A Theory of Solvability for Lossless Power Flow Equations

John W. Simpson-Porco

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This talk is based on these papers

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A Theory of Solvability for Lossless Power Flow Equations—Part I: Fixed-Point Power Flow

John W. Simpson-Porco, *Member, IEEE*

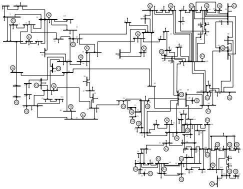
IEEE TRANSACTIONS ON CONTROL OF NETWORK SYSTEMS, VOL. 5, NO. 3, SEPTEMBER 2018 1373

A Theory of Solvability for Lossless Power Flow Equations—Part II: Conditions for Radial Networks

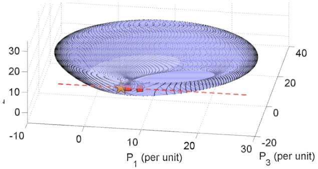
John W. Simpson-Porco, *Member, IEEE*

Problems in power system operations

Power Flow Analysis

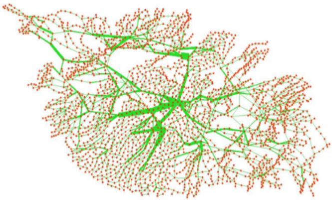


Optimal Power Flow



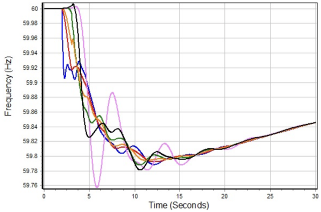
[Molzahn et al.]

Contingency Analysis



[Rezaei et al.]

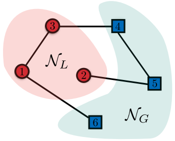
Transient Stability



[Overbye et al.]

Modeling AC power flow

- active power: $P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)$
- reactive power: $Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)$



❻ n **Loads** (●) **and** m **Generators** (■) $\mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G$

❼ **Load Model:** PQ bus constant P_i constant Q_i

❽ **Generator Model:** PV bus constant P_i constant V_i,

Power Flow Equations

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j), \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$

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$2n + m$ equations in variables $\theta \in \mathbb{T}^{n+m}$ and $V_L \in \mathbb{R}^n_{>0}$.

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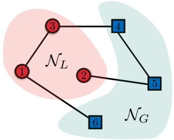

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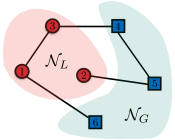

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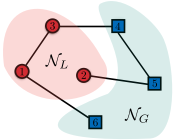

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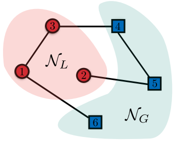

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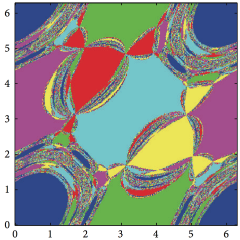
1. **Because it is interesting to do so**
2. **Numerical methods**
   - understand convergence, divergence, and initialization issues



[Deng et al.]

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   - **Newton iteration:**

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3. **Optimal power flow**
4. **Transient stability**

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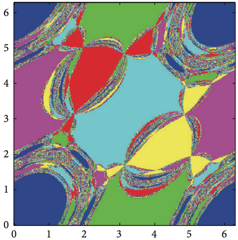

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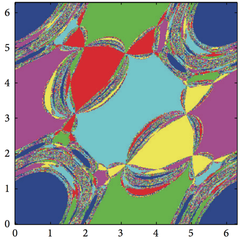

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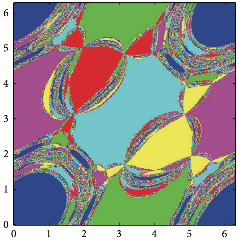

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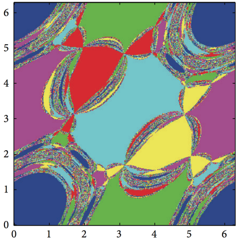

[Deng et al.]

