Understanding Power Flow Solutions: History, Practice, Theory, Progress

John W. Simpson-Porco
https://www.control.utoronto.ca/~jwsimpson/

The Edward S. Rogers Sr. Department
of Electrical & Computer Engineering
UNIVERSITY OF TORONTO

Centrum Wiskunde & Informatica

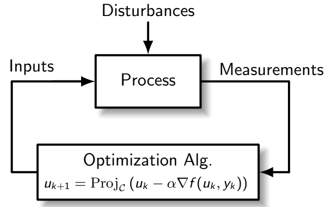
November 18, 2022

Prof. J. W. Simpson-Porco: control theory

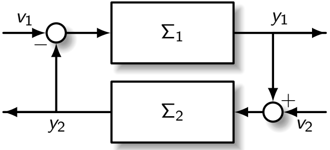
jwsimpson@ece.utoronto.ca

1. optimization and data-driven control
2. nonlinear systems
3. smart grid and energy systems

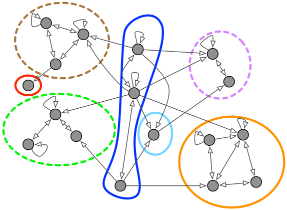
Feedback-Based Optimization



Nonlinear Systems



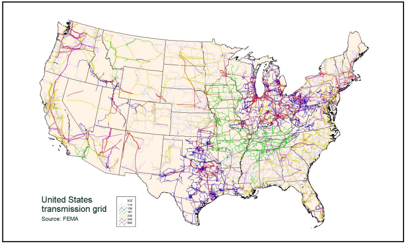
Network Dynamics & Control



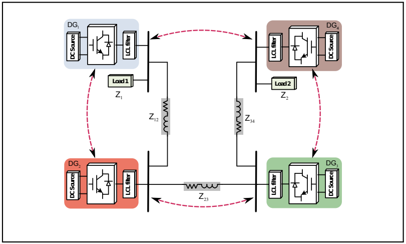
Prof. J. W. Simpson-Porco: energy systems

jwsimpson@ece.utoronto.ca

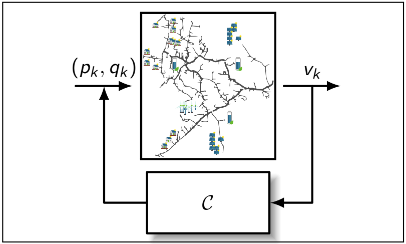
Power Flow Analysis & Algorithms



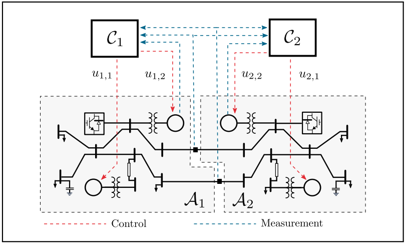
Microgrid Control & Optimization



Renewable Energy Integration

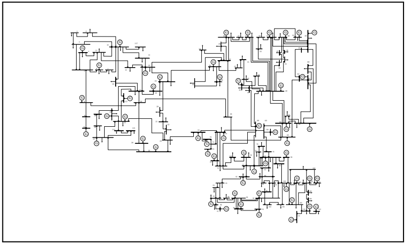


Next-Generation Hierarchical Control

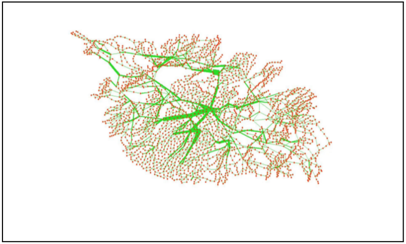


Problems in power system operations

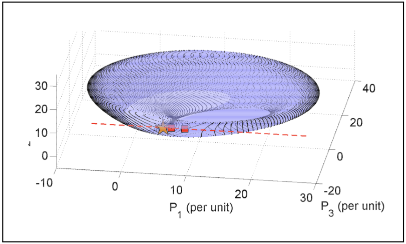
Power Flow Analysis



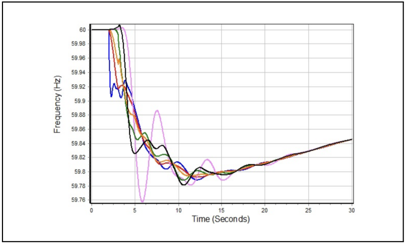
Contingency Analysis [Hines et al.]



Optimal Power Flow [Molzahn et al.]



Transient Stability [Overbye et al.]



Modeling AC power flow

1. **Network Graph:** $(\mathcal{N}, \mathcal{E})$, complex weights $y_{ij} = g_{ij} + \mathbf{j}b_{ij}$
2. **Nodal Variables:** voltage $V_i e^{\mathbf{j}\theta_i}$, power $S_i = P_i + \mathbf{j}Q_i$
3. **Coupling Laws:** Kirchhoff & Ohm
4. **Admittance Matrix:** $Y = G + \mathbf{j}B =$ Laplacian-like w/ weights y_{ij}
5. **Lossless Lines:** $G_{ij} = 0$

Modeling AC power flow

1. **Network Graph:** $(\mathcal{N}, \mathcal{E})$, complex weights $y_{ij} = g_{ij} + \mathbf{j} b_{ij}$
2. **Nodal Variables:** voltage $V_i e^{\mathbf{j}\theta_i}$, power $S_i = P_i + \mathbf{j} Q_i$
3. **Coupling Laws:** Kirchhoff & Ohm
4. **Admittance Matrix:** $Y = G + \mathbf{j}B$ = Laplacian-like w/ weights y_{ij}
5. **Lossless Lines:** $G_{ij} = 0$

Modeling AC power flow

1. **Network Graph:** $(\mathcal{N}, \mathcal{E})$, complex weights $y_{ij} = g_{ij} + \mathbf{j} b_{ij}$
2. **Nodal Variables:** voltage $V_i e^{\mathbf{j}\theta_i}$, power $S_i = P_i + \mathbf{j} Q_i$
3. **Coupling Laws:** Kirchhoff & Ohm

4. **Admittance Matrix:** $Y = G + \mathbf{j}B =$ Laplacian-like w/ weights y_{ij}
5. **Lossless Lines:** $G_{ij} = 0$

Modeling AC power flow

1. **Network Graph:** $(\mathcal{N}, \mathcal{E})$, complex weights $y_{ij} = g_{ij} + \mathbf{j} b_{ij}$
2. **Nodal Variables:** voltage $V_i e^{\mathbf{j}\theta_i}$, power $S_i = P_i + \mathbf{j} Q_i$
3. **Coupling Laws:** Kirchhoff & Ohm

4. **Admittance Matrix:** $Y = G + \mathbf{j} B =$ Laplacian-like w/ weights y_{ij}
5. **Lossless Lines:** $G_{ij} = 0$

- active power: $P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) + V_i V_j G_{ij} \cos(\theta_i - \theta_j)$
- reactive power: $Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j) + V_i V_j G_{ij} \sin(\theta_i - \theta_j)$

Modeling AC power flow

1. **Network Graph:** $(\mathcal{N}, \mathcal{E})$, complex weights $y_{ij} = g_{ij} + \mathbf{j} b_{ij}$
2. **Nodal Variables:** voltage $V_i e^{\mathbf{j}\theta_i}$, power $S_i = P_i + \mathbf{j} Q_i$
3. **Coupling Laws:** Kirchhoff & Ohm

4. **Admittance Matrix:** $Y = G + \mathbf{j}B =$ Laplacian-like w/ weights y_{ij}
5. **Lossless Lines:** $G_{ij} = 0$

- active power: $P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) + V_i V_j G_{ij} \cos(\theta_i - \theta_j)$
- reactive power: $Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j) + V_i V_j G_{ij} \sin(\theta_i - \theta_j)$

Modeling AC power flow

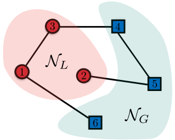
1. **Network Graph:** $(\mathcal{N}, \mathcal{E})$, complex weights $y_{ij} = g_{ij} + \mathbf{j} b_{ij}$
2. **Nodal Variables:** voltage $V_i e^{\mathbf{j}\theta_i}$, power $S_i = P_i + \mathbf{j} Q_i$
3. **Coupling Laws:** Kirchhoff & Ohm

4. **Admittance Matrix:** $Y = G + \mathbf{j} B$ = Laplacian-like w/ weights y_{ij}
5. **Lossless Lines:** $G_{ij} = 0$

- active power: $P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)$
- reactive power: $Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)$

Modeling AC power flow

- active power: $P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)$
- reactive power: $Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)$



1. **n Loads (●) and m Generators (■)** $\mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G$
2. **Load Model:** PQ bus constant P_i constant Q_i
3. **Generator Model:** PV bus constant P_i constant V_i,

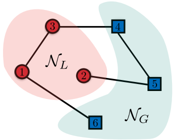
$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$

$$Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L$$

$2n + m$ equations in variables $\theta \in \mathbb{T}^{n+m}$ and $V_L \in \mathbb{R}^n_{>0}$.

Modeling AC power flow

- active power: $P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)$
- reactive power: $Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)$



6 **n Loads (●) and m Generators (■)** $\mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G$

7 **Load Model:** PQ bus constant P_i constant Q_i

8 **Generator Model:** PV bus constant P_i constant V_i,

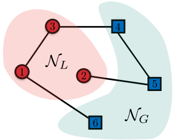
$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$

$$Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L$$

$2n + m$ equations in variables $\theta \in \mathbb{T}^{n+m}$ and $V_L \in \mathbb{R}^n_{>0}$.

Modeling AC power flow

- active power: $P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)$
- reactive power: $Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)$



6. **n Loads (●) and m Generators (■)** $\quad \mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G$
7. **Load Model:** PQ bus $\quad$ constant P_i $\quad$ constant Q_i
8. **Generator Model:** PV bus $\quad$ constant P_i $\quad$ constant V_i,

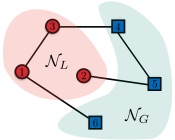
$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$

$$Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L$$

$2n + m$ equations in variables $\theta \in \mathbb{T}^{n+m}$ and $V_L \in \mathbb{R}^n_{>0}$.

Modeling AC power flow

- active power: $P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)$
- reactive power: $Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)$



6. **n Loads (●) and m Generators (■)** $\mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G$
7. **Load Model:** PQ bus constant P_i constant Q_i
8. **Generator Model:** PV bus constant P_i constant V_i,

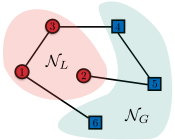
$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$

$$Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L$$

$2n + m$ equations in variables $\theta \in \mathbb{T}^{n+m}$ and $V_L \in \mathbb{R}^n_{>0}$.

Modeling AC power flow

- active power: $P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)$
- reactive power: $Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)$



6. ***n* Loads (●) and *m* Generators (■)** $\mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G$
7. **Load Model:** PQ bus constant P_i constant Q_i
8. **Generator Model:** PV bus constant P_i constant V_i,

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$

$$Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L$$

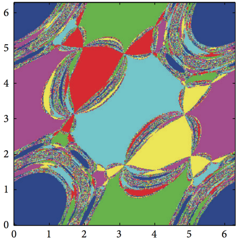
$2n + m$ equations in variables $\theta \in \mathbb{T}^{n+m}$ and $V_L \in \mathbb{R}^n_{>0}$.

Why study solvability of power flow problems?

1. **Because it is interesting to do so**
2. **Numerical methods**
   - understand convergence, divergence, and initialization issues

   - **State vector:** $x = (\theta, V)$
   - **Newton iteration:**

$$x^{k+1} = x^k - J(\theta^k, V^k)^{-1} f(x^k)$$



[Deng et al.]

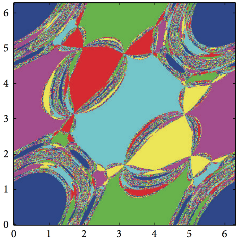
3. **Optimal power flow**
4. **Transient stability**

Why study solvability of power flow problems?

1. **Because it is interesting to do so**
2. **Numerical methods**
   - understand convergence, divergence, and initialization issues

- **State vector:** $x = (\theta, V)$
- **Newton iteration:**

$$x^{k+1} = x^k - J(\theta^k, V^k)^{-1} f(x^k)$$



[Deng et al.]

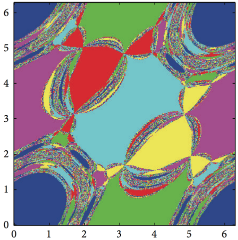
3. **Optimal power flow**
4. **Transient stability**

Why study solvability of power flow problems?

1. **Because it is interesting to do so**
2. **Numerical methods**
   - understand convergence, divergence, and initialization issues

- **State vector:** $x = (\theta, V)$
- **Newton iteration:**

$$x^{k+1} = x^k - J(\theta^k, V^k)^{-1} f(x^k)$$



[Deng et al.]

