A Theory of Solvability for Lossless Power Flow Equations

John W. Simpson-Porco

UNIVERSITY OF WATERLOO

Future Electric Power Systems and the Energy Transition
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This talk is based on these papers

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A Theory of Solvability for Lossless Power Flow Equations—Part I: Fixed-Point Power Flow

John W. Simpson-Porco, *Member, IEEE*

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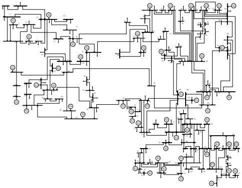
A Theory of Solvability for Lossless Power Flow Equations—Part II: Conditions for Radial Networks

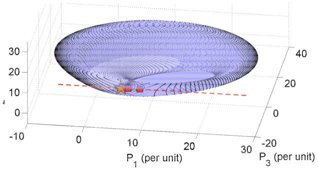
John W. Simpson-Porco, *Member, IEEE*

Problems in power system operations

Power Flow Analysis

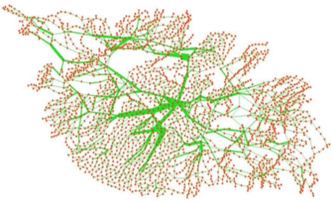


Optimal Power Flow



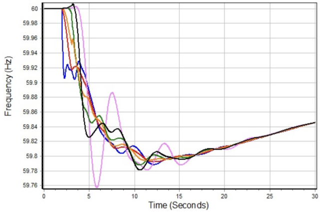
[Molzahn et al.]

Contingency Analysis



[Rezaei et al.]

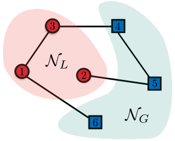
Transient Stability



[Overbye et al.]

Modeling AC power flow

- active power: $P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)$
- reactive power: $Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)$



6. n **Loads** (●) **and** m **Generators** (■) $\mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G$
7. **Load Model:** PQ bus constant P_i constant Q_i
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Power Flow Equations

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j), \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$

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$2n + m$ equations in variables $\theta \in \mathbb{T}^{n+m}$ and $V_L \in \mathbb{R}^n_{>0}$.

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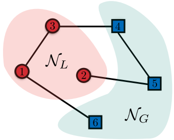

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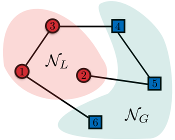

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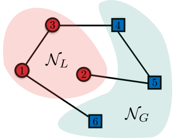

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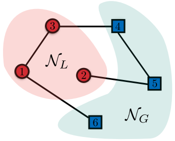

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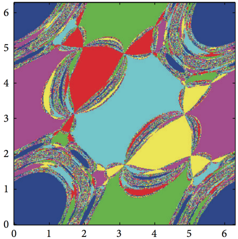
Why study solvability of power flow problems?

1. **Because it is interesting to do so**
2. **Numerical methods**
   - understand convergence, divergence, and initialization issues

   - **State vector:** $x = (\theta, V)$
   - **Newton iteration:**

$$x^{k+1} = x^k - J(\theta^k, V^k)^{-1} f(x^k)$$



[Deng et al.]

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4. **Transient stability**

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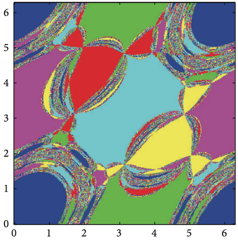

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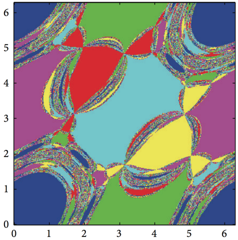

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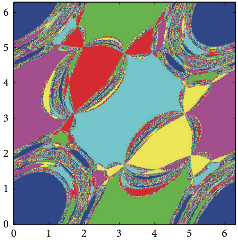

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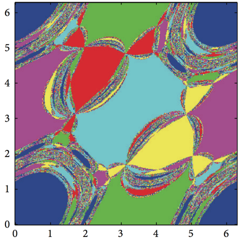

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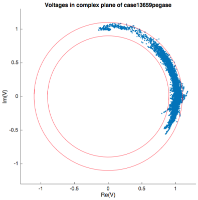
Intuition on power flow solutions

1. 'Normally', exists unique **high-voltage** soln:
   - voltage magnitude $V_i \simeq 1$
   - phase diff $|\theta_i - \theta_j| \ll 1$



[Josz et al.]