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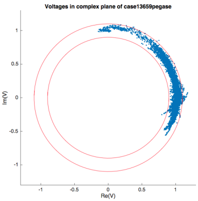
Intuition on power flow solutions

1. 'Normally', exists unique **high-voltage** soln:
   - voltage magnitude $V_i \simeq 1$
   - phase diff $|\theta_i - \theta_j| \ll 1$



[Josz et al.]

2. **Lightly loaded systems:** many **low-voltage** solutions

3. **Heavily loaded systems:** Few solutions or **infeasible**
   - saddle node bifurcations
   - maximum power transfer limit
   - non-convex feasible set in (P, Q)-space

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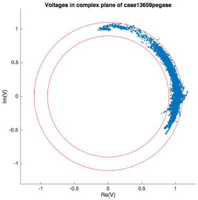

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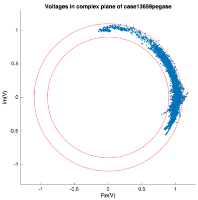

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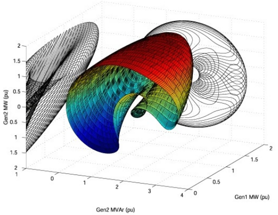

[Hiskens & Davy]

Mysteries of power flow

Given data: network topology, impedances, generation & loads

Q: $\exists$ "stable high-voltage" solution? unique? properties?

Partial answers from **45+ years** of literature:

- Jacobian singularity [Weedy '67]
- Multiple dynamically stable solutions [Korsak '72]
- Existence conditions [Wu & Kumagai '80, '82]
- Active power flow singularity [Araposthatis, Sastry & Varaiya, '81]
- Counting # of solutions [Baillieul and Byrnes '82]
- Properties of quadratic equations [Makarov, Hill & Hiskens '00]
- Optimization approaches [Cañizares '98], [Dvijotham, Low, Chertkov '15], [Molzahn]
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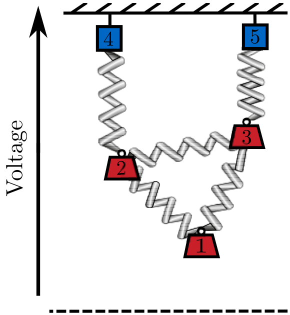
Partial answers from **45+ years** of literature:

Main insight: stiffness vs. loading

1. Stiff network + light loading $\Rightarrow$ feasible
2. Weak network + heavy loading $\Rightarrow$ infeasible

Q: How to quantify network stiffness vs. loading?



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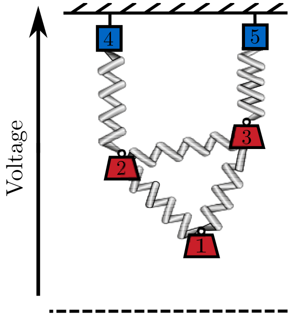

Solution of Two-Bus System

$$P_L = bV_G V_L \sin(-\eta)$$
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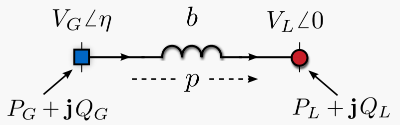
$$Q_L = bV_L^2 - bV_L V_G \cos(\eta)$$



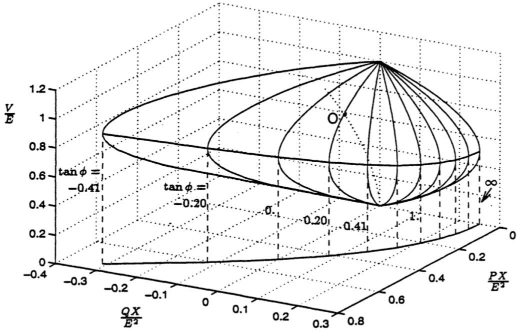


Figure 2.6 Voltage as a function of load active and reactive powers

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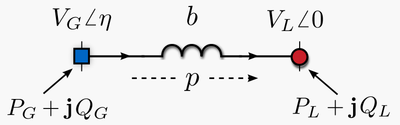

1. **Change Variables**

$$v := \frac{V_L}{V_G} \qquad \Gamma := \frac{p}{bV_G^2} \qquad \Delta := \frac{Q_L}{-\frac{1}{4}bV_G^2}$$