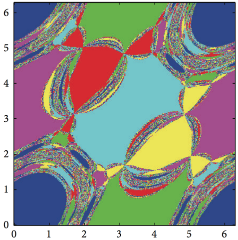
3. **Optimal power flow**
4. **Transient stability**

Why study solvability of power flow problems?

1. **Because it is interesting to do so**
2. **Numerical methods**
   - understand convergence, divergence, and initialization issues

- **State vector:** $x = (\theta, V)$
- **Newton iteration:**

$$x^{k+1} = x^k - J(\theta^k, V^k)^{-1} f(x^k)$$



[Deng et al.]

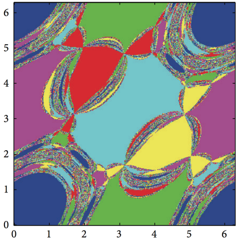
3. Optimal power flow
4. Transient stability

Why study solvability of power flow problems?

1. **Because it is interesting to do so**
2. **Numerical methods**
   - understand convergence, divergence, and initialization issues

- **State vector:** $x = (\theta, V)$
- **Newton iteration:**

$$x^{k+1} = x^k - J(\theta^k, V^k)^{-1} f(x^k)$$



[Deng et al.]

3. **Optimal power flow**
4. **Transient stability**

Motivation I: numerical methods for power flow

Power flow always solved with variant of Newton iteration

$$x = \begin{pmatrix} \theta & V_L \end{pmatrix}^\mathsf{T}, \qquad x^{k+1} = x^k - J(x^k)^{-1} f(x^k)\,.$$

- If **convergent**, may converge to "wrong" solution
- If **non-convergent**, several possibilities:

(a) No power flow solution exists

(b) Numerical instability (conditioning)

(c) x^0 not in any **region of convergence**

Motivation I: numerical methods for power flow

Power flow always solved with variant of Newton iteration

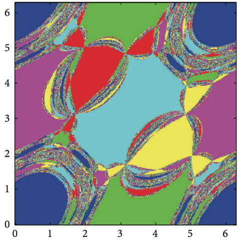
$$x = \begin{pmatrix} \theta & V_L \end{pmatrix}^{\mathsf{T}}, \qquad x^{k+1} = x^k - J(x^k)^{-1} f(x^k)\,.$$

- If **convergent**, may converge to "wrong" solution
- If **non-convergent**, several possibilities:

(a) No power flow solution exists

(b) Numerical instability (conditioning)

(c) x^0 not in any **region of convergence**



[Deng et al.]

Motivation I: numerical methods for power flow

Power flow always solved with variant of Newton iteration

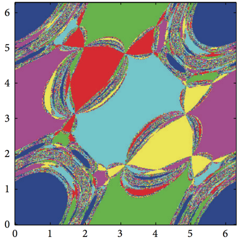
$$x = \begin{pmatrix}\theta & V_L\end{pmatrix}^{\mathsf{T}}\,, \qquad x^{k+1} = x^k - J(x^k)^{-1} f(x^k)\,.$$

- If **convergent**, may converge to “wrong” solution
- If **non-convergent**, several possibilities:

(a) No power flow solution exists

(b) Numerical instability (conditioning)

(c) x^0 not in any **region of convergence**



[Deng et al.]

Motivation I: numerical methods for power flow

Power flow always solved with variant of Newton iteration

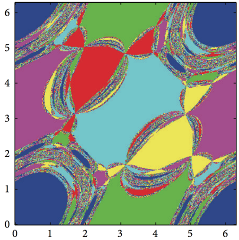
$$x = \begin{pmatrix} \theta & V_L \end{pmatrix}^{\mathsf{T}}, \qquad x^{k+1} = x^k - J(x^k)^{-1} f(x^k) \,.$$

- If **convergent**, may converge to "wrong" solution
- If **non-convergent**, several possibilities:

(a) No power flow solution exists

(b) Numerical instability (conditioning)

(c) x^0 not in any **region of convergence**



[Deng et al.]

Motivation I: numerical methods for power flow

Power flow always solved with variant of Newton iteration

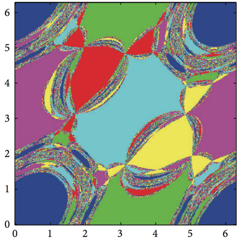
$$x = \begin{pmatrix} \theta & V_L \end{pmatrix}^{\mathsf{T}}, \qquad x^{k+1} = x^k - J(x^k)^{-1} f(x^k)\,.$$

- If **convergent**, may converge to "wrong" solution
- If **non-convergent**, several possibilities:

(a) No power flow solution exists

(b) Numerical instability (conditioning)

(c) x^0 not in any **region of convergence**



[Deng et al.]

Motivation I: numerical methods for power flow

Power flow always solved with variant of Newton iteration

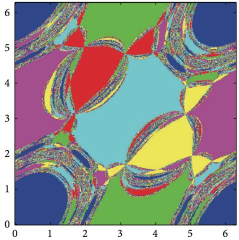
$$x = \begin{pmatrix} \theta & V_L \end{pmatrix}^{\mathsf{T}}, \qquad x^{k+1} = x^k - J(x^k)^{-1} f(x^k) .$$

- If **convergent**, may converge to "wrong" solution
- If **non-convergent**, several possibilities:

(a) No power flow solution exists

(b) Numerical instability (conditioning)

(c) x^0 not in any **region of convergence**



[Deng et al.]

Motivation I: numerical methods for power flow

Power flow always solved with variant of Newton iteration

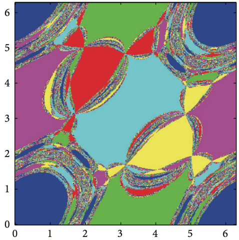
$$x = \begin{pmatrix}\theta & V_L\end{pmatrix}^{\mathsf{T}}, \qquad x^{k+1} = x^k - J(x^k)^{-1} f(x^k).$$

- If **convergent**, may converge to "wrong" solution
- If **non-convergent**, several possibilities:

(a) No power flow solution exists

(b) Numerical instability (conditioning)

(c) x^0 not in any **region of convergence**



[Deng et al.]

Motivation I: numerical methods for power flow

Power flow always solved with variant of Newton iteration

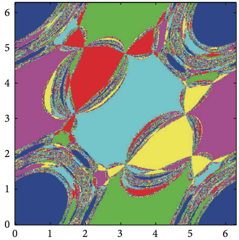
$$x = \begin{pmatrix} \theta & V_L \end{pmatrix}^{\mathsf{T}}, \qquad x^{k+1} = x^k - J(x^k)^{-1} f(x^k).$$

- If **convergent**, may converge to "wrong" solution
- If **non-convergent**, several possibilities:

(a) No power flow solution exists

(b) Numerical instability (conditioning)

(c) x^0 not in any **region of convergence**



[Deng et al.]

To differentiate, need theory of power flow solvability

Motivation II: multimachine transient stability

Constrained Swing Dynamics

$$\text{Gen}: \begin{cases} \dot{\theta}_i = \omega_i \\ M_i\dot{\omega}_i = -D_i\omega_i + P_i - \sum_j V_iV_jB_{ij}\sin(\theta_i - \theta_j) \end{cases}$$

$$\text{Load}: \begin{cases} D_i\dot{\theta}_i = P_i - \sum_j V_iV_jB_{ij}\sin(\theta_i - \theta_j) \\ Q_i = -\sum_j V_iV_jB_{ij}\cos(\theta_i - \theta_j) \end{cases}$$

Challenge: Characterize equilibria, stability, basin of attraction

Approaches: Energy functions, nearest unstable eq. point, S.O.S., ...

Motivation II: multimachine transient stability

Constrained Swing Dynamics

$$\text{Gen}: \begin{cases} \dot{\theta}_i = \omega_i \\ M_i \dot{\omega}_i = -D_i \omega_i + P_i - \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \end{cases}$$

$$\text{Load}: \begin{cases} D_i \dot{\theta}_i = P_i - \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \\ Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j) \end{cases}$$

Challenge: Characterize equilibria, stability, basin of attraction

Approaches: Energy functions, nearest unstable eq. point, S.O.S., . . .

Motivation II: multimachine transient stability

Constrained Swing Dynamics

$$\text{Gen}: \begin{cases} \dot{\theta}_i = \omega_i \\ M_i \dot{\omega}_i = -D_i \omega_i + P_i - \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \end{cases}$$

$$\text{Load}: \begin{cases} D_i \dot{\theta}_i = P_i - \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \\ Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j) \end{cases}$$

Challenge: Characterize equilibria, stability, basin of attraction

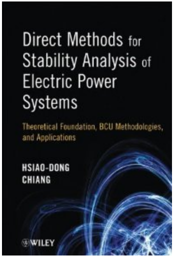
Approaches: Energy functions, nearest unstable eq. point, S.O.S., ...

Motivation II: multimachine transient stability

Constrained Swing Dynamics

$$\text{Gen}: \begin{cases} \dot{\theta}_i = \omega_i \\ M_i \dot{\omega}_i = -D_i \omega_i + P_i - \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \end{cases}$$

$$\text{Load}: \begin{cases} D_i \dot{\theta}_i = P_i - \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \\ Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j) \end{cases}$$



Challenge: Characterize equilibria, stability, basin of attraction

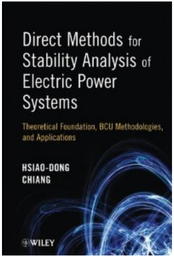
Approaches: Energy functions, nearest unstable eq. point, S.O.S., ...

Motivation II: multimachine transient stability

Constrained Swing Dynamics

$$\text{Gen}: \begin{cases} \dot{\theta}_i = \omega_i \\ M_i \dot{\omega}_i = -D_i \omega_i + P_i - \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \end{cases}$$

$$\text{Load}: \begin{cases} D_i \dot{\theta}_i = P_i - \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \\ Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j) \end{cases}$$



Challenge: Characterize equilibria, stability, basin of attraction

Approaches: Energy functions, nearest unstable eq. point, S.O.S., ...

$$\{\text{Equilibria}\} = \{\text{Power Flow Solutions}\}$$

Motivation III: optimal power flow

Idea: Optimally match supply and demand (with constraints)

Motivation III: optimal power flow

Idea: Optimally match supply and demand (with constraints)

$$\begin{aligned}
\underset{\theta, V_L, P_G}{\text{minimize}} \quad & \sum\nolimits_{i \in \mathcal{N}_G} f_i(P_i) \\
\text{subject to} \quad & P_i = \sum\nolimits_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) && i \in \mathcal{N}_L \cup \mathcal{N}_G \,, \\
& Q_i = -\sum\nolimits_j V_i V_j B_{ij} \cos(\theta_i - \theta_j) && i \in \mathcal{N}_L \,, \\
& V_i^{\min} \leq V_i \leq V_i^{\max} && i \in \mathcal{N}_L \,, \\
& S_i^{\min} \leq |P_i + \mathbf{j} Q_i| \leq S_i^{\max} && i \in \mathcal{N}_G \,, \\
& s_{ij}^{\min} \leq |p_{i \to j} + \mathbf{j} q_{i \to j}| \leq s_{ij}^{\max} && (i,j) \in \mathcal{E} \,,
\end{aligned}$$

Motivation III: optimal power flow

Idea: Optimally match supply and demand (with constraints)

$$
\begin{aligned}
\underset{\theta, V_L, P_G}{\text{minimize}} \quad & \sum\nolimits_{i \in \mathcal{N}_G} f_i(P_i) & \\
\text{subject to} \quad & P_i = \sum\nolimits_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) & i \in \mathcal{N}_L \cup \mathcal{N}_G \,, \\
& Q_i = -\sum\nolimits_j V_i V_j B_{ij} \cos(\theta_i - \theta_j) & i \in \mathcal{N}_L \,, \\
& V_i^{\min} \leq V_i \leq V_i^{\max} & i \in \mathcal{N}_L \,, \\
& S_i^{\min} \leq |P_i + \mathbf{j} Q_i| \leq S_i^{\max} & i \in \mathcal{N}_G \,, \\
& s_{ij}^{\min} \leq |p_{i\to j} + \mathbf{j} q_{i \to j}| \leq s_{ij}^{\max} & (i,j) \in \mathcal{E} \,,
\end{aligned}
$$

- non-convex, solved every 5-15 min. via linearization, ($$$)