2. **Lightly loaded systems:** many **low-voltage** solutions

3. **Heavily loaded systems:** Few solutions or **infeasible**
   - saddle node bifurcations
   - maximum power transfer limit
   - non-convex feasible set in (P, Q)-space

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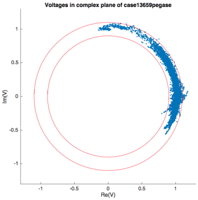

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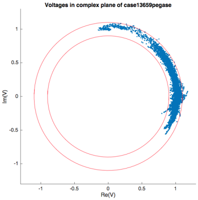

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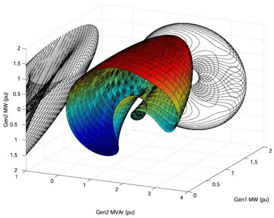

[Hiskens & Davy]

Mysteries of power flow

Given data: network topology, impedances, generation & loads

Q: ∃ "stable high-voltage" solution? unique? properties?

Partial answers from **45+ years** of literature:

- Jacobian singularity [Weedy '67]
- Multiple dynamically stable solutions [Korsak '72]
- Existence conditions [Wu & Kumagai '80, '82]
- Active power flow singularity [Araposthatis, Sastry & Varaiya, '81]
- Counting # of solutions [Baillieul and Byrnes '82]
- Properties of quadratic equations [Makarov, Hill & Hiskens '00]
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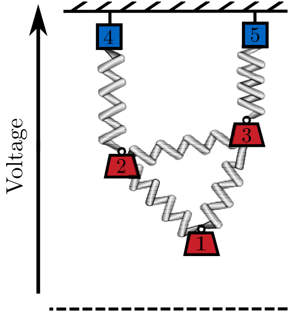
Partial answers from **45+ years** of literature:

Main insight: stiffness vs. loading

1. Stiff network + light loading ⇒ feasible
2. Weak network + heavy loading ⇒ infeasible

Q: How to quantify network stiffness vs. loading?



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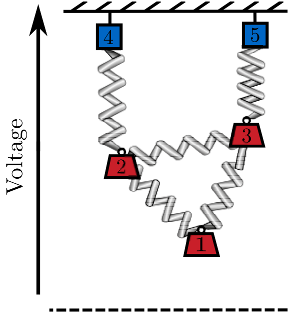

Solution of Two-Bus System

$$P_L = bV_G V_L \sin(-\eta)$$
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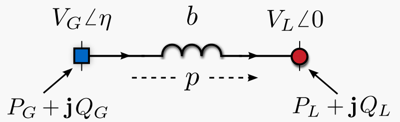

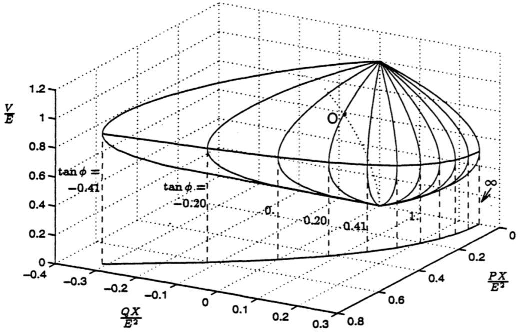

Figure 2.6 Voltage as a function of load active and reactive powers

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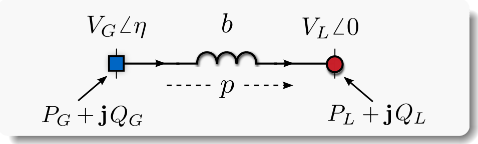

1. **Change Variables**

$$v := \frac{V_L}{V_G} \qquad \Gamma := \frac{p}{bV_G^2} \qquad \Delta := \frac{Q_L}{-\frac{1}{4}bV_G^2}$$

2. **Square equations, add, and solve quadratic in v^2**

$$v_\pm = \sqrt{\frac{1}{2}\left(1 - \frac{\Delta}{2} \pm \sqrt{1 - (4\Gamma^2 + \Delta)}\right)}$$

3. **Nec. & Suff. Condition**

$$4\Gamma^2 + \Delta < 1$$

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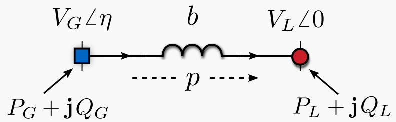