2. **Square equations, add, and solve quadratic in v^2**

$$v_\pm = \sqrt{\frac{1}{2}\left(1 - \frac{\Delta}{2} \pm \sqrt{1 - (4\Gamma^2 + \Delta)}\right)}$$

3. **Nec. & Suff. Condition**

$$4\Gamma^2 + \Delta < 1$$

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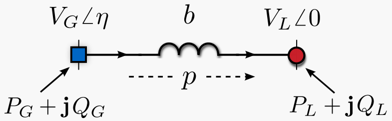

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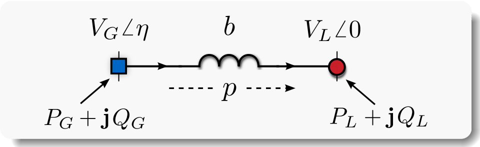

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1. **High-voltage** solution
   $v_+ \in [\frac{1}{2}, 1)$
2. **Low-voltage** solution
   $v_- \in [0, \frac{1}{\sqrt{2}})$

Angle: $\sin(\eta_\mp) = \Gamma / v_\pm$

1. **Small-angle** solution
   $\eta_- \in [0, \pi/4)$
2. **Large-angle** solution
   $\eta_+ \in [0, \pi/2)$

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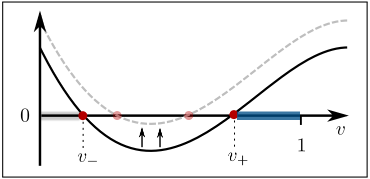

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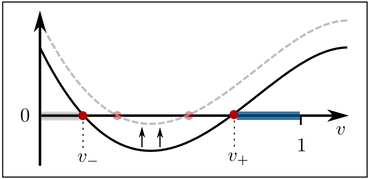
   $v_- \in [0, \frac{1}{\sqrt{2}})$



Angle: $\sin(\eta_\mp) = \Gamma / v_\pm$

1. **Small-angle** solution
   $\eta_- \in [0, \pi/4)$
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Solution of Two-Bus System

$$\Gamma = v\sin(\eta)$$
$$\Delta = -4v^2 + 4v\cos(\eta)$$

$$v := \frac{V_L}{V_G} \quad \Gamma := \frac{p}{bV_G^2} \quad \Delta := \frac{Q_L}{-\frac{1}{4}bV_G^2}$$
$$4\Gamma^2 + \Delta < 1$$

1. **High-voltage** solution
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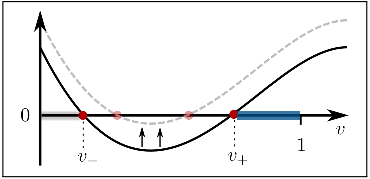

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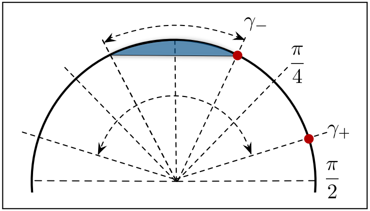

Solution of Two-Bus System IV

- Squaring and adding equations **does not generalize** to networks.
- Is there any hope then?

$$\Gamma = v\sin(\eta)$$
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- Use $\cos(\eta) = \sqrt{1-\sin^2(\eta)} \implies \Delta = -4v^2 + 4v\sqrt{1-(\Gamma/v)^2}$
- Rearrange to get *fixed-point equation*

$$v = f(v) := -\frac{1}{4}\frac{\Delta}{v} + \sqrt{1-\left(\frac{\Gamma}{v}\right)^2}$$

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Network Notation I: Branches Between Bus Types

Power Flow Equations

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$

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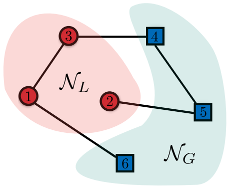

