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$$\mathbb{0}_c = C^{\mathsf{T}}\,\mathbf{arcsin}(\psi(v, p_c))\,.$$

where

$$u(v, p_c) \triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi}$$

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Main Modeling Result

Fixed-Point Power Flow: Meshed Networks

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New Approximate Power Flow Solution

- The model says $v = f(v, p_c)$, and $\mathbf{sin}(A^\mathsf{T}\theta) = \psi(v, p_c)$.
- By construction, when $P = Q_L = \mathbb{0}$, a solution is

$$v = \mathbb{1}_n, \quad p_c = \mathbb{0}_c\,, \quad A^\mathsf{T}\theta = \mathbb{0}_{|\mathcal{E}|}\,.$$

- **Taylor expand** FPPF model around this solution

$$A^\mathsf{T}\theta_{\mathrm{approx}} = A^\mathsf{T} L^\dagger P$$

$$v_{\mathrm{approx}} \simeq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}Q_L + \frac{1}{8}\mathsf{S}^{-1}|A|_L \mathsf{D}[A^\mathsf{T}\mathsf{L}^\dagger P]A^\mathsf{T}\mathsf{L}^\dagger P$$

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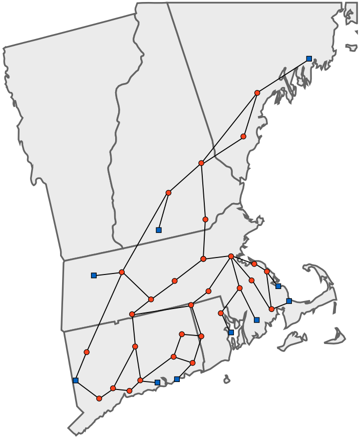

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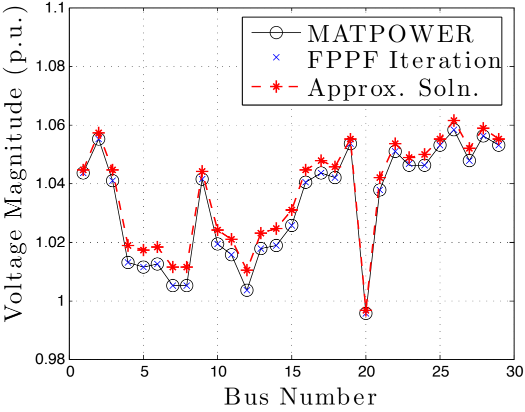

Numerical Results

$$\delta_{\max} = \|v - v_{\text{approx}}\|_\infty , \quad \delta_{\text{avg}} = \frac{1}{n}\|v - v_{\text{approx}}\|_1$$

	Base Load			High Load	
Test Case	FPPF Iters.	$\delta_{\max}$ (p.u.)	δ_{avg} (p.u.)	FPPF Iters.	$\delta_{\max}$ (p.u.)
New England 39	4	0.006	0.004	8	0.086
57 bus system	5	0.011	0.003	8	0.118
RTS '96 (3 area)	4	0.003	0.001	8	0.084
118 bus system	3	0.001	0.000	7	0.054
300 bus system	6	0.022	0.004	8	0.059
PEGASE 1,354	5	0.011	0.001	8	0.070
Polish 2,383 wp	4	0.003	0.000	8	0.078
PEGASE 2,869	5	0.015	0.002	8	0.098
PEGASE 9,241	6	0.063	0.003	9	0.133

FPPF Simplifies for Acyclic Networks

Fixed-Point Power Flow: Radial Networks

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On what invariant set is f a **contraction**?

