Low-Gain Stability of Projected Integral Control for Input-Constrained Discrete-Time Nonlinear Systems

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Group Meeting, Summer 2022

Motivation & Contributions

Integral Control:

- required for robust output regulation w/ constant exog. signals
- often in practice, process is complex/uncertain but **internally stable**
- in this case, **low-gain integral control** is effective & practical

Actuator Limits:

- degrade perf., **may prevent** perfect asymptotic output regulation
- Implicit Approach: Anti-windup, reference governors/modifiers
- Explicit Approach: RHC, **misc. nonlinear methods**

Contribution: A constrained integral controller w/ proof of low-gain stability for nonlinear systems.

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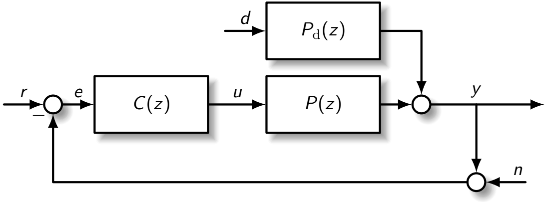

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- d, r = **constant**

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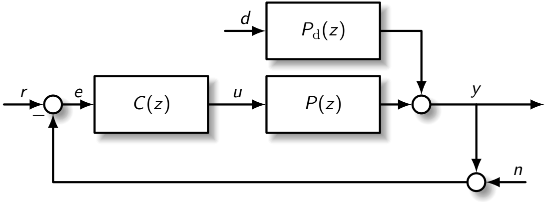

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EDWARD J. DAVISON, MEMBER, IEEE

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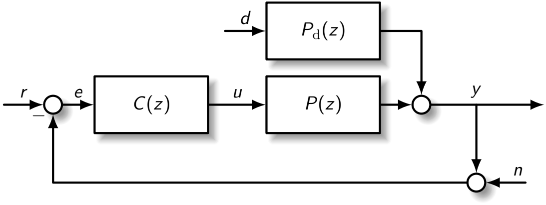

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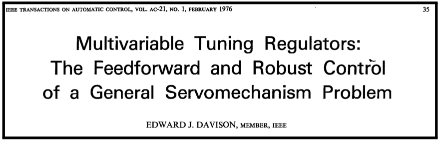
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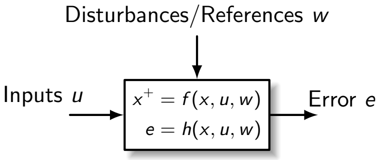
Low-Gain Stability (Davison '76): If $-P(1)K$ is Hurwitz stable, then $\exists T_i^\star > 0$ s.t. $\forall T_i \in (T_i^\star, \infty)$ the closed-loop system is exp. stable and $\lim_{k\to\infty} e_k = 0$.

Model Assumptions



- DT nonlinear system
- sample period T_s
- $u \in \mathbb{R}^m$, $e \in \mathbb{R}^p$
- $p \leq m$

Assumptions: There exist domains $\mathcal{X}, \mathcal{U}, \mathcal{W}$ such that

1. **Model Regularity:** f, h cts, f is C^1 in (x, u), $f, h, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}$ are Lipschitz cts. in (x, u) uniformly in $w \in \mathcal{W}$
2. **Steady-State:** $\exists$ a $\mathcal{C}^1$ map $\pi_x : \mathcal{U} \times \mathcal{W} \to \mathcal{X}$ which is Lipschitz on $\mathcal{U} \times \mathcal{W}$ and satisfies $\pi_x(u, w) = f(\pi_x(u, w), u, w)$ for all (u, w)
3. **Stability:** the equilibrium $\pi_x(u, w)$ is locally exponentially stable, uniformly in the inputs (u, w)

Equilibrium I/O Map: $\pi(\bar{u}, w) \triangleq h(\pi_x(\bar{u}, w), \bar{u}, w).$

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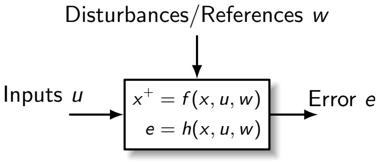

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Constrained Error-Zeroing Specification

- Let $\mathcal{C} \subseteq \mathcal{U}$ be a closed, convex input constraint set
- If $u_k = K\eta_k$, then integral state η must live in

$$\Gamma \triangleq \{\eta \in \mathbb{R}^p \;:\; K\eta \in \mathcal{C}\} \qquad \text{(also closed, convex)}$$

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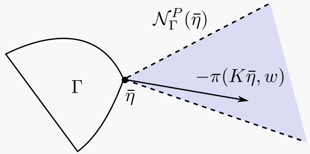
Find $\bar{\eta}$ s.t.

$$\forall \eta \in \Gamma : \quad \langle \bar{e}, \eta - \bar{\eta} \rangle_P \geq 0$$

where $\bar{e} = \pi(K\bar{\eta}, w)$



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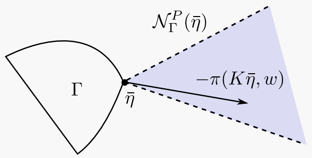

- spec. is a **variational ineq.**; could arise from minimization problem
- if $\bar{\eta} \in \mathrm{interior}(\Gamma)$, then spec. is simply $\bar{e} = 0$