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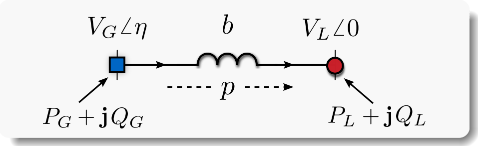

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1. **High-voltage** solution
   $v_+ \in [\frac{1}{2}, 1)$
2. **Low-voltage** solution
   $v_- \in [0, \frac{1}{\sqrt{2}})$

Angle: $\sin(\eta_\mp) = \Gamma / v_\pm$

1. **Small-angle** solution
   $\eta_- \in [0, \pi/4)$
2. **Large-angle** solution
   $\eta_+ \in [0, \pi/2)$

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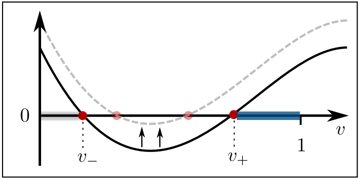

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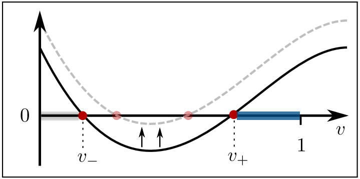

Angle: $\sin(\eta_\mp) = \Gamma / v_\pm$

1. **Small-angle** solution
   $\eta_- \in [0, \pi/4)$
2. **Large-angle** solution
   $\eta_+ \in [0, \pi/2)$

Solution of Two-Bus System

$$\Gamma = v \sin(\eta)$$
$$\Delta = -4v^2 + 4v\cos(\eta)$$

$$v := \frac{V_L}{V_G} \quad \Gamma := \frac{p}{bV_G^2} \quad \Delta := \frac{Q_L}{-\frac{1}{4}bV_G^2}$$
$$4\Gamma^2 + \Delta < 1$$

1. **High-voltage** solution
   $v_+ \in [\frac{1}{2}, 1)$
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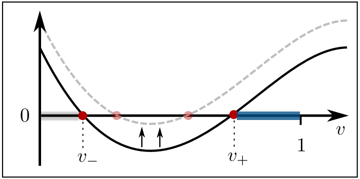

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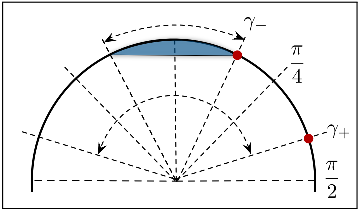

Solution of Two-Bus System IV

- Squaring and adding equations **does not generalize** to networks.
- Is there any hope then?

$$\Gamma = v \sin(\eta)$$
$$\Delta = -4v^2 + 4v\cos(\eta)$$

- Use $\cos(\eta) = \sqrt{1 - \sin^2(\eta)} \implies \Delta = -4v^2 + 4v\sqrt{1 - (\Gamma/v)^2}$
- Rearrange to get *fixed-point equation*

$$v = f(v) := -\frac{1}{4}\frac{\Delta}{v} + \sqrt{1 - \left(\frac{\Gamma}{v}\right)^2}$$

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Network Notation I: Branches Between Bus Types

Power Flow Equations

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$

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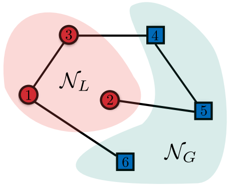

- Bus partitioning $\mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G$ induces **branch partitioning**

$$\mathcal{E} = \mathcal{E}^{\ell\ell} \cup \mathcal{E}^{g\ell} \cup \mathcal{E}^{gg}\,, \quad A = \left(\begin{array}{c} A_L \\ \hline A_G \end{array}\right) = \left(\begin{array}{c|c|c} A_L^{\ell\ell} & A_L^{g\ell} & \mathbb{0} \\ \hline \mathbb{0} & A_G^{g\ell} & A_G^{gg} \end{array}\right).$$

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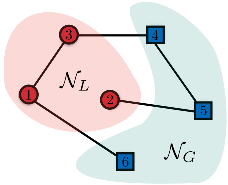

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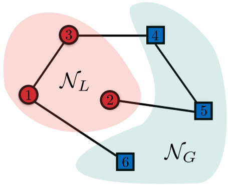

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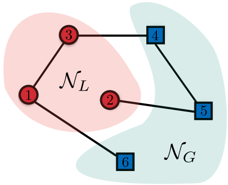