- Bus partitioning $\mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G$ induces **branch partitioning**

$$\mathcal{E} = \mathcal{E}^{\ell\ell} \cup \mathcal{E}^{g\ell} \cup \mathcal{E}^{gg}\,, \quad A = \left(\begin{array}{c} A_L \\ \hline A_G \end{array}\right) = \left(\begin{array}{c|c|c} A_L^{\ell\ell} & A_L^{g\ell} & \mathbb{0} \\ \hline \mathbb{0} & A_G^{g\ell} & A_G^{gg} \end{array}\right).$$

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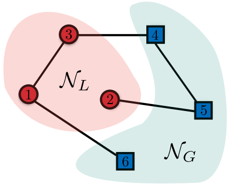

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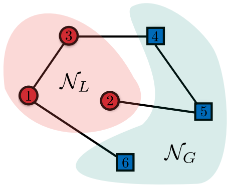

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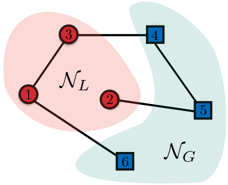

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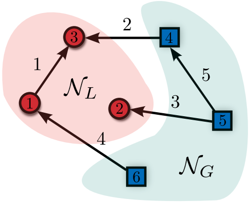

$$A = \overset{\quad\mathcal{E}^{\ell\ell}\qquad\qquad \mathcal{E}^{g\ell}\qquad\qquad \mathcal{E}^{gg}}{\left(\begin{array}{c|ccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)} \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\0\\-1\end{array}}\right\} \mathcal{N}_L \\ \left.\vphantom{\begin{array}{c}1\\0\\-1\end{array}}\right\} \mathcal{N}_G \end{array}$$

Network Notation II: Open-Circuit Voltages

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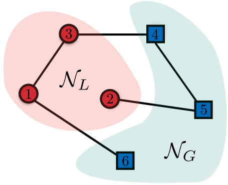

Partitioned Variables

$$V = \begin{pmatrix} V_L \\ \hline V_G \end{pmatrix}, \qquad B = \left(\begin{array}{c|c} B_{LL} & B_{LG} \\ \hline B_{GL} & B_{GG} \end{array}\right)$$

Open-circuit voltages

$$V_L^* \triangleq \underbrace{-B_{LL}^{-1} B_{LG}}_{\text{Generators}\to\text{Loads}} \cdot V_G$$

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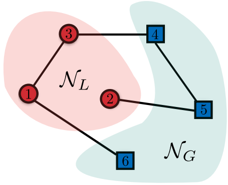

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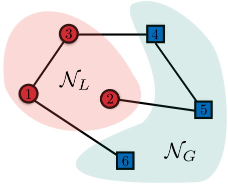

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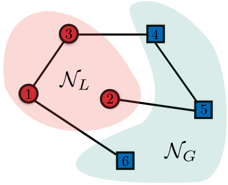

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- Need to **non-dimensionalize** power flow equations
- **Stiffness matrices** quantify **grid strength** in **units of power**

1. **Nodal** stiffness matrix $\quad \mathrm{S} \triangleq \frac{1}{4}[V_L^*] \cdot B_{LL} \cdot [V_L^*]$
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Main Modeling Result

Fixed-Point Power Flow: Meshed Networks

(θ, V_L) is a power flow solution iff (v, p_c) solves the FPPF

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(θ, V_L) is a power flow solution iff (v, p_c) solves the FPPF

$$\begin{aligned} v = f(v, p_c) &\triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n \\ &\qquad + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\,u(v, p_c)\,, \\ \mathbb{0}_c &= C^{\mathsf{T}}\,\mathbf{arcsin}(\psi(v, p_c))\,. \end{aligned}$$

where

$$\begin{aligned} u(v, p_c) &\triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi} \\ \psi(v, p_c) &= [h(v)]^{-1}\left(A^{\mathsf{T}}\mathsf{L}^{\dagger}P + \mathsf{D}^{-1}Cp_c\right)\,, \end{aligned}$$

with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)\,.$

New Approximate Power Flow Solution

- The model says $v = f(v, p_c)$, and $\mathbf{sin}(A^{\mathsf{T}}\theta) = \psi(v, p_c)$.
- By construction, when $P = Q_L = \mathbb{0}$, a solution is

$$v = \mathbb{1}_n, \quad p_c = \mathbb{0}_c\,, \quad A^{\mathsf{T}}\theta = \mathbb{0}_{|\mathcal{E}|}\,.$$