Damped-Projected Integral Control

- Projection operator onto closed, convex set Γ in P-norm:

$$\mathrm{Proj}_\Gamma^P : \mathbb{R}^m \to \Gamma, \quad \mathrm{Proj}_\Gamma^P(\eta) = \underset{\nu \in \Gamma}{\operatorname{argmin}} \; \|\eta - \nu\|_P$$

The proposed **damped-projected integral controller** is

$$\begin{aligned} \eta_{k+1} &= (1-\lambda)\eta_k + \lambda \mathrm{Proj}_\Gamma^P(\eta_k - \tfrac{T_\mathrm{s}}{T_\mathrm{i}} e_k) \\ u_k &= K\eta_k \end{aligned}$$

where $\lambda \in (0, 1)$ is the damping parameter

1. **Equilibrium:** closed-loop equil. $\Longleftrightarrow$ error-zeroing spec.
2. **Constraint Satisfaction:** $u_k \in \mathcal{C}$ for all $k \geq 0$; wind-up impossible
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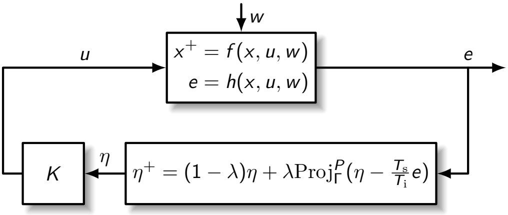

- **Assume**: $\exists P \succ 0$ and constants $\mu, L > 0$ s.t. $\eta \mapsto \pi(K\eta, w)$ is μ-strongly monotone and L-Lipschitz continuous on Γ with respect to $\langle \cdot, \cdot \rangle_P$, uniformly in w
- Pick any integral time $T_{\mathrm{i}} \in (T_{\mathrm{s}} L^2/2\mu, \infty)$

Then $\exists \lambda^\star \in (0,1)$ such that $\forall \lambda \in (0, \lambda^\star), \forall w \in \mathcal{W}$:

1. the C.L.S. possesses an exp. stable equil. point $(\bar{x}, \bar{\eta}) \in \mathcal{X} \times \Gamma$
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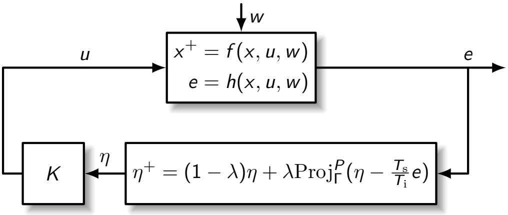

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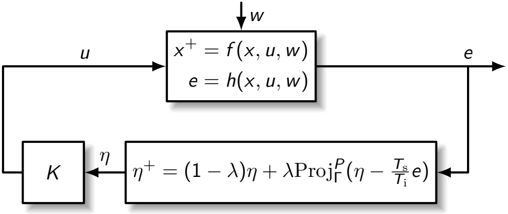

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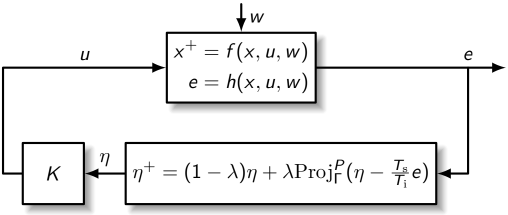

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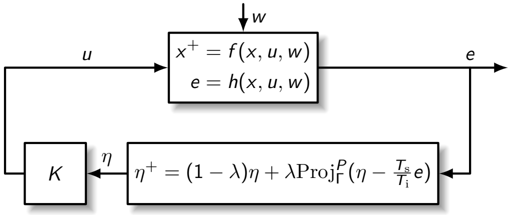

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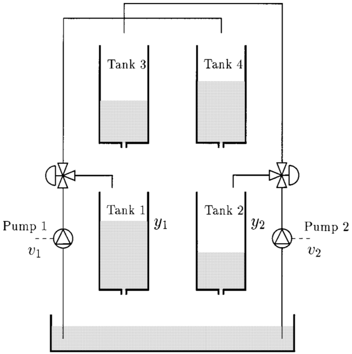
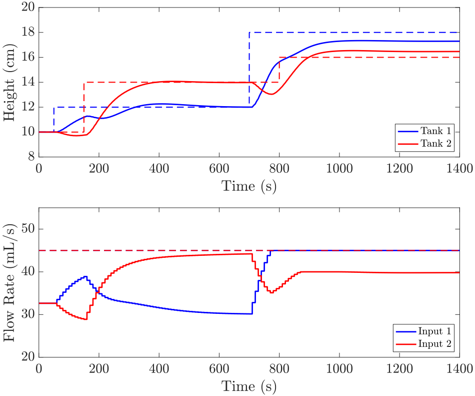
Example: Four-Tank Process (Johannson, TCST, 2000)



- state $h \in \mathbb{R}^4_{>0}$ = water levels in tanks
- input $u \in \mathbb{R}^2_{\geq 0}$ = pump flow rates w/ constraints $u_1, u_2 \in [0, 45]$, $u_1 + u_2 \leq 85$
- error $e = (h_1 - r_1, h_2 - r_2) \in \mathbb{R}^2$



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