Motivation III: optimal power flow

Idea: Optimally match supply and demand (with constraints)

$$\begin{aligned}
\underset{\theta, V_L, P_G}{\text{minimize}} \quad & \sum\nolimits_{i \in \mathcal{N}_G} f_i(P_i) & \\
\text{subject to} \quad & P_i = \sum\nolimits_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) & i \in \mathcal{N}_L \cup \mathcal{N}_G\,, \\
& Q_i = -\sum\nolimits_j V_i V_j B_{ij} \cos(\theta_i - \theta_j) & i \in \mathcal{N}_L\,, \\
& V_i^{\min} \leq V_i \leq V_i^{\max} & i \in \mathcal{N}_L\,, \\
& S_i^{\min} \leq |P_i + \mathbf{j} Q_i| \leq S_i^{\max} & i \in \mathcal{N}_G\,, \\
& s_{ij}^{\min} \leq |p_{i\to j} + \mathbf{j} q_{i\to j}| \leq s_{ij}^{\max} & (i,j) \in \mathcal{E}\,,
\end{aligned}$$

- non-convex, solved every 5-15 min. via linearization, ($$$)

> *"Today, 50 years after the problem was formulated, we still do not have a fast, robust solution technique for the full ACOPF."*
>
> — Richard P. O'Neill (Chief Economic Advisor, FERC, 2016)

Motivation III: optimal power flow

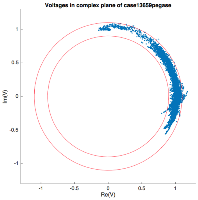
Idea: Optimally match supply and demand (with constraints)

$$
\begin{aligned}
\underset{\theta, V_L, P_G}{\text{minimize}} \quad & \sum\nolimits_{i \in \mathcal{N}_G} f_i(P_i) & \\
\text{subject to} \quad & P_i = \sum\nolimits_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) & i \in \mathcal{N}_L \cup \mathcal{N}_G\,, \\
& Q_i = -\sum\nolimits_j V_i V_j B_{ij} \cos(\theta_i - \theta_j) & i \in \mathcal{N}_L\,, \\
& V_i^{\min} \leq V_i \leq V_i^{\max} & i \in \mathcal{N}_L\,, \\
& S_i^{\min} \leq |P_i + \mathbf{j} Q_i| \leq S_i^{\max} & i \in \mathcal{N}_G\,, \\
& s_{ij}^{\min} \leq |p_{i\to j} + \mathbf{j} q_{i\to j}| \leq s_{ij}^{\max} & (i,j) \in \mathcal{E}\,,
\end{aligned}
$$

- non-convex, solved every 5-15 min. via linearization, ($$$)

> *"Today, 50 years after the problem was formulated, we still do not have a fast, robust solution technique for the full ACOPF."*
>
> — Richard P. O'Neill (Chief Economic Advisor, FERC, 2016)

Intuition on power flow solutions



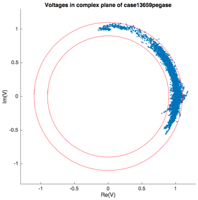
[Josz et al.]

1. 'Normally', exists unique **high-voltage** soln:
    - voltage magnitude $V_i \simeq 1$
    - phase diff $|\theta_i - \theta_j| \ll 1$
    - current flows from high V to low V!

2. **Lightly loaded systems:** many **low-voltage** solutions

3. **Heavily loaded systems:** Few solutions or **infeasible**
    - saddle node bifurcations
    - maximum power transfer limit
    - non-convex feasible set in (P, Q)-space

Intuition on power flow solutions



[Josz et al.]

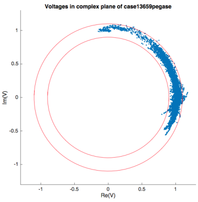
1. 'Normally', exists unique **high-voltage** soln:
   - voltage magnitude $V_i \simeq 1$
   - phase diff $|\theta_i - \theta_j| \ll 1$
   - current flows from high V to low V!

2. **Lightly loaded systems:** many **low-voltage** solutions

3. **Heavily loaded systems:** Few solutions or **infeasible**
   - saddle node bifurcations
   - maximum power transfer limit
   - non-convex feasible set in (P, Q)-space

Intuition on power flow solutions

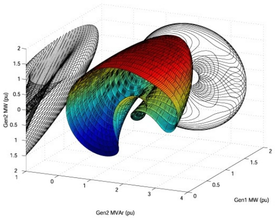
1. ‘Normally’, exists unique **high-voltage** soln:
   - voltage magnitude $V_i \simeq 1$
   - phase diff $|\theta_i - \theta_j| \ll 1$
   - current flows from high V to low V!



[Josz et al.]

2. **Lightly loaded systems:** many **low-voltage** solutions

3. **Heavily loaded systems:** Few solutions or **infeasible**
   - saddle node bifurcations
   - maximum power transfer limit
   - non-convex feasible set in (P, Q)-space



[Hiskens & Davy]

Literature on Power Flow Solvability

Given data: network topology, impedances, generation & loads

Q: $\exists$ "stable high-voltage" solution? unique? properties?

Many approaches over **45+ years** of literature:

- [Weedy '67]: Jacobian singularity
- [Korsak '72]: Multiple "stable" solutions
- [Wu & Kumagai '77, '80, '82]: Fixed-point analysis of existence
- [Araposthatis, Sastry & Varaiya, '81]: Jacobian analysis
- [Baillieul and Byrnes '82]: Counting # of solutions, Bezout/Morse analysis
- [Ilic '86, '92]: "no-gain" results, nonlinear resistive networks
- [Makarov, Hill & Hiskens '00]: Solution insights for general quadratic equations
- [Dörfler, Chertkov & Bullo '12]: Existence/uniqueness for lossless P/θ problem
- [JWSP, Dörfler & Bullo '15]: Existence/uniqueness for lossless Q/V problem
- [Bolognani & Zampieri '16, Nguyen *et al.* '17, Wang *et al.* '17, ...]: Distribution networks
- [JWSP '16, '17]: Lossy P/θ, coupled power flow conditions
- [Delabays, Jafarpour, Bullo '21]: Effect of cycles in P/θ problem
- ...

Literature on Power Flow Solvability

Given data: network topology, impedances, generation & loads

Q: $\exists$ "stable high-voltage" solution? unique? properties?

Many approaches over **45+ years** of literature:

- [Weedy '67]: Jacobian singularity
- [Korsak '72]: Multiple "stable" solutions
- [Wu & Kumagai '77, '80, '82]: Fixed-point analysis of existence
- [Araposthatis, Sastry & Varaiya, '81]: Jacobian analysis
- [Baillieul and Byrnes '82]: Counting # of solutions, Bezout/Morse analysis
- [Ilic '86, '92]: "no-gain" results, nonlinear resistive networks
- [Makarov, Hill & Hiskens '00]: Solution insights for general quadratic equations
- [Dörfler, Chertkov & Bullo '12]: Existence/uniqueness for lossless P/θ problem
- [JWSP, Dörfler & Bullo '15]: Existence/uniqueness for lossless Q/V problem
- [Bolognani & Zampieri '16, Nguyen *et al.* '17, Wang *et al.* '17, ...]: Distribution networks
- [JWSP '16, '17]: Lossy P/θ, coupled power flow conditions
- [Delabays, Jafarpour, Bullo '21]: Effect of cycles in P/θ problem
- ...

Literature on Power Flow Solvability

Given data: network topology, impedances, generation & loads

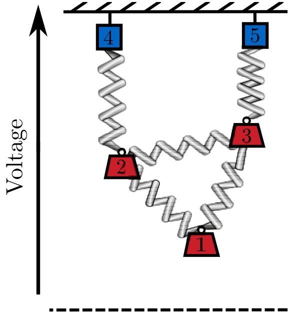
Q: $\exists$ "stable high-voltage" solution? unique? properties?

Many approaches over **45+ years** of literature:

Main insight: stiffness vs. loading

1. Stiff network + light loading $\Rightarrow$ feasible
2. Weak network + heavy loading $\Rightarrow$ infeasible

Q: How to quantify network stiffness vs. loading?



Literature on Power Flow Solvability

Given data: network topology, impedances, generation & loads

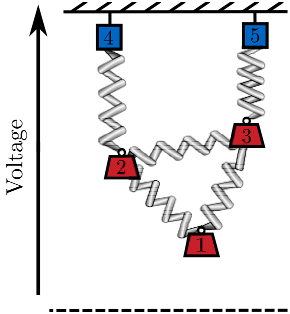
Q: $\exists$ "stable high-voltage" solution? unique? properties?

Many approaches over **45+ years** of literature:

Main insight: stiffness vs. loading

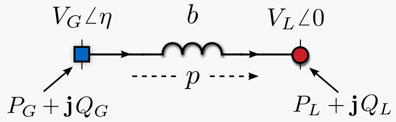
1. Stiff network + light loading $\Rightarrow$ feasible
2. Weak network + heavy loading $\Rightarrow$ infeasible

Q: How to quantify network stiffness vs. loading?



Solution of two-bus system

$$P_L = bV_G V_L \sin(-\eta)$$
$$P_G = bV_G V_L \sin(\eta)$$
$$Q_L = bV_L^2 - bV_L V_G \cos(\eta)$$



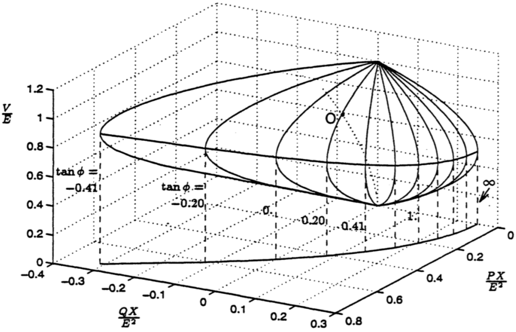
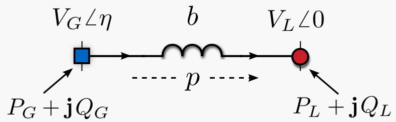


Figure 2.6 Voltage as a function of load active and reactive powers

Solution of two-bus system

$$p = bV_G V_L \sin(\eta)$$
$$Q_L = bV_L^2 - bV_L V_G \cos(\eta)$$



1. **Change Variables**

$$v := \frac{V_L}{V_G} \qquad \Gamma := \frac{p}{bV_G^2} \qquad \Delta := \frac{Q_L}{-\frac{1}{4}bV_G^2}$$

2. **Square equations, add, and solve quadratic in v^2**

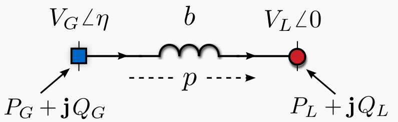
$$v_{\pm} = \sqrt{\frac{1}{2}\left(1 - \frac{\Delta}{2} \pm \sqrt{1 - (4\Gamma^2 + \Delta)}\right)}$$

3. **Nec. & Suff. Condition**

$$4\Gamma^2 + \Delta < 1$$

Solution of two-bus system

$$p = bV_G V_L \sin(\eta)$$
$$Q_L = bV_L^2 - bV_L V_G \cos(\eta)$$



1. **Change Variables**

$$v := \frac{V_L}{V_G} \qquad \Gamma := \frac{p}{bV_G^2} \qquad \Delta := \frac{Q_L}{-\frac{1}{4}bV_G^2}$$

2. **Square equations, add, and solve quadratic in v^2**

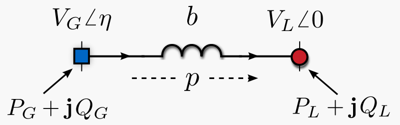
$$v_\pm = \sqrt{\frac{1}{2}\left(1 - \frac{\Delta}{2} \pm \sqrt{1 - (4\Gamma^2 + \Delta)}\right)}$$

3. **Nec. & Suff. Condition**

$$4\Gamma^2 + \Delta < 1$$

Solution of two-bus system

$$p = bV_G V_L \sin(\eta)$$
$$Q_L = bV_L^2 - bV_L V_G \cos(\eta)$$



1. **Change Variables**

$$v := \frac{V_L}{V_G} \qquad \Gamma := \frac{p}{bV_G^2} \qquad \Delta := \frac{Q_L}{-\frac{1}{4}bV_G^2}$$

2. **Square equations, add, and solve quadratic in v^2**

$$v_\pm = \sqrt{\frac{1}{2}\left(1 - \frac{\Delta}{2} \pm \sqrt{1 - (4\Gamma^2 + \Delta)}\right)}$$

3. **Nec. & Suff. Condition**

$$\boxed{4\Gamma^2 + \Delta < 1}$$

Solution of two-bus system

$$\Gamma = v\sin(\eta)$$
$$\Delta = -4v^2 + 4v\cos(\eta)$$

$$v := \frac{V_L}{V_G} \quad \Gamma := \frac{p}{bV_G^2} \quad \Delta := \frac{Q_L}{-\frac{1}{4}bV_G^2}$$
$$4\Gamma^2 + \Delta < 1$$