- **Taylor expand** FPPF model around this solution

$$\begin{aligned} A^{\mathsf{T}}\theta_{\mathrm{approx}} &= A^{\mathsf{T}}L^{\dagger}P \\ v_{\mathrm{approx}} &\simeq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}Q_L + \frac{1}{8}\mathsf{S}^{-1}|A|_L\mathsf{D}[A^{\mathsf{T}}\mathsf{L}^{\dagger}P]A^{\mathsf{T}}\mathsf{L}^{\dagger}P \\ p_{c,\mathrm{approx}} &= \mathbb{0} \end{aligned}$$

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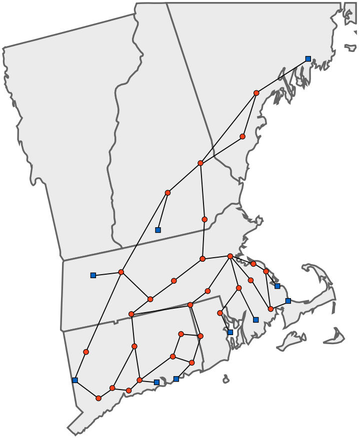

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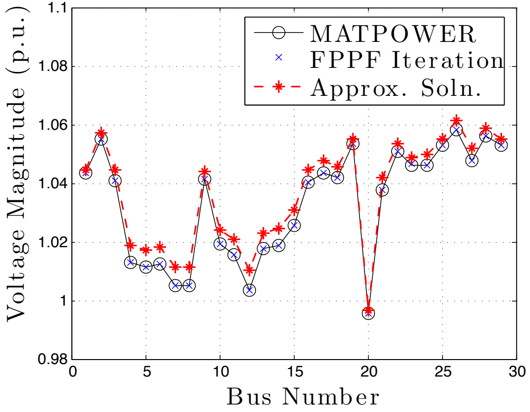


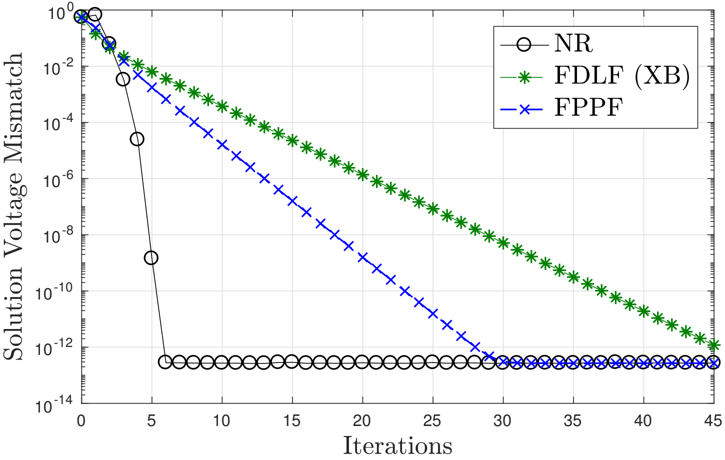
Numerical Results I

$$\delta_{\max} = \|v - v_{\text{approx}}\|_\infty \,, \quad \delta_{\text{avg}} = \frac{1}{n}\|v - v_{\text{approx}}\|_1$$

	Base Load			High Load	
Test Case	FPPF Iters.	$\delta_{\max}$ (p.u.)	δ_{avg} (p.u.)	FPPF Iters.	$\delta_{\max}$ (p.u.)
New England 39	4	0.006	0.004	8	0.086
57 bus system	5	0.011	0.003	8	0.118
RTS '96 (3 area)	4	0.003	0.001	8	0.084
118 bus system	3	0.001	0.000	7	0.054
300 bus system	6	0.022	0.004	8	0.059
PEGASE 1,354	5	0.011	0.001	8	0.070
Polish 2,383 wp	4	0.003	0.000	8	0.078
PEGASE 2,869	5	0.015	0.002	8	0.098
PEGASE 9,241	6	0.063	0.003	9	0.133