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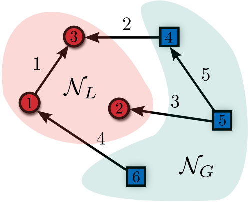

$$A = \begin{array}{c} \overbrace{}^{\mathcal{E}^{\ell\ell}} \;\; \overbrace{}^{\mathcal{E}^{g\ell}} \;\; \overbrace{}^{\mathcal{E}^{gg}} \\ \left(\begin{array}{c|ccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \mathcal{N}_L \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \mathcal{N}_G \end{array}$$

Network Notation II: Open-Circuit Voltages

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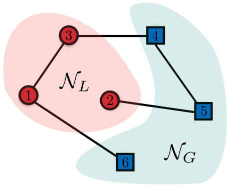

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$$V = \left(\frac{V_L}{V_G}\right), \qquad B = \left(\begin{array}{c|c} B_{LL} & B_{LG} \\ \hline B_{GL} & B_{GG} \end{array}\right)$$

Open-circuit voltages

$$V_L^* \triangleq \underbrace{-B_{LL}^{-1} B_{LG}}_{\text{Generators}\rightarrow\text{Loads}} \cdot V_G$$

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$$v_i \triangleq V_i / V_i^*$$

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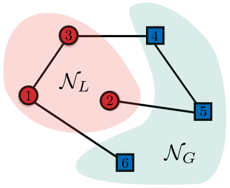

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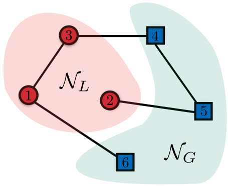

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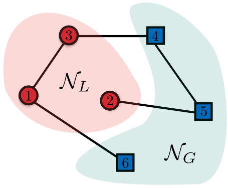

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- Need to **non-dimensionalize** power flow equations
- **Stiffness matrices** quantify **grid strength** in **units of power**

1. **Nodal** stiffness matrix $\quad S \triangleq \frac{1}{4}[V_L^*] \cdot B_{LL} \cdot [V_L^*]$
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Main Modeling Result

Fixed-Point Power Flow: Meshed Networks

(θ, V_L) is a power flow solution iff (v, p_c) solves the FPPF

$$v = f(v, p_c) \triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L\mathsf{D}\,[h(v)]\,u(v, p_c)\,,$$

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$$\begin{aligned} v = f(v, p_c) &\triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n \\ &\qquad + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L\mathsf{D}\,[h(v)]\,u(v, p_c)\,, \\ \mathbb{0}_c &= C^{\mathsf{T}}\,\mathbf{arcsin}(\psi(v, p_c))\,. \end{aligned}$$

where

$$\begin{aligned} u(v, p_c) &\triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi} \\ \psi(v, p_c) &= [h(v)]^{-1}\left(A^{\mathsf{T}}\mathsf{L}^{\dagger}P + \mathsf{D}^{-1}Cp_c\right)\,, \end{aligned}$$

with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)\,.$

New Approximate Power Flow Solution

- The model says $v = f(v, p_c)$, and $\mathbf{sin}(A^\mathsf{T}\theta) = \psi(v, p_c)$.
- By construction, when $P = Q_L = \mathbb{0}$, a solution is

$$v = \mathbb{1}_n, \quad p_c = \mathbb{0}_c\,, \quad A^\mathsf{T}\theta = \mathbb{0}_{|\mathcal{E}|}\,.$$

- **Taylor expand** FPPF model around this solution

$$A^\mathsf{T}\theta_{\text{approx}} = A^\mathsf{T} L^\dagger P$$

$$v_{\text{approx}} \simeq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}Q_L + \frac{1}{8}\mathsf{S}^{-1}|A|_L \mathsf{D}[A^\mathsf{T}\mathsf{L}^\dagger P]A^\mathsf{T}\mathsf{L}^\dagger P$$

$$p_{c,\text{approx}} = \mathbb{0}$$

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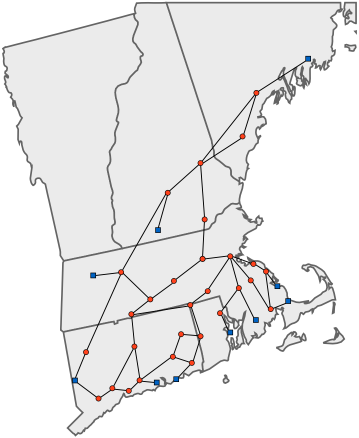