1. **High-voltage** solution
   $v_+ \in [\frac{1}{2}, 1)$
2. **Low-voltage** solution
   $v_- \in [0, \frac{1}{\sqrt{2}})$

Angle: $\sin(\eta_\mp) = \Gamma / v_\pm$

1. **Small-angle** solution
   $\eta_- \in [0, \pi/4)$
2. **Large-angle** solution
   $\eta_+ \in [0, \pi/2)$

Solution of two-bus system

$$\Gamma = v\sin(\eta)$$
$$\Delta = -4v^2 + 4v\cos(\eta)$$

$$v := \frac{V_L}{V_G} \quad \Gamma := \frac{p}{bV_G^2} \quad \Delta := \frac{Q_L}{-\frac{1}{4}bV_G^2}$$
$$4\Gamma^2 + \Delta < 1$$

1. **High-voltage** solution
   $v_+ \in [\frac{1}{2}, 1)$
2. **Low-voltage** solution
   $v_- \in [0, \frac{1}{\sqrt{2}})$

Angle: $\sin(\eta_\mp) = \Gamma / v_\pm$

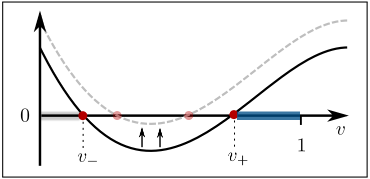
1. **Small-angle** solution
   $\eta_- \in [0, \pi/4)$
2. **Large-angle** solution
   $\eta_+ \in [0, \pi/2)$

Solution of two-bus system

$$\Gamma = v\sin(\eta)$$
$$\Delta = -4v^2 + 4v\cos(\eta)$$

$$v := \frac{V_L}{V_G} \quad \Gamma := \frac{p}{bV_G^2} \quad \Delta := \frac{Q_L}{-\frac{1}{4}bV_G^2}$$
$$4\Gamma^2 + \Delta < 1$$

1. **High-voltage** solution
   $v_+ \in [\frac{1}{2}, 1)$
2. **Low-voltage** solution
   $v_- \in [0, \frac{1}{\sqrt{2}})$



Angle: $\sin(\eta_\mp) = \Gamma / v_\pm$

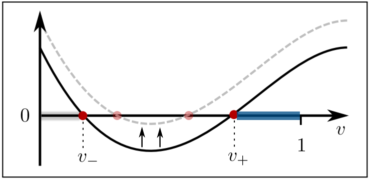
1. **Small-angle** solution
   $\eta_- \in [0, \pi/4)$
2. **Large-angle** solution
   $\eta_+ \in [0, \pi/2)$

Solution of two-bus system

$$\Gamma = v \sin(\eta)$$
$$\Delta = -4v^2 + 4v\cos(\eta)$$

$$v := \frac{V_L}{V_G} \quad \Gamma := \frac{p}{bV_G^2} \quad \Delta := \frac{Q_L}{-\frac{1}{4}bV_G^2}$$
$$4\Gamma^2 + \Delta < 1$$

1. **High-voltage** solution
   $v_+ \in [\frac{1}{2}, 1)$
2. **Low-voltage** solution
   $v_- \in [0, \frac{1}{\sqrt{2}})$



Angle: $\sin(\eta_\mp) = \Gamma / v_\pm$

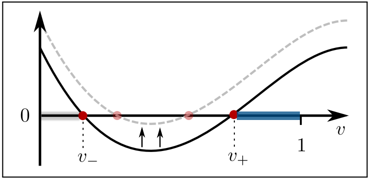
1. **Small-angle** solution
   $\eta_- \in [0, \pi/4)$
2. **Large-angle** solution
   $\eta_+ \in [0, \pi/2)$

Solution of two-bus system

$$\Gamma = v\sin(\eta)$$
$$\Delta = -4v^2 + 4v\cos(\eta)$$

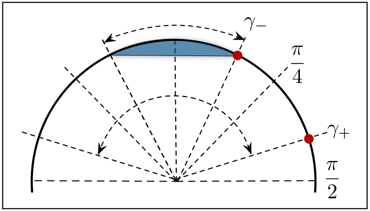
$$v := \frac{V_L}{V_G} \quad \Gamma := \frac{p}{bV_G^2} \quad \Delta := \frac{Q_L}{-\frac{1}{4}bV_G^2}$$
$$4\Gamma^2 + \Delta < 1$$

1. **High-voltage** solution
   $v_+ \in [\frac{1}{2}, 1)$
2. **Low-voltage** solution
   $v_- \in [0, \frac{1}{\sqrt{2}})$



Angle: $\sin(\eta_\mp) = \Gamma / v_\pm$

1. **Small-angle** solution
   $\eta_- \in [0, \pi/4)$
2. **Large-angle** solution
   $\eta_+ \in [0, \pi/2)$



Solution of two-bus system

- Squaring and adding equations **does not generalize** to networks.
- Is there any hope then?

$$\Gamma = v\sin(\eta)$$
$$\Delta = -4v^2 + 4v\cos(\eta)$$

- Use $\cos(\eta) = \sqrt{1-\sin^2(\eta)} \implies \Delta = -4v^2 + 4v\sqrt{1-(\Gamma/v)^2}$
- Rearrange to get *fixed-point equation*

$$v = f(v) := -\frac{1}{4}\frac{\Delta}{v} + \sqrt{1-\left(\frac{\Gamma}{v}\right)^2}$$

This generalizes! Leverage intuition.

Solution of two-bus system

- Squaring and adding equations **does not generalize** to networks.
- Is there any hope then?

$$\Gamma = v\sin(\eta)$$
$$\Delta = -4v^2 + 4v\cos(\eta)$$

- Use $\cos(\eta) = \sqrt{1-\sin^2(\eta)} \implies \Delta = -4v^2 + 4v\sqrt{1-(\Gamma/v)^2}$
- Rearrange to get *fixed-point equation*

$$v = f(v) := -\frac{1}{4}\frac{\Delta}{v} + \sqrt{1-\left(\frac{\Gamma}{v}\right)^2}$$

This generalizes! Leverage intuition.

Solution of two-bus system

- Squaring and adding equations **does not generalize** to networks.
- Is there any hope then?

$$\Gamma = v\sin(\eta)$$
$$\Delta = -4v^2 + 4v\cos(\eta)$$

- Use $\cos(\eta) = \sqrt{1-\sin^2(\eta)} \implies \Delta = -4v^2 + 4v\sqrt{1-(\Gamma/v)^2}$
- Rearrange to get *fixed-point equation*

$$v = f(v) := -\frac{1}{4}\frac{\Delta}{v} + \sqrt{1-\left(\frac{\Gamma}{v}\right)^2}$$

This generalizes! Leverage intuition.

Solution of two-bus system

- Squaring and adding equations **does not generalize** to networks.
- Is there any hope then?

$$\Gamma = v\sin(\eta)$$
$$\Delta = -4v^2 + 4v\cos(\eta)$$

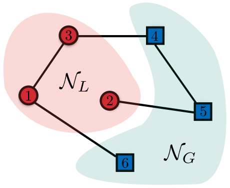
- Use $\cos(\eta) = \sqrt{1-\sin^2(\eta)} \implies \Delta = -4v^2 + 4v\sqrt{1-(\Gamma/v)^2}$
- Rearrange to get *fixed-point equation*

$$v = f(v) := -\frac{1}{4}\frac{\Delta}{v} + \sqrt{1-\left(\frac{\Gamma}{v}\right)^2}$$

This generalizes! Leverage intuition.

Notation I: branches and bus types

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$
$$Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)\,, \quad i \in \mathcal{N}_L$$

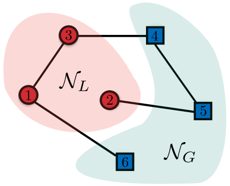


- Bus partitioning $\mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G$ induces **branch partitioning**

$$\mathcal{E} = \mathcal{E}^{\ell\ell} \cup \mathcal{E}^{g\ell} \cup \mathcal{E}^{gg}\,, \quad A = \left(\begin{array}{c} A_L \\ \hline A_G \end{array}\right) = \left(\begin{array}{c|c|c} A_L^{\ell\ell} & A_L^{g\ell} & \mathbb{0} \\ \hline \mathbb{0} & A_G^{g\ell} & A_G^{gg} \end{array}\right).$$

Notation I: branches and bus types

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$
$$Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)\,, \quad i \in \mathcal{N}_L$$

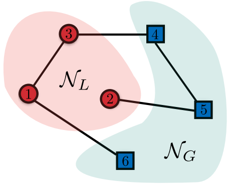


- Bus partitioning $\mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G$ induces **branch partitioning**

$$\mathcal{E} = \mathcal{E}^{\ell\ell} \cup \mathcal{E}^{g\ell} \cup \mathcal{E}^{gg}\,, \quad A = \left(\begin{array}{c} A_L \\ \hline A_G \end{array}\right) = \left(\begin{array}{c|c|c} A_L^{\ell\ell} & A_L^{g\ell} & \mathbb{0} \\ \hline \mathbb{0} & A_G^{g\ell} & A_G^{gg} \end{array}\right).$$

Notation I: branches and bus types

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$
$$Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)\,, \quad i \in \mathcal{N}_L$$

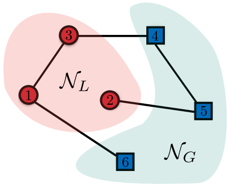


- Bus partitioning $\mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G$ induces **branch partitioning**

$$\mathcal{E} = \mathcal{E}^{\ell\ell} \cup \mathcal{E}^{g\ell} \cup \mathcal{E}^{gg}\,, \quad A = \left(\begin{array}{c} A_L \\ \hline A_G \end{array}\right) = \left(\begin{array}{c|c|c} A_L^{\ell\ell} & A_L^{g\ell} & \mathbb{0} \\ \hline \mathbb{0} & A_G^{g\ell} & A_G^{gg} \end{array}\right).$$

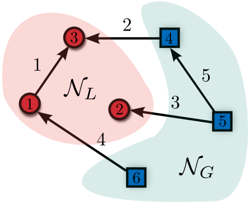
Notation I: branches and bus types

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$
$$Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)\,, \quad i \in \mathcal{N}_L$$



- Bus partitioning $\mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G$ induces **branch partitioning**

$$\mathcal{E} = \mathcal{E}^{\ell\ell} \cup \mathcal{E}^{g\ell} \cup \mathcal{E}^{gg}\,, \quad A = \left(\begin{array}{c} A_L \\ \hline A_G \end{array}\right) = \left(\begin{array}{c|c|c} A_L^{\ell\ell} & A_L^{g\ell} & \mathbb{0} \\ \hline \mathbb{0} & A_G^{g\ell} & A_G^{gg} \end{array}\right).$$



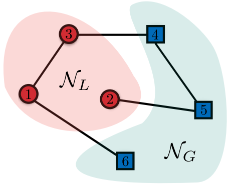
$$A = \begin{array}{c} \begin{array}{ccc} \overbrace{\quad}^{\mathcal{E}^{\ell\ell}} & \overbrace{\qquad\qquad\qquad}^{\mathcal{E}^{g\ell}} & \overbrace{\quad}^{\mathcal{E}^{gg}} \end{array} \\ \left(\begin{array}{c|ccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\0\\-1\end{array}}\right\} \mathcal{N}_L \\ \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} \mathcal{N}_G \end{array}$$

Notation II: open-circuit voltages

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$

$$Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)\,, \quad i \in \mathcal{N}_L$$

- **Generators $\mathcal{N}_G$:** V_i fixed
- **Loads $\mathcal{N}_L$:** V_i free



Partitioned Variables

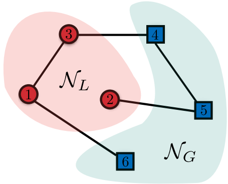
$$V = \left(\frac{V_L}{V_G}\right), \qquad B = \left(\begin{array}{c|c} B_{LL} & B_{LG} \\ \hline B_{GL} & B_{GG} \end{array}\right)$$

Notation II: open-circuit voltages

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$

$$Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)\,, \quad i \in \mathcal{N}_L$$

- **Generators $\mathcal{N}_G$:** V_i fixed
- **Loads $\mathcal{N}_L$:** V_i free



Partitioned Variables

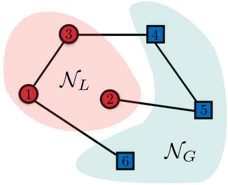
$$V = \left(\begin{array}{c} V_L \\ \hline V_G \end{array}\right), \qquad B = \left(\begin{array}{c|c} B_{LL} & B_{LG} \\ \hline B_{GL} & B_{GG} \end{array}\right)$$

Notation II: open-circuit voltages

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$

$$Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)\,, \quad i \in \mathcal{N}_L$$