Numerical Results II – Convergence Rates

- IEEE 300 bus system under heavy loading



Numerical Results III – Sensitivity to Initialization

- perturb voltage magnitude initialization randomly
- IEEE 118 bus system, base case

IC Spread (α)	NR	FDLF	FPPF
0.05	0.98	0.98	1.00
0.10	0.53	0.53	1.00
0.15	0.18	0.18	1.00
0.2	0.03	0.03	1.00
0.3	0.00	0.00	1.00
0.5	0.00	0.00	1.00
0.7	0.00	0.00	0.99
0.9	0.00	0.00	0.99

- extreme insensitivity to initialization (contraction)

FPPF Simplifies for Acyclic Networks

Fixed-Point Power Flow: Radial Networks

(θ, V_L) is a power flow solution iff v is a fixed point of

$$f(v) \triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L\mathsf{D}\,[h(v)]\,u(v)\,,$$

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On what invariant set is f a **contraction**?

Solvability Results for Different Acyclic Topologies

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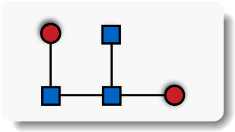
PQ buses have one PV bus neighbor



Sufficient + Necessary
Existence + Uniqueness

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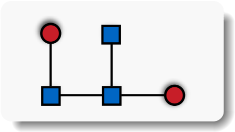

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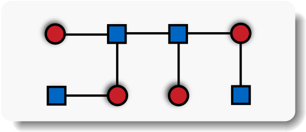
PQ buses have many PV bus neighbors



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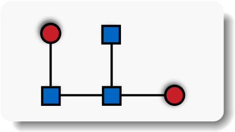

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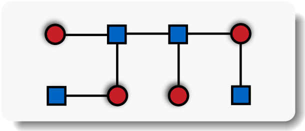

Sufficient + Tight
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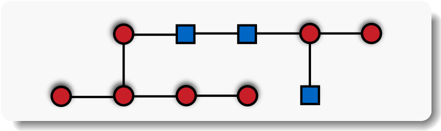
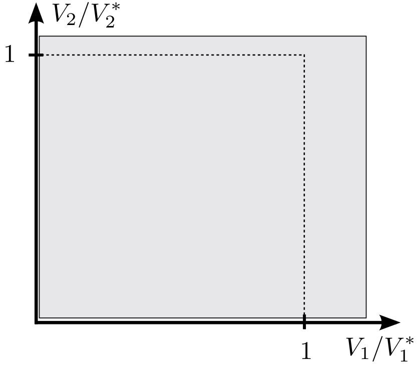
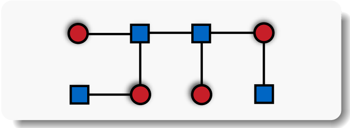
General interconnections



Sufficient
Existence

Partitioning of Voltage Space





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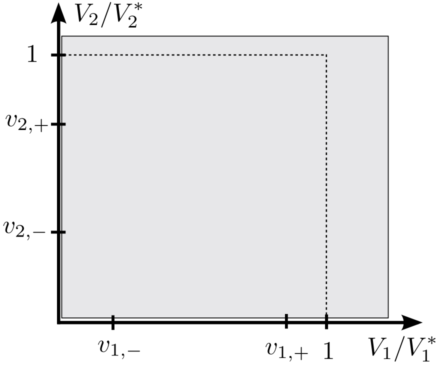

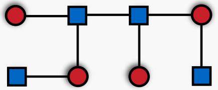

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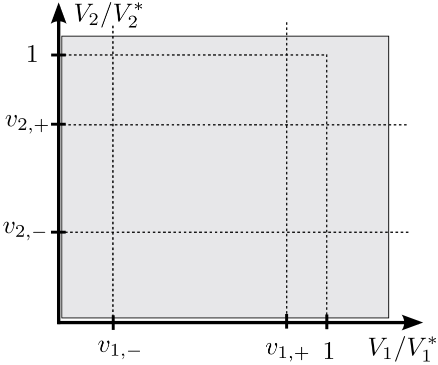