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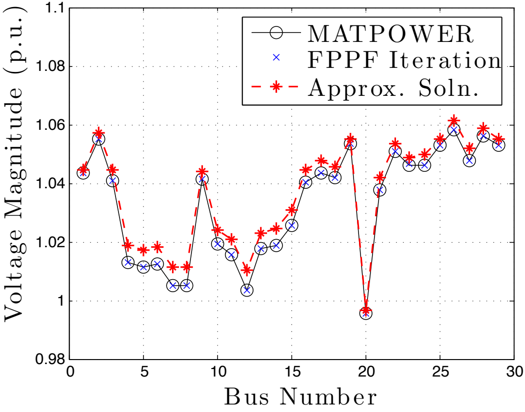


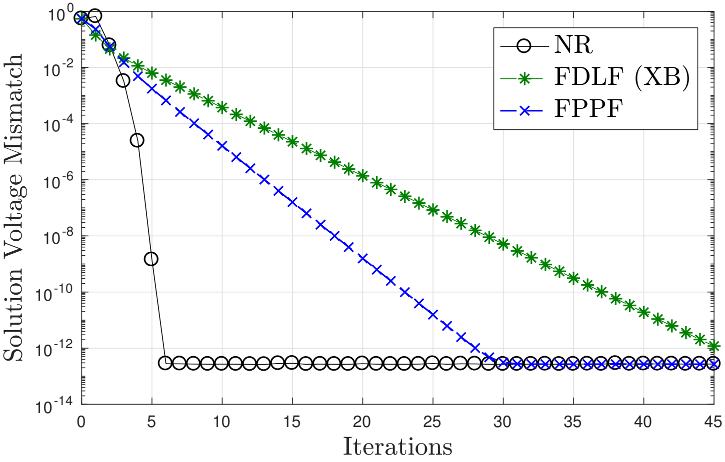
Numerical Results I

$$\delta_{\max} = \|v - v_{\text{approx}}\|_{\infty}\,, \quad \delta_{\text{avg}} = \frac{1}{n}\|v - v_{\text{approx}}\|_{1}$$

	Base Load			High Load	
Test Case	FPPF Iters.	$\delta_{\max}$ (p.u.)	δ_{avg} (p.u.)	FPPF Iters.	$\delta_{\max}$ (p.u.)
New England 39	4	0.006	0.004	8	0.086
57 bus system	5	0.011	0.003	8	0.118
RTS '96 (3 area)	4	0.003	0.001	8	0.084
118 bus system	3	0.001	0.000	7	0.054
300 bus system	6	0.022	0.004	8	0.059
PEGASE 1,354	5	0.011	0.001	8	0.070
Polish 2,383 wp	4	0.003	0.000	8	0.078
PEGASE 2,869	5	0.015	0.002	8	0.098
PEGASE 9,241	6	0.063	0.003	9	0.133

Numerical Results II – Convergence Rates

- IEEE 300 bus system under heavy loading



Numerical Results III – Sensitivity to Initialization

- perturb voltage magnitude initialization randomly
- IEEE 118 bus system, base case

IC Spread (α)	NR	FDLF	FPPF
0.05	0.98	0.98	1.00
0.10	0.53	0.53	1.00
0.15	0.18	0.18	1.00
0.2	0.03	0.03	1.00
0.3	0.00	0.00	1.00
0.5	0.00	0.00	1.00
0.7	0.00	0.00	0.99
0.9	0.00	0.00	0.99

- extreme insensitivity to initialization (contraction)

FPPF Simplifies for Acyclic Networks

Fixed-Point Power Flow: Radial Networks

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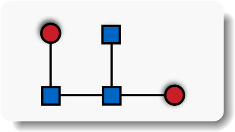
On what invariant set is f a **contraction**?

Solvability Results for Different Acyclic Topologies

Solvability Results for Different Acyclic Topologies

PQ buses have one PV bus neighbor



Sufficient + Necessary
Existence + Uniqueness

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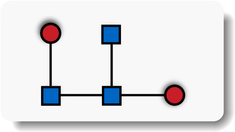

Sufficient + Necessary
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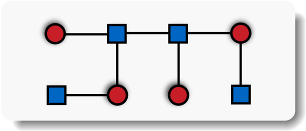
PQ buses have many PV bus neighbors



Sufficient + Tight
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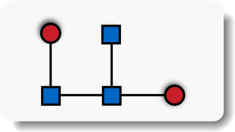

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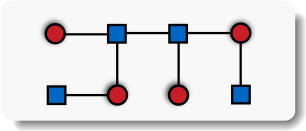