- **Generators $\mathcal{N}_G$:** V_i fixed
- **Loads $\mathcal{N}_L$:** V_i free



Partitioned Variables

$$V = \begin{pmatrix} V_L \\ \hline V_G \end{pmatrix}, \qquad B = \left(\begin{array}{c|c} B_{LL} & B_{LG} \\ \hline B_{GL} & B_{GG} \end{array}\right)$$

Open-circuit voltages

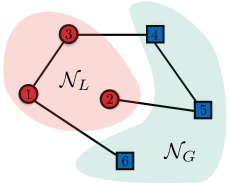
$$V_L^* \triangleq \underbrace{-B_{LL}^{-1} B_{LG}}_{Generators \to Loads} \cdot V_G$$

Notation II: open-circuit voltages

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \mathcal{N}_L \cup \mathcal{N}_G$$

$$Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j)\,, \quad i \in \mathcal{N}_L$$

- **Generators $\mathcal{N}_G$:** V_i fixed
- **Loads $\mathcal{N}_L$:** V_i free



Partitioned Variables

$$V = \begin{pmatrix} V_L \\ \hline V_G \end{pmatrix}, \qquad B = \left(\begin{array}{c|c} B_{LL} & B_{LG} \\ \hline B_{GL} & B_{GG} \end{array}\right)$$

Open-circuit voltages

$$V_L^* \triangleq \underbrace{-B_{LL}^{-1} B_{LG}}_{\text{Generators}\to\text{Loads}} \cdot V_G$$

$$v_i \triangleq V_i / V_i^*$$

Notation III: stiffness matrices

$$V = \left(\frac{V_L}{V_G}\right), \quad B = \left(\begin{array}{c|c} B_{LL} & B_{LG} \\ \hline B_{GL} & B_{GG} \end{array}\right), \quad V_L^* = -B_{LL}^{-1} B_{LG} V_G$$

- Need to **non-dimensionalize** power flow equations
- **Stiffness matrices** quantify **grid strength** in **units of power**

1. **Nodal** stiffness matrix $\quad \mathsf{S} \triangleq \frac{1}{4}[V_L^*] \cdot B_{LL} \cdot [V_L^*]$
2. **Branch** stiffness matrix $\quad \mathsf{D} \triangleq [V_i^* V_j^* B_{ij}]_{(i,j)\in\mathcal{E}}$
3. **Laplacian** stiffness matrix $\quad \mathsf{L} \triangleq A\mathsf{D}A^\mathsf{T}$

Notation III: stiffness matrices

$$V = \begin{pmatrix} V_L \\ \hline V_G \end{pmatrix}, \quad B = \left(\begin{array}{c|c} B_{LL} & B_{LG} \\ \hline B_{GL} & B_{GG} \end{array}\right), \quad V_L^* = -B_{LL}^{-1} B_{LG} V_G$$

- Need to **non-dimensionalize** power flow equations
- **Stiffness matrices** quantify **grid strength** in **units of power**

1. **Nodal** stiffness matrix

$$S \triangleq \frac{1}{4} [V_L^*] \cdot B_{LL} \cdot [V_L^*]$$

2. **Branch** stiffness matrix

$$D \triangleq [V_i^* V_j^* B_{ij}]_{(i,j)\in\mathcal{E}}$$

3. **Laplacian** stiffness matrix

$$L \triangleq ADA^{\mathsf{T}}$$

Notation III: stiffness matrices

$$V = \begin{pmatrix} V_L \\ \hline V_G \end{pmatrix}, \quad B = \left(\begin{array}{c|c} B_{LL} & B_{LG} \\ \hline B_{GL} & B_{GG} \end{array}\right), \quad V_L^* = -B_{LL}^{-1} B_{LG} V_G$$

- Need to **non-dimensionalize** power flow equations
- **Stiffness matrices** quantify **grid strength** in **units of power**

1. **Nodal** stiffness matrix $\quad \mathsf{S} \triangleq \frac{1}{4}[V_L^*] \cdot B_{LL} \cdot [V_L^*]$
2. **Branch** stiffness matrix $\quad \mathsf{D} \triangleq [V_i^* V_j^* B_{ij}]_{(i,j)\in\mathcal{E}}$
3. **Laplacian** stiffness matrix $\quad \mathsf{L} \triangleq A\mathsf{D}A^{\mathsf{T}}$

Notation III: stiffness matrices

$$V = \left(\frac{V_L}{V_G}\right), \quad B = \left(\begin{array}{c|c} B_{LL} & B_{LG} \\ \hline B_{GL} & B_{GG} \end{array}\right), \quad V_L^* = -B_{LL}^{-1} B_{LG} V_G$$

- Need to **non-dimensionalize** power flow equations
- **Stiffness matrices** quantify **grid strength** in **units of power**

1. **Nodal** stiffness matrix

$$S \triangleq \frac{1}{4} [V_L^*] \cdot B_{LL} \cdot [V_L^*]$$

2. **Branch** stiffness matrix

$$D \triangleq [V_i^* V_j^* B_{ij}]_{(i,j) \in \mathcal{E}}$$

3. **Laplacian** stiffness matrix

$$L \triangleq ADA^T$$

Notation III: stiffness matrices

$$V = \left(\frac{V_L}{V_G}\right), \quad B = \left(\begin{array}{c|c} B_{LL} & B_{LG} \\ \hline B_{GL} & B_{GG} \end{array}\right), \quad V_L^* = -B_{LL}^{-1} B_{LG} V_G$$

- Need to **non-dimensionalize** power flow equations
- **Stiffness matrices** quantify **grid strength** in **units of power**

1. **Nodal** stiffness matrix $$\mathsf{S} \triangleq \frac{1}{4} [V_L^*] \cdot B_{LL} \cdot [V_L^*]$$
2. **Branch** stiffness matrix $$\mathsf{D} \triangleq [V_i^* V_j^* B_{ij}]_{(i,j)\in\mathcal{E}}$$
3. **Laplacian** stiffness matrix $$\mathsf{L} \triangleq A\mathsf{D}A^\mathsf{T}$$

Active power flow reformulation

Notation:

$$h_e(v) = \begin{cases} v_i v_j & \text{if } e = (i,j) \in \mathcal{E}^{\ell\ell} \\ v_j & \text{if } e = (i,j) \in \mathcal{E}^{g\ell} \\ 1 & \text{if } e = (i,j) \in \mathcal{E}^{gg} \end{cases}$$

Active Power:

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)$$

$$P = \underbrace{A}_{\text{Incidence}} \underbrace{\mathrm{D}}_{\text{Branch Stiff.}} \underbrace{[h(v)]}_{\text{Voltages}} \underbrace{\mathbf{sin}(A^{\mathsf{T}}\theta)}_{\sin(\theta_i - \theta_j)}$$

- Let columns of C be a basis for $\ker(A)$, let $p_c \in \mathbb{R}^c$

Semi-Explicit Solution

$$\mathbf{sin}(A^{\mathsf{T}}\theta) = \psi(v, p_c) \triangleq [h(v)]^{-1} \left(A^{\mathsf{T}} \mathrm{L}^{\dagger} P + \mathrm{D}^{-1} C p_c \right)$$

$$\mathbb{0} = C^{\mathsf{T}} \mathbf{arcsin}(\psi)$$

Active power flow reformulation

Notation:

$$h_e(v) = \begin{cases} v_i v_j & \text{if } e = (i,j) \in \mathcal{E}^{\ell\ell} \\ v_j & \text{if } e = (i,j) \in \mathcal{E}^{g\ell} \\ 1 & \text{if } e = (i,j) \in \mathcal{E}^{gg} \end{cases}$$

Active Power:

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)$$

$$P = \underbrace{A}_{\text{Incidence}} \; \underbrace{\mathrm{D}}_{\text{Branch Stiff.}} \; \underbrace{[h(v)]}_{\text{Voltages}} \; \underbrace{\mathbf{sin}(A^{\mathsf{T}}\theta)}_{\sin(\theta_i - \theta_j)}$$

- Let columns of C be a basis for $\ker(A)$, let $p_c \in \mathbb{R}^c$

Semi-Explicit Solution

$$\mathbf{sin}(A^{\mathsf{T}}\theta) = \psi(v, p_c) \triangleq [h(v)]^{-1}\left(A^{\mathsf{T}}\mathrm{L}^{\dagger}P + \mathrm{D}^{-1}Cp_c\right)$$

$$\mathbb{0} = C^{\mathsf{T}}\mathbf{arcsin}(\psi)$$

Active power flow reformulation

Notation:

$$h_e(v) = \begin{cases} v_i v_j & \text{if } e = (i,j) \in \mathcal{E}^{\ell\ell} \\ v_j & \text{if } e = (i,j) \in \mathcal{E}^{g\ell} \\ 1 & \text{if } e = (i,j) \in \mathcal{E}^{gg} \end{cases}$$

Active Power:

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)$$

$$P = \underbrace{A}_{\text{Incidence}} \quad \underbrace{\mathrm{D}}_{\text{Branch Stiff.}} \quad \underbrace{[h(v)]}_{\text{Voltages}} \underbrace{\mathbf{sin}(A^{\mathsf{T}}\theta)}_{\sin(\theta_i - \theta_j)}$$

- Let columns of C be a basis for $\ker(A)$, let $p_c \in \mathbb{R}^c$

Semi-Explicit Solution

$$\mathbf{sin}(A^{\mathsf{T}}\theta) = \psi(v, p_c) \triangleq [h(v)]^{-1}\left(A^{\mathsf{T}}\mathrm{L}^{\dagger}P + \mathrm{D}^{-1}Cp_c\right)$$

$$\mathbb{0} = C^{\mathsf{T}}\mathbf{arcsin}(\psi)$$

Active power flow reformulation

Notation:

$$h_e(v) = \begin{cases} v_i v_j & \text{if } e = (i,j) \in \mathcal{E}^{\ell\ell} \\ v_j & \text{if } e = (i,j) \in \mathcal{E}^{g\ell} \\ 1 & \text{if } e = (i,j) \in \mathcal{E}^{gg} \end{cases}$$

Active Power:

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)$$

$$P = \underbrace{A}_{\text{Incidence}} \underbrace{\mathrm{D}}_{\text{Branch Stiff.}} \underbrace{[h(v)]}_{\text{Voltages}} \underbrace{\mathbf{sin}(A^{\mathsf{T}}\theta)}_{\sin(\theta_i - \theta_j)}$$

- Let columns of C be a basis for $\ker(A)$, let $p_c \in \mathbb{R}^c$

Semi-Explicit Solution

$$\mathbf{sin}(A^{\mathsf{T}}\theta) = \psi(v, p_c) \triangleq [h(v)]^{-1} \left(A^{\mathsf{T}} \mathrm{L}^{\dagger} P + \mathrm{D}^{-1} C p_c \right)$$

$$\mathbb{0} = C^{\mathsf{T}} \mathbf{arcsin}(\psi)$$

Active power flow reformulation

Notation:

$$h_e(v) = \begin{cases} v_i v_j & \text{if } e = (i,j) \in \mathcal{E}^{\ell\ell} \\ v_j & \text{if } e = (i,j) \in \mathcal{E}^{g\ell} \\ 1 & \text{if } e = (i,j) \in \mathcal{E}^{gg} \end{cases}$$

Active Power:

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)$$

$$P = \underbrace{A}_{\text{Incidence}} \underbrace{\mathrm{D}}_{\text{Branch Stiff.}} \underbrace{[h(v)]}_{\text{Voltages}} \underbrace{\mathbf{sin}(A^{\mathsf{T}}\theta)}_{\sin(\theta_i - \theta_j)}$$

- Let columns of C be a basis for $\ker(A)$, let $p_c \in \mathbb{R}^c$

Semi-Explicit Solution

$$\mathbf{sin}(A^{\mathsf{T}}\theta) = \psi(v, p_c) \triangleq [h(v)]^{-1} \left(A^{\mathsf{T}} \mathrm{L}^{\dagger} P + \mathrm{D}^{-1} C p_c \right)$$

$$\mathbb{0} = C^{\mathsf{T}} \mathbf{arcsin}(\psi)$$

Reactive power flow reformulation

Skipping some details . . .

$$Q_L = -4[v]\mathsf{S}(v - \mathbb{1}_n) + |A|_L \mathsf{D}\,[h(v)](\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^\mathsf{T}\theta))\,.$$

- Rearrange for v

$$v = f(v,\theta) = \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\,(\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^\mathsf{T}\theta))\,,$$

- Now plug in $\cos(z) = \sqrt{1-\sin^2(z)}$!