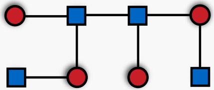

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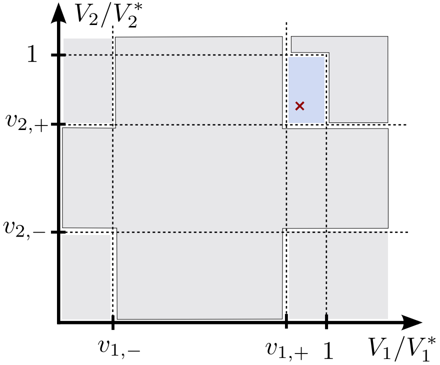

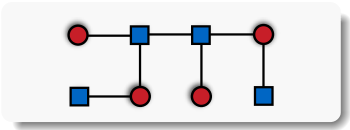

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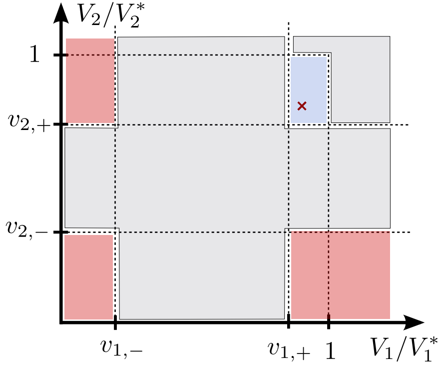

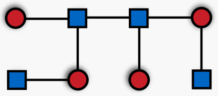

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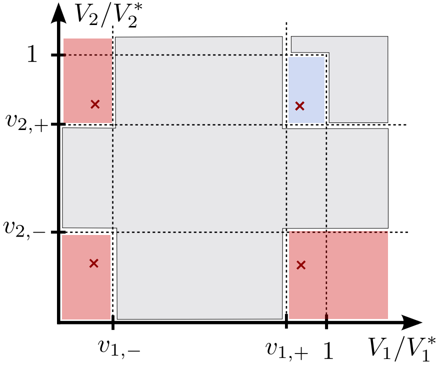

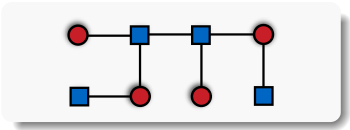

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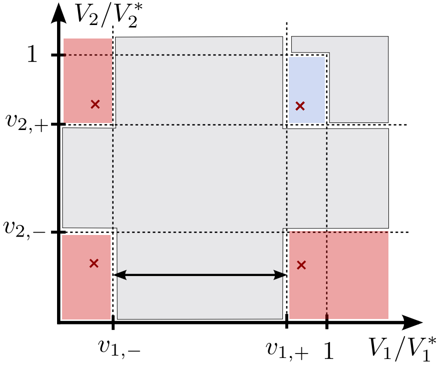

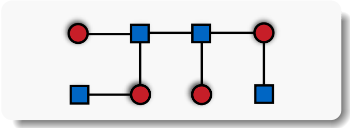

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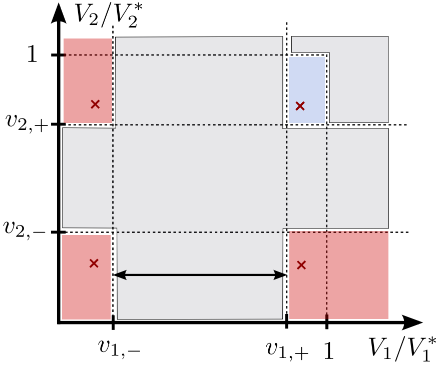

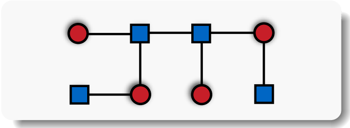

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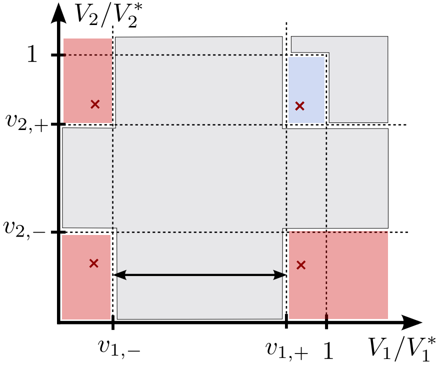
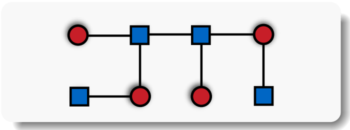
Partitioning of Voltage Space