Solvability Results for Different Acyclic Topologies

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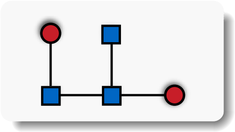
PQ buses have one PV bus neighbor



Sufficient + Necessary
Existence + Uniqueness

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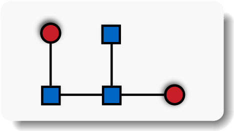

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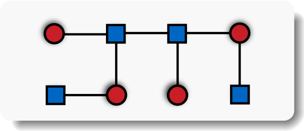
PQ buses have many PV bus neighbors



Sufficient + Tight
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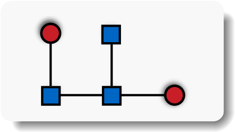

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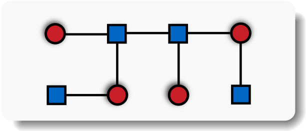

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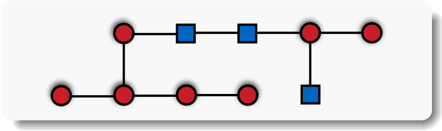
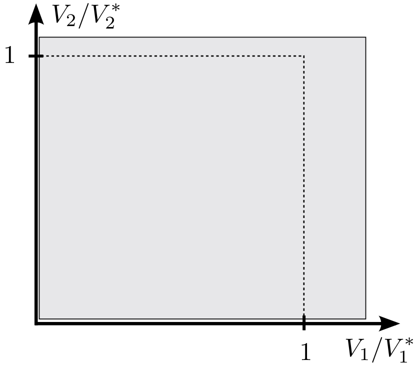
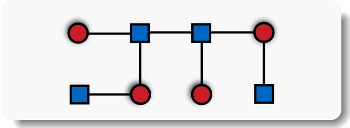
General interconnections



Sufficient
Existence

Partitioning of Voltage Space





$$\max_{i\in\mathcal{N}_L} \Delta_i + 4\Gamma_i^2 < 1$$

$$\max_{(i,j)\in\mathcal{E}^{gg}} \Gamma_{ij} < 1\,,$$

$$v_{i,\pm} \triangleq \sqrt{\frac{1}{2}\left(1 - \frac{\Delta_i}{2} \pm \sqrt{1-(\Delta_i + 4\Gamma_i^2)}\right)}$$

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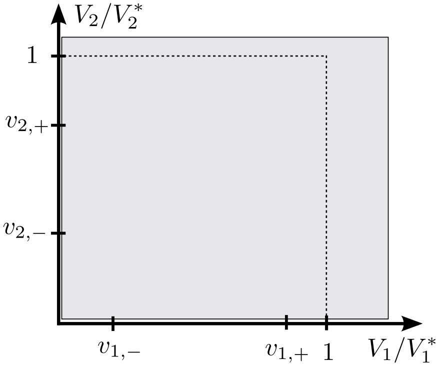

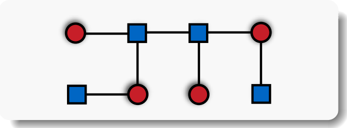

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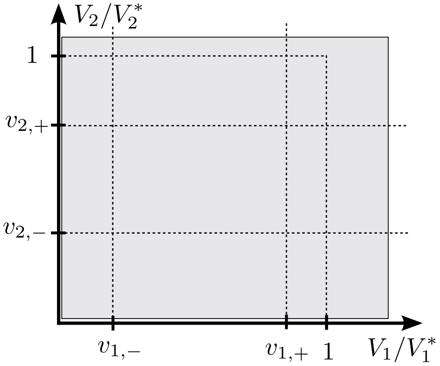

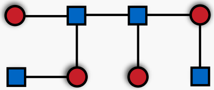

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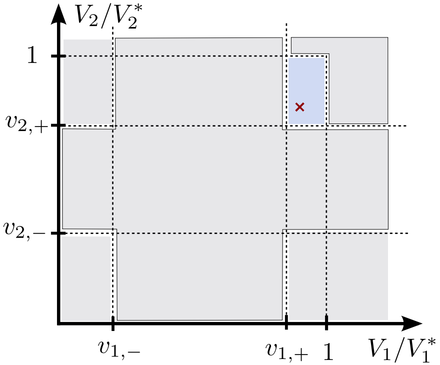

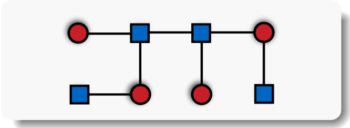

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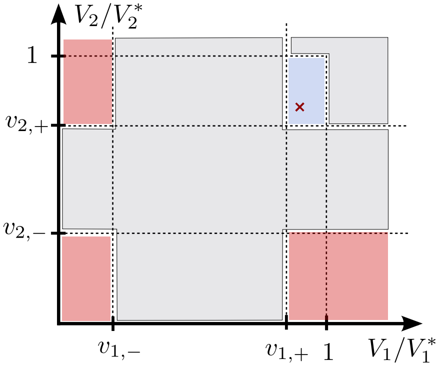