Reactive power flow reformulation

Skipping some details ...

$$Q_L = -4[v] \underbrace{\mathsf{S}}_{\text{Nodal stiff.}} (v - \mathbb{1}_n) + |A|_L \mathsf{D}\,[h(v)](\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^\mathsf{T}\theta))\,.$$

- Rearrange for v

$$\begin{aligned} v = f(v,\theta) = \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n \\ + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\,(\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^\mathsf{T}\theta))\,, \end{aligned}$$

- Now plug in $\cos(z) = \sqrt{1 - \sin^2(z)}$!

Reactive power flow reformulation

Skipping some details ...

$$Q_L = -4[v]\mathsf{S}(v - \mathbb{1}_n) + \underbrace{|A|_L}_{\text{Abs. Incidence Blk.}} \mathsf{D}\,[h(v)](\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^{\mathsf{T}}\theta))\,.$$

- Rearrange for v

$$\begin{aligned} v = f(v,\theta) = \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n \\ + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L\mathsf{D}\,[h(v)]\,(\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^{\mathsf{T}}\theta))\,, \end{aligned}$$

- Now plug in $\cos(z) = \sqrt{1-\sin^2(z)}$!

Reactive power flow reformulation

Skipping some details ...

$$Q_L = -4[v]\mathsf{S}(v - \mathbb{1}_n) + |A|_L \underbrace{\mathsf{D}}_{\text{Branch Stiff.}} [h(v)](\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^{\mathsf{T}}\theta)).$$

- Rearrange for v

$$\begin{aligned} v = f(v,\theta) &= \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n \\ &\quad + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L\mathsf{D}\,[h(v)]\,(\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^{\mathsf{T}}\theta)), \end{aligned}$$

- Now plug in $\cos(z) = \sqrt{1 - \sin^2(z)}$!

Reactive power flow reformulation

Skipping some details . . .

$$Q_L = -4[v]\mathsf{S}(v - \mathbb{1}_n) + |A|_L \mathsf{D}\,[h(v)](\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^{\mathsf{T}}\theta))\,.$$

- Rearrange for v

$$\begin{aligned} v = f(v,\theta) = \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n \\ + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\,(\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^{\mathsf{T}}\theta))\,, \end{aligned}$$

- Now plug in $\cos(z) = \sqrt{1 - \sin^2(z)}$!

Reactive power flow reformulation

Skipping some details . . .

$$Q_L = -4[v]\mathsf{S}(v - \mathbb{1}_n) + |A|_L \mathsf{D}\,[h(v)](\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^{\mathsf{T}}\theta))\,.$$

- Rearrange for v

$$\begin{aligned} v = f(v,\theta) = \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n \\ + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\,(\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^{\mathsf{T}}\theta))\,, \end{aligned}$$

- Now plug in $\cos(z) = \sqrt{1 - \sin^2(z)}$!

Reactive power flow reformulation

Skipping some details . . .

$$Q_L = -4[v]\mathsf{S}(v - \mathbb{1}_n) + |A|_L \mathsf{D}\,[h(v)](\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^\mathsf{T}\theta))\,.$$

- Rearrange for v

$$\begin{aligned} v = f(v,\theta) = \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n \\ + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\,(\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^\mathsf{T}\theta))\,, \end{aligned}$$

- Now plug in $\cos(z) = \sqrt{1 - \sin^2(z)}$!

Reactive power flow reformulation

Skipping some details ...

$$Q_L = -4[v]\mathsf{S}(v - \mathbb{1}_n) + |A|_L \mathsf{D}\,[h(v)](\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^\mathsf{T}\theta))\,.$$

- Rearrange for v

$$\begin{aligned} v = f(v,\theta) = \mathbb{1}_n &- \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n \\ &+ \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\,(\mathbb{1}_{|\mathcal{E}|} - \mathbf{cos}(A^\mathsf{T}\theta))\,, \end{aligned}$$

- Now plug in $\cos(z) = \sqrt{1-\sin^2(z)}$!

Fixed-Point Power Flow: Meshed Networks

(θ, V_L) is a power flow solution iff (v, p_c) solves the FPPF

$$v = f(v, p_c) \triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\left[h(v)\right] u(v, p_c)\,,$$

$$\mathbb{0}_c = C^{\mathsf{T}}\mathbf{arcsin}(\psi(v, p_c))\,.$$

where

$$u(v, p_c) \triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi}$$

$$\psi(v, p_c) = [h(v)]^{-1}\left(A^{\mathsf{T}}\mathsf{L}^{\dagger}P + \mathsf{D}^{-1}Cp_c\right)\,,$$

with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)\,.$

Fixed-Point Power Flow: Meshed Networks

(θ, V_L) is a power flow solution iff (v, p_c) solves the FPPF

$$v = f(v, p_c) \triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\,u(v, p_c)\,,$$

$$\mathbb{0}_c = C^{\mathsf{T}}\mathbf{arcsin}(\psi(v, p_c))\,.$$

where

$$u(v, p_c) \triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi}$$

$$\psi(v, p_c) = [h(v)]^{-1}\left(A^{\mathsf{T}}\mathsf{L}^{\dagger}P + \mathsf{D}^{-1}Cp_c\right)\,,$$

with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)\,.$

Fixed-Point Power Flow: Meshed Networks

(θ, V_L) is a power flow solution iff (v, p_c) solves the FPPF

$$v = f(v, p_c) \triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\,u(v, p_c)\,,$$

$$\mathbb{0}_c = C^{\mathsf{T}}\mathbf{arcsin}(\psi(v, p_c))\,.$$

where

$$u(v, p_c) \triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi}$$

$$\psi(v, p_c) = [h(v)]^{-1}\left(A^{\mathsf{T}}\mathsf{L}^{\dagger}P + \mathsf{D}^{-1}Cp_c\right)\,,$$

with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)\,.$

Fixed-Point Power Flow: Meshed Networks

(θ, V_L) is a power flow solution iff (v, p_c) solves the FPPF

$$
\begin{aligned}
v = f(v, p_c) &\triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n \\
&\qquad + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\,u(v, p_c)\,,
\end{aligned}
$$

$$\mathbb{0}_c = C^{\mathsf{T}}\mathbf{arcsin}(\psi(v, p_c))\,.$$

where

$$u(v, p_c) \triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi}$$

$$\psi(v, p_c) = [h(v)]^{-1}\left(A^{\mathsf{T}}\mathsf{L}^{\dagger}P + \mathsf{D}^{-1}Cp_c\right)\,,$$

with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)$.

Fixed-Point Power Flow: Meshed Networks

(θ, V_L) is a power flow solution iff (v, p_c) solves the FPPF

$$v = f(v, p_c) \triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\,u(v, p_c)\,,$$

$$\mathbb{0}_c = C^{\mathsf{T}}\mathbf{arcsin}(\psi(v, p_c))\,.$$

where

$$u(v, p_c) \triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi}$$
$$\psi(v, p_c) = [h(v)]^{-1}\left(A^{\mathsf{T}}\mathsf{L}^{\dagger}P + \mathsf{D}^{-1}Cp_c\right)\,,$$

with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)\,.$

Fixed-Point Power Flow: Meshed Networks

(θ, V_L) is a power flow solution iff (v, p_c) solves the FPPF

$$
\begin{aligned}
v = f(v, p_c) &\triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n \\
&\qquad + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\, u(v, p_c)\,, \\
\mathbb{0}_c &= C^{\mathsf{T}}\mathbf{arcsin}(\psi(v, p_c))\,.
\end{aligned}
$$

where

$$
\begin{aligned}
u(v, p_c) &\triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi} \\
\psi(v, p_c) &= [h(v)]^{-1}\left(A^{\mathsf{T}}\mathsf{L}^{\dagger}P + \mathsf{D}^{-1}Cp_c\right)\,,
\end{aligned}
$$

with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)$.

Fixed-Point Power Flow: Meshed Networks

(θ, V_L) is a power flow solution iff (v, p_c) solves the FPPF

$$
\begin{aligned}
v = f(v, p_c) &\triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n \\
&\qquad + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\, u(v, p_c)\,, \\
\mathbb{0}_c &= C^{\mathsf{T}}\mathbf{arcsin}(\psi(v, p_c))\,.
\end{aligned}
$$

where

$$
\begin{aligned}
u(v, p_c) &\triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi} \\
\psi(v, p_c) &= [h(v)]^{-1}\left(A^{\mathsf{T}}\mathsf{L}^{\dagger}P + \mathsf{D}^{-1}Cp_c\right)\,,
\end{aligned}
$$

with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)$.

Fixed-Point Power Flow: Meshed Networks

(θ, V_L) is a power flow solution iff (v, p_c) solves the FPPF

$$v = f(v, p_c) \triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\,u(v, p_c)\,,$$

$$\mathbb{0}_c = C^{\mathsf{T}}\mathbf{arcsin}(\psi(v, p_c))\,.$$

where

$$u(v, p_c) \triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi}$$

$$\psi(v, p_c) = [h(v)]^{-1}\left(A^{\mathsf{T}}\mathsf{L}^{\dagger}P + \mathsf{D}^{-1}Cp_c\right)\,,$$

with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)\,.$

An approximate power flow solution

- The model says $v = f(v, p_c)$, and $\mathbf{sin}(A^\mathsf{T}\theta) = \psi(v, p_c)$.
- By construction, when $P = Q_L = \mathbb{0}$, a solution is

$$v = \mathbb{1}_n, \quad p_c = \mathbb{0}_c\,, \quad A^\mathsf{T}\theta = \mathbb{0}_{|\mathcal{E}|}\,.$$

- **Taylor expand** FPPF model around this solution

$$\begin{aligned} A^\mathsf{T}\theta_{\text{approx}} &= A^\mathsf{T}L^\dagger P \\ v_{\text{approx}} &\simeq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}Q_L + \frac{1}{8}\mathsf{S}^{-1}|A|_L\mathsf{D}[A^\mathsf{T}\mathsf{L}^\dagger P]A^\mathsf{T}\mathsf{L}^\dagger P \\ p_{c,\text{approx}} &= \mathbb{0} \end{aligned}$$

An approximate power flow solution

- The model says $v = f(v, p_c)$, and $\mathbf{sin}(A^\mathsf{T}\theta) = \psi(v, p_c)$.
- By construction, when $P = Q_L = \mathbb{0}$, a solution is

$$v = \mathbb{1}_n, \quad p_c = \mathbb{0}_c\,, \quad A^\mathsf{T}\theta = \mathbb{0}_{|\mathcal{E}|}\,.$$

- **Taylor expand** FPPF model around this solution

$$\begin{aligned} A^\mathsf{T}\theta_{\mathrm{approx}} &= A^\mathsf{T}L^\dagger P \\ v_{\mathrm{approx}} &\simeq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}Q_L + \frac{1}{8}\mathsf{S}^{-1}|A|_L\mathsf{D}[A^\mathsf{T}\mathsf{L}^\dagger P]A^\mathsf{T}\mathsf{L}^\dagger P \\ p_{c,\mathrm{approx}} &= \mathbb{0} \end{aligned}$$

An approximate power flow solution

- The model says $v = f(v, p_c)$, and $\mathbf{sin}(A^\mathsf{T}\theta) = \psi(v, p_c)$.
- By construction, when $P = Q_L = \mathbb{0}$, a solution is

$$v = \mathbb{1}_n, \quad p_c = \mathbb{0}_c\,, \quad A^\mathsf{T}\theta = \mathbb{0}_{|\mathcal{E}|}\,.$$

- **Taylor expand** FPPF model around this solution

$$\begin{aligned} A^\mathsf{T}\theta_{\mathrm{approx}} &= A^\mathsf{T} L^\dagger P \\ v_{\mathrm{approx}} &\simeq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1} Q_L + \frac{1}{8}\mathsf{S}^{-1}|A|_L \mathsf{D}[A^\mathsf{T}\mathsf{L}^\dagger P] A^\mathsf{T}\mathsf{L}^\dagger P \\ p_{c,\mathrm{approx}} &= \mathbb{0} \end{aligned}$$

An approximate power flow solution

- The model says $v = f(v, p_c)$, and $\mathbf{sin}(A^\mathsf{T}\theta) = \psi(v, p_c)$.
- By construction, when $P = Q_L = \mathbb{0}$, a solution is

$$v = \mathbb{1}_n, \quad p_c = \mathbb{0}_c\,, \quad A^\mathsf{T}\theta = \mathbb{0}_{|\mathcal{E}|}\,.$$