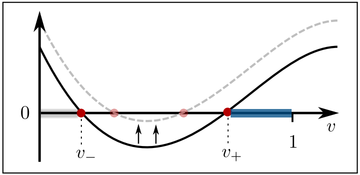




$$\max_{i \in \mathcal{N}_L} \ \Delta_i + 4\Gamma_i^2 < 1$$

$$\max_{(i,j) \in \mathcal{E}^{gg}} \ \Gamma_{ij} < 1\,,$$

$$v_{i,\pm} \triangleq \sqrt{\frac{1}{2}\left(1 - \frac{\Delta_i}{2} \pm \sqrt{1 - (\Delta_i + 4\Gamma_i^2)}\right)}$$



$$v_{i,+}^2 - v_{i,-}^2 = \sqrt{1 - (\Delta_i + 4\Gamma_i^2)}\,.$$

What About Networks with Losses?

IEEE TRANSACTIONS ON POWER SYSTEMS, VOL. 33, NO. 3, MAY 2018 2477

Lossy DC Power Flow

John W. Simpson-Porco, *Member, IEEE*

Focus just on **active power balance** (minus slack bus)

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) + G_{ii} V_i^2 + \sum_j V_i V_j G_{ij}$$

Key ideas for analysis:

1. The case $G = 0$ is (somewhat) well understood
2. (Solvability with $G = 0$) + $\|G/B\| \leq \rho \quad \Longrightarrow \quad$ Solution?

Decoupled Active Power Flow on Radial Networks

Let A_r be the (reduced) graph incidence matrix, and define

$\Gamma := \|A_r^{\mathsf{T}} L_r^{-1} P_r\|_\infty$ (lossless loading margin)

$\rho := \|\mathrm{diag}(B_{ij})^{-1} A_r^{-1} |A_r| \mathrm{diag}(G_{ij})\|_\infty$ (r/x ratio)

If $\Gamma^2 + 2\Gamma\rho < 1$, then

1. $\exists!$ soln. θ^* satisfying $|\theta_i^* - \theta_j^*| \leq \arcsin(\beta_-) \in [0, \frac{\pi}{2})$, where

$$\beta_\pm = \frac{\Gamma + \rho}{1 + \rho^2} \pm \frac{\rho}{1 + \rho^2}\sqrt{1 - (\Gamma^2 + 2\Gamma\rho)} \;\in\; [0, 1].$$

2. $\nexists$ soln. θ' satisfying

$$\arcsin(\beta_-) < |\theta_i' - \theta_j'| < \arcsin(\beta_+).$$

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Conclusions

Framework for studying **Lossless Power Flow**:

1. Fixed-Point Power Flow
2. Approximate solution

A Theory of Solvability for Lossless Power Flow Equations — Part I: Fixed-Point Power Flow
John W. Simpson-Porco, *Member, IEEE*

New **conditions for power flow solvability**:

3. Contractive iteration
4. Existence/uniqueness
5. Generalizes known results

What's next?

1. **Lossless meshed** case unresolved
2. **Lossy radial** case unresolved
3. **Lossy meshed (!!)** (be afraid)

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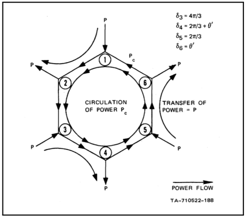

Questions

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appendix

Power flow and grid connectivity

"[Power flow feasibility] is one question which is unresolved in power systems analysis, but which is of basic theoretical and practical importance . . . **is a given network structurally susceptible to unfeasibility***? What type and what value of injections are most likely to result in unfeasible situations?"*

– F. D. Galiana, 1975

"The power systems theory needs to be pushed further in the direction of exploiting **structural features of the networks** *. . . realistic power systems models have at least two different types of node dynamics (generators, loads) and the directional power flows between them play a major role."*

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