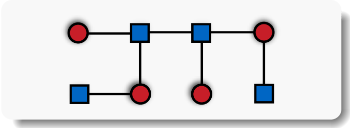

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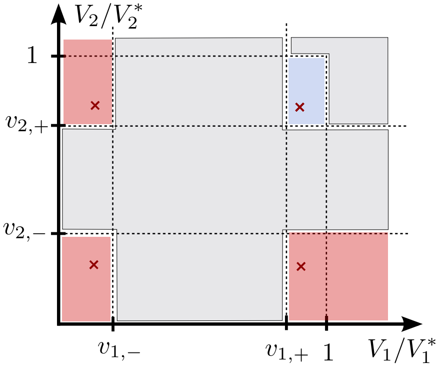

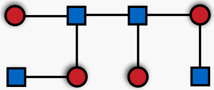

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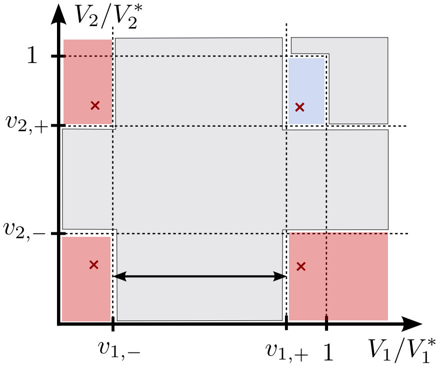
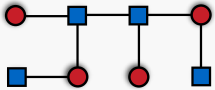
Partitioning of Voltage Space





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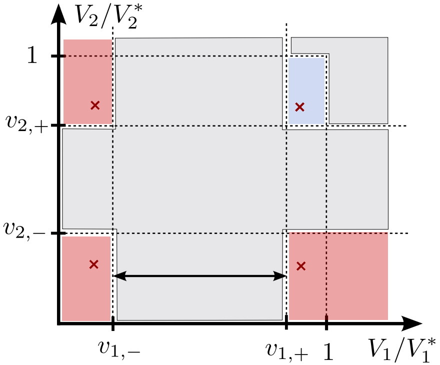

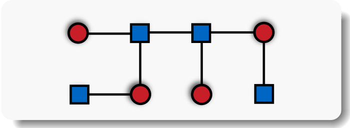

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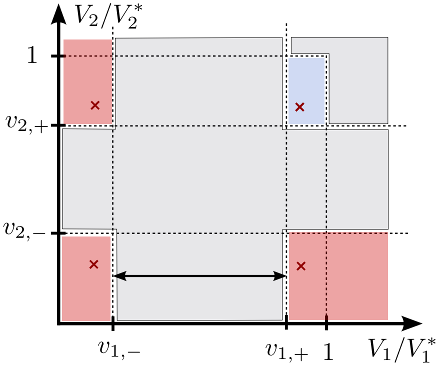

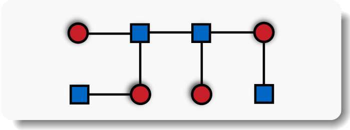

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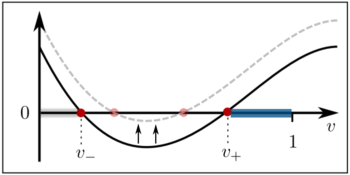

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Framework for studying **Lossless Power Flow**:

1. Fixed-Point Power Flow
2. Approximate solution

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John W. Simpson-Porco, *Member, IEEE*

New **conditions for power flow solvability**:

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4. Existence/uniqueness
5. Generalizes known results

What's next?

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2. **Lossy** networks (TPWRS: JWSP '17)
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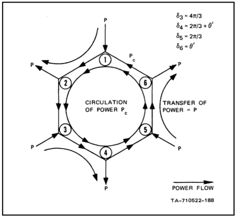

Questions

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appendix

Power flow and grid connectivity

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