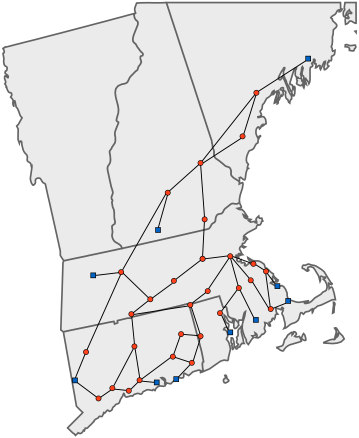
- **Taylor expand** FPPF model around this solution

$$\begin{aligned} A^\mathsf{T}\theta_{\text{approx}} &= A^\mathsf{T} L^\dagger P \\ v_{\text{approx}} &\simeq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}Q_L + \frac{1}{8}\mathsf{S}^{-1}|A|_L\mathsf{D}[A^\mathsf{T}\mathsf{L}^\dagger P]A^\mathsf{T}\mathsf{L}^\dagger P \\ p_{c,\text{approx}} &= \mathbb{0} \end{aligned}$$

An approximate power flow solution

$$A^{\mathsf{T}}\theta_{\text{approx}} = A^{\mathsf{T}}L^{\dagger}P$$

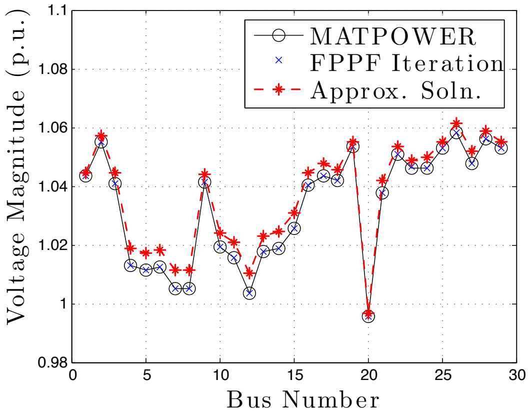
$$v_{\text{approx}} \simeq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}Q_L + \frac{1}{8}\mathsf{S}^{-1}|A|_L\mathsf{D}[A^{\mathsf{T}}\mathsf{L}^{\dagger}P]A^{\mathsf{T}}\mathsf{L}^{\dagger}P$$



An approximate power flow solution

$$A^{\mathsf{T}}\theta_{\text{approx}} = A^{\mathsf{T}} L^{\dagger} P$$

$$v_{\text{approx}} \simeq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1} Q_L + \frac{1}{8}\mathsf{S}^{-1}|A|_L \mathsf{D}[A^{\mathsf{T}}\mathsf{L}^{\dagger}P]A^{\mathsf{T}}\mathsf{L}^{\dagger}P$$



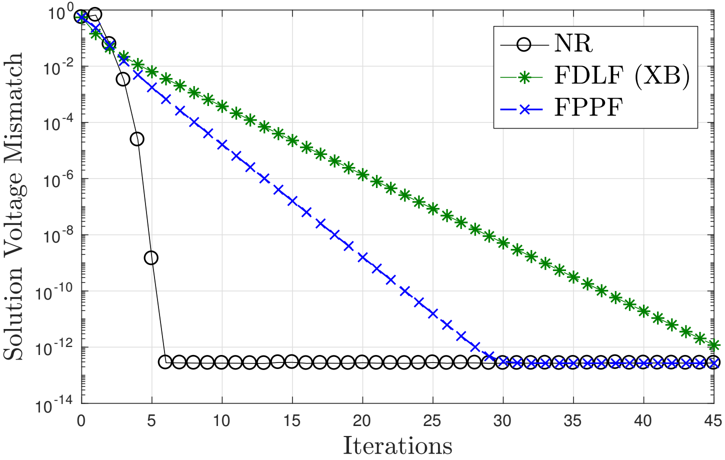
Numerical results I

$$\delta_{\max} = \|v - v_{\text{approx}}\|_\infty \,, \quad \delta_{\text{avg}} = \frac{1}{n}\|v - v_{\text{approx}}\|_1$$

	Base Load			High Load	
Test Case	FPPF Iters.	$\delta_{\max}$ (p.u.)	δ_{avg} (p.u.)	FPPF Iters.	$\delta_{\max}$ (p.u.)
New England 39	4	0.006	0.004	8	0.086
57 bus system	5	0.011	0.003	8	0.118
RTS '96 (3 area)	4	0.003	0.001	8	0.084
118 bus system	3	0.001	0.000	7	0.054
300 bus system	6	0.022	0.004	8	0.059
PEGASE 1,354	5	0.011	0.001	8	0.070
Polish 2,383 wp	4	0.003	0.000	8	0.078
PEGASE 2,869	5	0.015	0.002	8	0.098
PEGASE 9,241	6	0.063	0.003	9	0.133

Numerical results II – convergence rates

- IEEE 300 bus system under heavy loading



Numerical results III – sensitivity to initialization

- perturb voltage magnitude initialization randomly
- IEEE 118 bus system, base case

IC Spread (α)	NR	FDLF	FPPF
0.05	0.98	0.98	1.00
0.10	0.53	0.53	1.00
0.15	0.18	0.18	1.00
0.2	0.03	0.03	1.00
0.3	0.00	0.00	1.00
0.5	0.00	0.00	1.00
0.7	0.00	0.00	0.99
0.9	0.00	0.00	0.99

- extreme insensitivity to initialization (contraction)

Fixed-Point Power Flow: Radial Networks

(θ, V_L) is a power flow solution iff v is a fixed point of

$$f(v) \triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\,u(v)\,,$$

where

$$u(v) \triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi}$$
$$\psi(v) = [h(v)]^{-1}\mathsf{D}^{-1}p$$
$$p = (A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}P$$

with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)$.

Fixed-Point Power Flow: Radial Networks

(θ, V_L) is a power flow solution iff v is a fixed point of

$$f(v) \triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L\mathsf{D}\,[h(v)]\,u(v)\,,$$

where

$$u(v) \triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi}$$
$$\psi(v) = [h(v)]^{-1}\mathsf{D}^{-1}p$$
$$p = (A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}P$$

with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)$.

Fixed-Point Power Flow: Radial Networks

(θ, V_L) is a power flow solution iff v is a fixed point of

$$f(v) \triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\,[h(v)]\,u(v)\,,$$

where

$$u(v) \triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi}$$

$$\psi(v) = [h(v)]^{-1}\mathsf{D}^{-1}p$$

$$p = (A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}P$$

with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)\,.$

Fixed-Point Power Flow: Radial Networks

(θ, V_L) is a power flow solution iff v is a fixed point of

$$f(v) \triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L\mathsf{D}\left[h(v)\right]u(v)\,,$$

where

$$u(v) \triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi}$$

$$\psi(v) = [h(v)]^{-1}\mathsf{D}^{-1}p$$

$$p = (A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}P$$

with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)$.

Fixed-Point Power Flow: Radial Networks

(θ, V_L) is a power flow solution iff v is a fixed point of

$$f(v) \triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L \mathsf{D}\left[h(v)\right] u(v)\,,$$

where

$$u(v) \triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi}$$
$$\psi(v) = [h(v)]^{-1}\mathsf{D}^{-1}p$$
$$p = (A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}P$$

with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)$.

Fixed-Point Power Flow: Radial Networks

(θ, V_L) is a power flow solution iff v is a fixed point of

$$f(v) \triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L\mathsf{D}\,[h(v)]\,u(v)\,,$$

where

$$u(v) \triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi}$$
$$\psi(v) = [h(v)]^{-1}\mathsf{D}^{-1}p$$
$$p = (A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}P$$

with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)\,.$

Fixed-Point Power Flow: Radial Networks

(θ, V_L) is a power flow solution iff v is a fixed point of

$$f(v) \triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L\mathsf{D}\,[h(v)]\,u(v)\,,$$

where

$$u(v) \triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi}$$
$$\psi(v) = [h(v)]^{-1}\mathsf{D}^{-1}p$$
$$p = (A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}P$$

with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)\,.$

Fixed-Point Power Flow: Radial Networks

(θ, V_L) is a power flow solution iff v is a fixed point of

$$f(v) \triangleq \mathbb{1}_n - \frac{1}{4}\mathsf{S}^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4}\mathsf{S}^{-1}[v]^{-1}|A|_L\mathsf{D}\,[h(v)]\,u(v)\,,$$

where

$$u(v) \triangleq \mathbb{1} - \sqrt{\mathbb{1} - [\psi]\psi}$$
$$\psi(v) = [h(v)]^{-1}\mathsf{D}^{-1}p$$
$$p = (A^{\mathsf{T}}A)^{-1}A^{\mathsf{T}}P$$

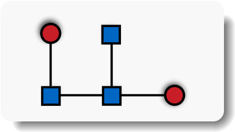
with the phase angles $A^{\mathsf{T}}\theta = \mathbf{arcsin}(\psi)$.

On what invariant set is f a **contraction**?

Solvability results for different tree topologies

Solvability results for different tree topologies

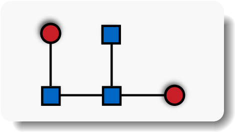
PQ buses have one PV bus neighbor



Sufficient + Necessary
Existence + Uniqueness

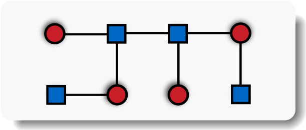
Solvability results for different tree topologies

PQ buses have one PV bus neighbor



Sufficient + Necessary
Existence + Uniqueness

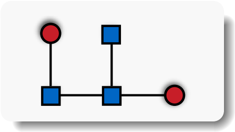
PQ buses have many PV bus neighbors



Sufficient + Tight
Existence + Uniqueness

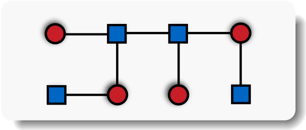
Solvability results for different tree topologies

PQ buses have one PV bus neighbor



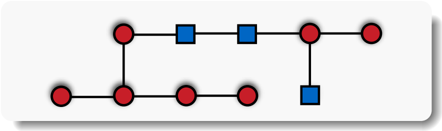
Sufficient + Necessary
Existence + Uniqueness

PQ buses have many PV bus neighbors



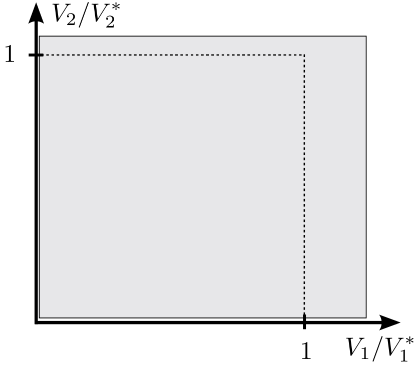
Sufficient + Tight
Existence + Uniqueness

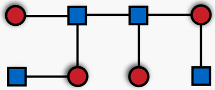
General interconnections



Sufficient
Existence

Partitioning of voltage space





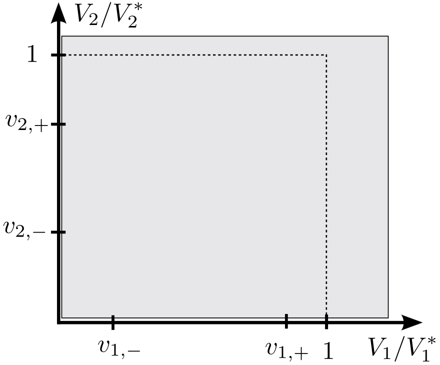
$$\max_{i\in\mathcal{N}_L} \quad \Delta_i + 4\Gamma_i^2 < 1$$

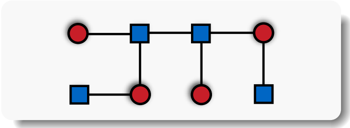
$$\max_{(i,j)\in\mathcal{E}^{gg}} \; \Gamma_{ij} < 1\,,$$

$$v_{i,\pm} \triangleq \sqrt{\frac{1}{2}\left(1 - \frac{\Delta_i}{2} \pm \sqrt{1-(\Delta_i + 4\Gamma_i^2)}\right)}$$

$$v_{i,+}^2 - v_{i,-}^2 = \sqrt{1-(\Delta_i + 4\Gamma_i^2)}\,.$$

Partitioning of voltage space





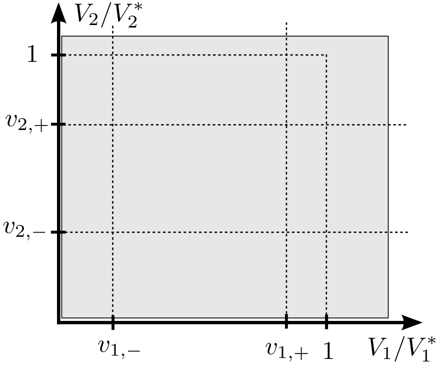
$$\max_{i \in \mathcal{N}_L} \; \Delta_i + 4\Gamma_i^2 < 1$$