Sufficient + Tight
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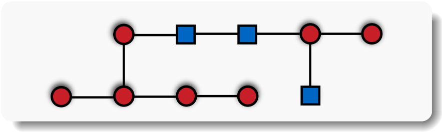
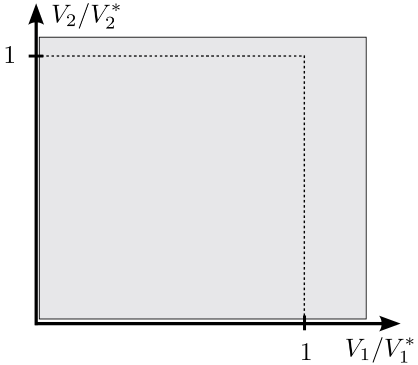
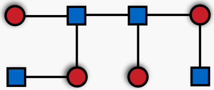
General interconnections



Sufficient
Existence

Partitioning of Voltage Space





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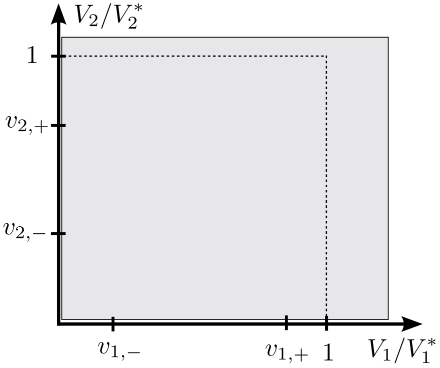

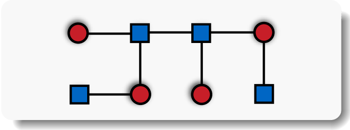

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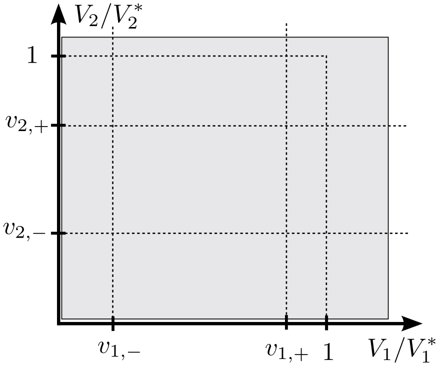

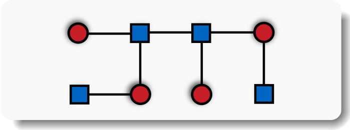

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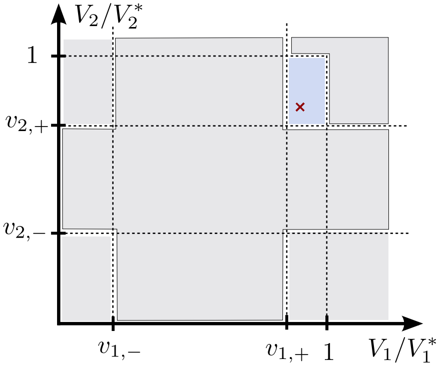

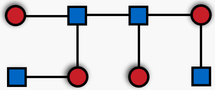

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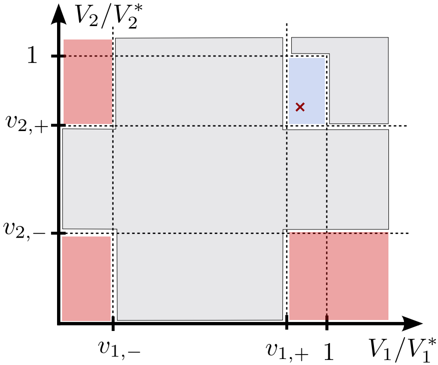

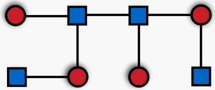

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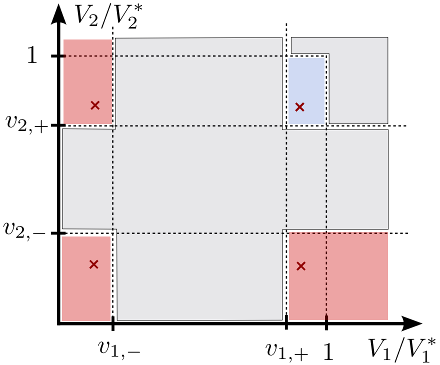