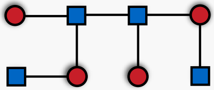
$$\max_{(i,j) \in \mathcal{E}^{gg}} \; \Gamma_{ij} < 1\,,$$

$$v_{i,\pm} \triangleq \sqrt{\frac{1}{2}\left(1 - \frac{\Delta_i}{2} \pm \sqrt{1 - (\Delta_i + 4\Gamma_i^2)}\right)}$$

$$v_{i,+}^2 - v_{i,-}^2 = \sqrt{1 - (\Delta_i + 4\Gamma_i^2)}\,.$$

Partitioning of voltage space





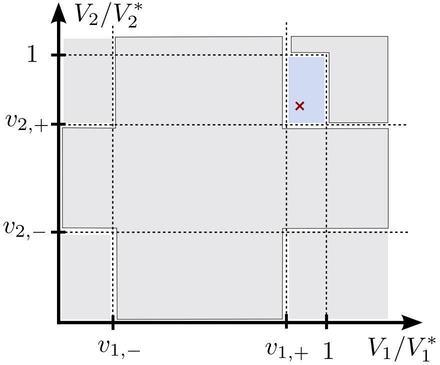
$$\max_{i \in \mathcal{N}_L} \; \Delta_i + 4\Gamma_i^2 < 1$$

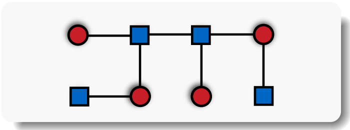
$$\max_{(i,j) \in \mathcal{E}^{gg}} \; \Gamma_{ij} < 1 \,,$$

$$v_{i,\pm} \triangleq \sqrt{\frac{1}{2}\left(1 - \frac{\Delta_i}{2} \pm \sqrt{1 - (\Delta_i + 4\Gamma_i^2)}\right)}$$

$$v_{i,+}^2 - v_{i,-}^2 = \sqrt{1 - (\Delta_i + 4\Gamma_i^2)} \,.$$

Partitioning of voltage space





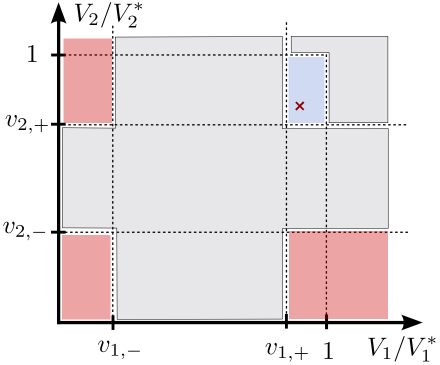
$$\max_{i \in \mathcal{N}_L} \; \Delta_i + 4\Gamma_i^2 < 1$$

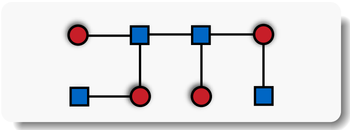
$$\max_{(i,j) \in \mathcal{E}^{gg}} \; \Gamma_{ij} < 1\,,$$

$$v_{i,\pm} \triangleq \sqrt{\frac{1}{2}\left(1 - \frac{\Delta_i}{2} \pm \sqrt{1 - (\Delta_i + 4\Gamma_i^2)}\right)}$$

$$v_{i,+}^2 - v_{i,-}^2 = \sqrt{1 - (\Delta_i + 4\Gamma_i^2)}\,.$$

Partitioning of voltage space





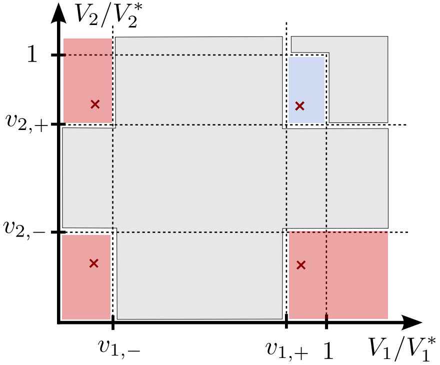
$$\max_{i \in \mathcal{N}_L} \; \Delta_i + 4\Gamma_i^2 < 1$$

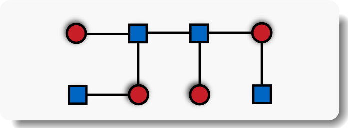
$$\max_{(i,j) \in \mathcal{E}^{gg}} \; \Gamma_{ij} < 1 \,,$$

$$v_{i,\pm} \triangleq \sqrt{\frac{1}{2}\left(1 - \frac{\Delta_i}{2} \pm \sqrt{1 - (\Delta_i + 4\Gamma_i^2)}\right)}$$

$$v_{i,+}^2 - v_{i,-}^2 = \sqrt{1 - (\Delta_i + 4\Gamma_i^2)} \,.$$

Partitioning of voltage space





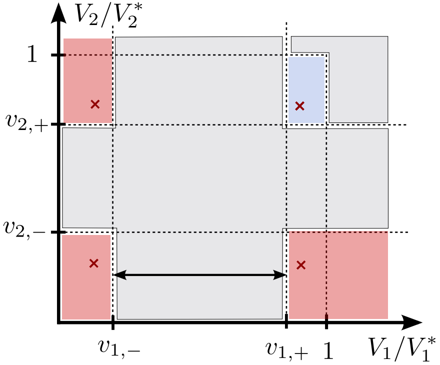
$$\max_{i \in \mathcal{N}_L} \; \Delta_i + 4\Gamma_i^2 < 1$$

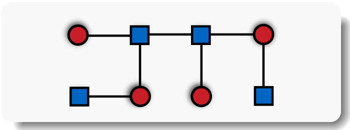
$$\max_{(i,j) \in \mathcal{E}^{gg}} \; \Gamma_{ij} < 1\,,$$

$$v_{i,\pm} \triangleq \sqrt{\frac{1}{2}\left(1 - \frac{\Delta_i}{2} \pm \sqrt{1 - (\Delta_i + 4\Gamma_i^2)}\right)}$$

$$v_{i,+}^2 - v_{i,-}^2 = \sqrt{1 - (\Delta_i + 4\Gamma_i^2)}\,.$$

Partitioning of voltage space





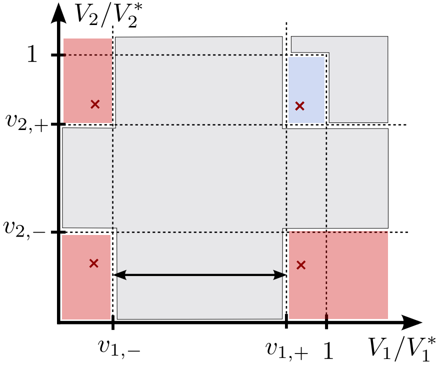
$$\max_{i \in \mathcal{N}_L} \Delta_i + 4\Gamma_i^2 < 1$$

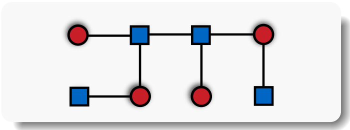
$$\max_{(i,j) \in \mathcal{E}^{gg}} \Gamma_{ij} < 1\,,$$

$$v_{i,\pm} \triangleq \sqrt{\frac{1}{2}\left(1 - \frac{\Delta_i}{2} \pm \sqrt{1 - (\Delta_i + 4\Gamma_i^2)}\right)}$$

$$v_{i,+}^2 - v_{i,-}^2 = \sqrt{1 - (\Delta_i + 4\Gamma_i^2)}\,.$$

Partitioning of voltage space





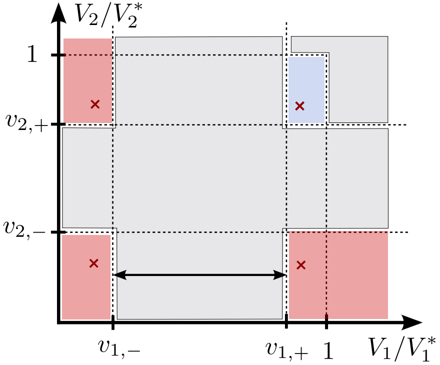
$$\max_{i\in\mathcal{N}_L} \; \Delta_i + 4\Gamma_i^2 < 1$$

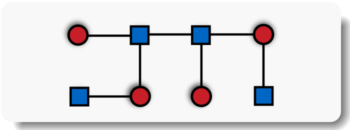
$$\max_{(i,j)\in\mathcal{E}^{gg}} \; \Gamma_{ij} < 1\,,$$

$$v_{i,\pm} \triangleq \sqrt{\frac{1}{2}\left(1 - \frac{\Delta_i}{2} \pm \sqrt{1-(\Delta_i + 4\Gamma_i^2)}\right)}$$

$$v_{i,+}^2 - v_{i,-}^2 = \sqrt{1-(\Delta_i + 4\Gamma_i^2)}\,.$$

Partitioning of voltage space

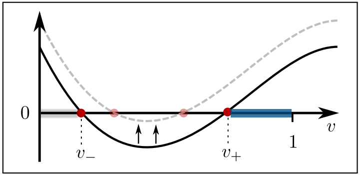




$$\max_{i\in\mathcal{N}_L} \; \Delta_i + 4\Gamma_i^2 < 1$$

$$\max_{(i,j)\in\mathcal{E}^{gg}} \; \Gamma_{ij} < 1\,,$$

$$v_{i,\pm} \triangleq \sqrt{\frac{1}{2}\left(1 - \frac{\Delta_i}{2} \pm \sqrt{1-(\Delta_i + 4\Gamma_i^2)}\right)}$$



$$v_{i,+}^2 - v_{i,-}^2 = \sqrt{1-(\Delta_i + 4\Gamma_i^2)}\,.$$

Summary

Framework for studying **Lossless Power Flow**:

1. Fixed-Point Power Flow
2. Approximate solution

A Theory of Solvability for Lossless Power Flow Equations — Part I: Fixed-Point Power Flow John W. Simpson-Porco, *Member, IEEE*

New **conditions for power flow solvability**:

3. Contractive iteration
4. Existence/uniqueness
5. Generalizes known results

What's unresolved?

1. **Lossless meshed** case
2. **Lossy meshed** case; algorithms in Chen & JWSP 2022, but theory is hard

Summary

Framework for studying **Lossless Power Flow**:

1. Fixed-Point Power Flow
2. Approximate solution

A Theory of Solvability for Lossless Power Flow Equations — Part I: Fixed-Point Power Flow
John W. Simpson-Porco, *Member, IEEE*

New **conditions for power flow solvability**:

3. Contractive iteration
4. Existence/uniqueness
5. Generalizes known results

A Theory of Solvability for Lossless Power Flow Equations — Part II: Existence and Uniqueness
John W. Simpson-Porco, *Member, IEEE*

What's unresolved?

1. **Lossless meshed** case
2. **Lossy meshed** case; algorithms in Chen & JWSP 2022, but theory is hard

Summary

Framework for studying **Lossless Power Flow**:

1. Fixed-Point Power Flow
2. Approximate solution

A Theory of Solvability for Lossless Power Flow Equations — Part I: Fixed-Point Power Flow

John W. Simpson-Porco, *Member, IEEE*

New **conditions for power flow solvability**:

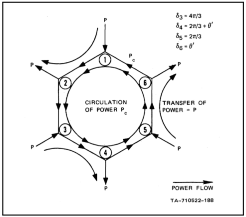
3. Contractive iteration
4. Existence/uniqueness
5. Generalizes known results

A Theory of Solvability for Lossless Power Flow Equations — Part II: Existence and Uniqueness

John W. Simpson-Porco, *Member, IEEE*

What's unresolved?

1. **Lossless meshed** case
2. **Lossy meshed** case; algorithms in Chen & JWSP 2022, but theory is hard



Final Thoughts

1. Power engineers have incredible intuitive insight into how the grid works; *put in the effort to work with them*.
2. Opportunity to embed practitioner knowledge and intuition within advanced optimization methods like moment-SOS hierarchy
3. Decades of literature on power flow theory, lots of important insights, still many poorly understood aspects

Final Thoughts

1. Power engineers have incredible intuitive insight into how the grid works; *put in the effort to work with them*.
2. Opportunity to embed practitioner knowledge and intuition within advanced optimization methods like moment-SOS hierarchy
3. Decades of literature on power flow theory, lots of important insights, still many poorly understood aspects

Final Thoughts

1. Power engineers have incredible intuitive insight into how the grid works; *put in the effort to work with them*.
2. Opportunity to embed practitioner knowledge and intuition within advanced optimization methods like moment-SOS hierarchy
3. Decades of literature on power flow theory, lots of important insights, still many poorly understood aspects

Questions

The Edward S. Rogers Sr. Department of Electrical & Computer Engineering
UNIVERSITY OF TORONTO

https://www.control.utoronto.ca/~jwsimpson/
jwsimpson@ece.utoronto.ca