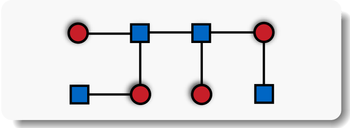

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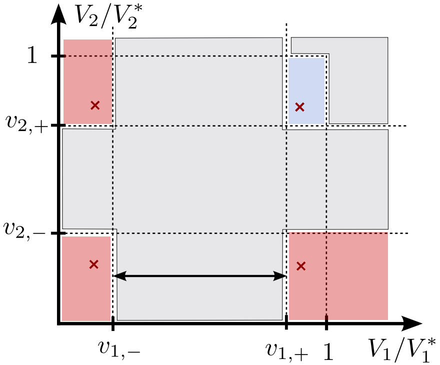

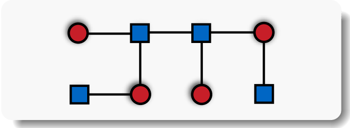

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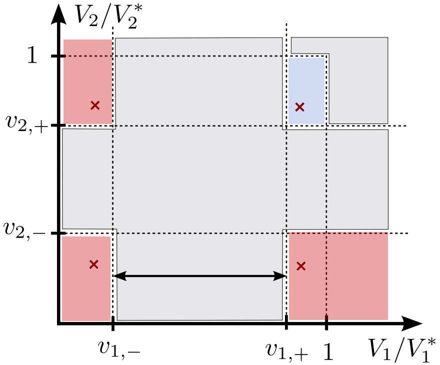

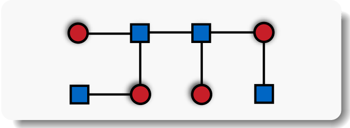

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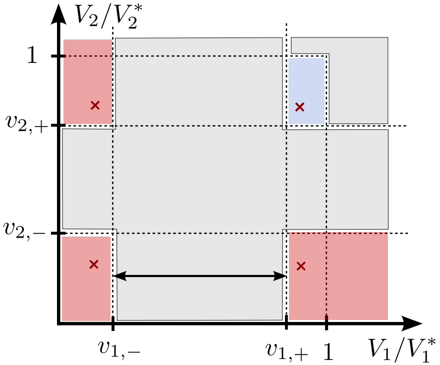
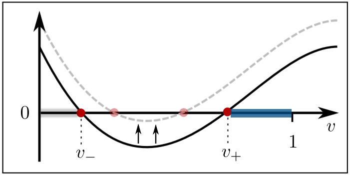
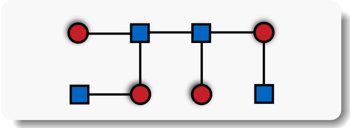
Partitioning of Voltage Space







$$\max_{i \in \mathcal{N}_L} \Delta_i + 4\Gamma_i^2 < 1$$

$$\max_{(i,j) \in \mathcal{E}^{gg}} \Gamma_{ij} < 1\,,$$

$$v_{i,\pm} \triangleq \sqrt{\frac{1}{2}\left(1 - \frac{\Delta_i}{2} \pm \sqrt{1 - (\Delta_i + 4\Gamma_i^2)}\right)}$$

$$v_{i,+}^2 - v_{i,-}^2 = \sqrt{1 - (\Delta_i + 4\Gamma_i^2)}\,.$$

Conclusions

Framework for studying **Lossless Power Flow**:

1. Fixed-Point Power Flow
2. Approximate solution

A Theory of Solvability for Lossless Power Flow Equations — Part I: Fixed-Point Power Flow

John W. Simpson-Porco, *Member, IEEE*

New **conditions for power flow solvability**:

3. Contractive iteration
4. Existence/uniqueness
5. Generalizes known results

What's next?

1. Analysis for **meshed networks** unresolved
2. **Lossy** networks (TPWRS: JWSP '17)
3. Applications ($n-1$, opt/control)

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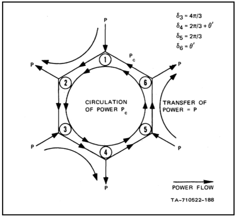

Questions

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appendix

Power flow and grid connectivity

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*"***Root causes of [the northeastern] blackout: lack of basic understanding of power systems . . . theoretical understanding of nonlinear power system dynamics is inadequate.** *It is time for more theoretical research to develop alternatives to complement scenario-based simulation paradigm: mathematical theory to understand the complex dynamic behavior of large-scale interconnected power systems utilizing modern nonlinear mathematics."*

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