Low-Gain Stabilizers for Linear-Convex Optimal Steady-State Control

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UNIVERSITY OF TORONTO

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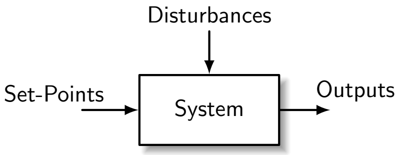
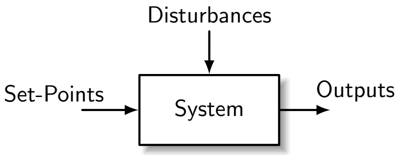
Offline vs. Online Optimization



Goal: Real-time regulation of system to an **optimal constrained operating point**.

Offline vs. Online Optimization



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Example: Optimal dispatch of generation in AC electric power systems, subject to frequency and voltage regulation constraints.

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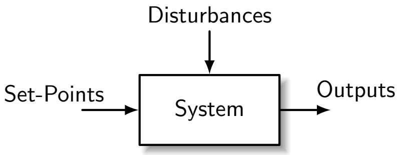

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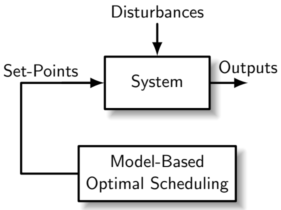
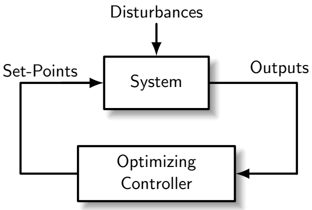


Online Optimization



Optimal Steady-State Control

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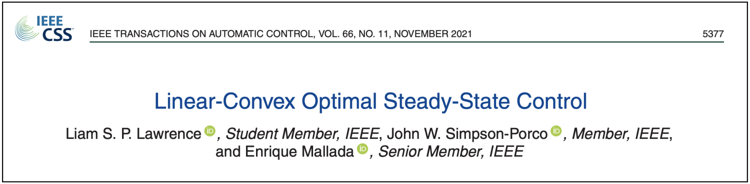
Linear-Convex Optimal Steady-State Control

Liam S. P. Lawrence, *Student Member, IEEE*, John W. Simpson-Porco, *Member, IEEE*, and Enrique Mallada, *Senior Member, IEEE*

Optimal Steady-State Control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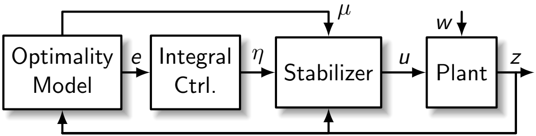
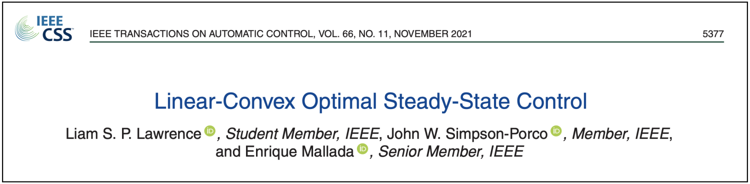


Classical LTI output regulation fused with **optimization**.

Optimal Steady-State Control



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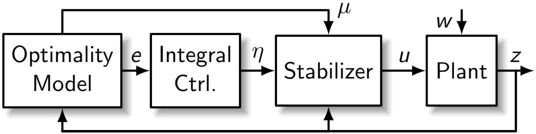
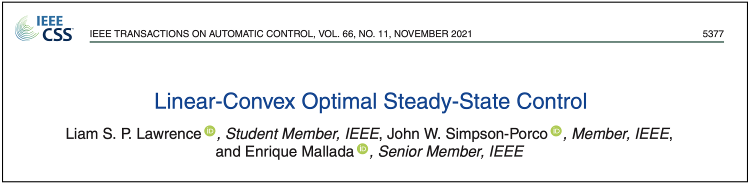


Classical LTI output regulation fused with **optimization**.

- Dynamic **optimality model** encodes **KKT conditions**
- Integral control regulates KKT error to zero; stabilizer stabilizes

Optimal Steady-State Control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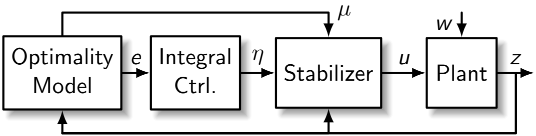

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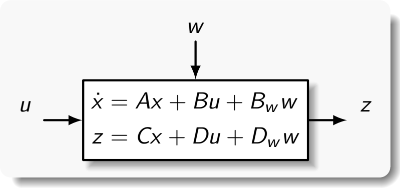
New Contribution: *Low-gain* stabilizer designs for important case of **exponentially stable LTI** plants.

OSS Control Problem Setup



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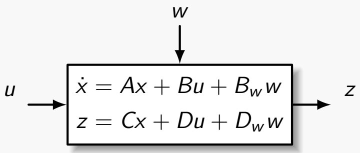
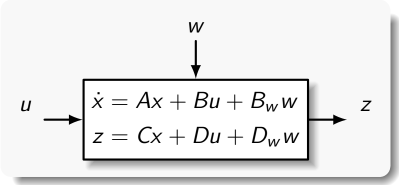


- A Hurwitz, w constant
- $G_u \triangleq -CA^{-1}B + D$
- $G_w \triangleq -CA^{-1}B_w + D_w$

OSS Control Problem Setup



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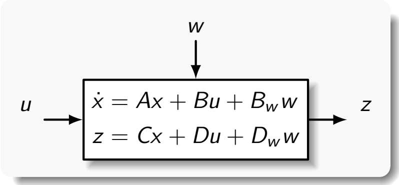

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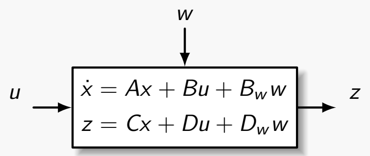

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- $f_0 : \mathcal{U} \to \mathbb{R}$ convex, diff.
- $g_0 : \mathcal{Z} \to \mathbb{R}$ convex, diff.
- strictly feasible, $\exists$ opt. solution
- z and $H_z z + H_u u + H_w w$ measured
- $H_z G_u + H_u$ full row rank

Examples of Optimality Models

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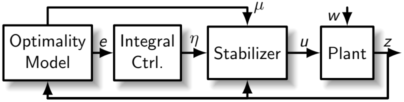

Optimality Model #1

Optimality Model #2

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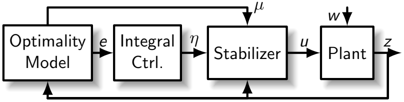

Optimality Models $\approx$ KKT conditions driven by measurements.

Optimality Model #1

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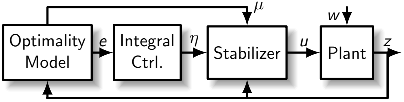

Optimality Models ≈ KKT conditions driven by measurements.

Optimality Model #1

- Dualize engineering constraint

$$\tau\dot{\mu} = H_z z + H_u u + H_w w$$

$$e = \nabla f_0(u) + G_u^{\mathsf{T}} \nabla g_0(z) + (H_z G_u + H_u)^{\mathsf{T}} \mu$$

Optimality Model #2

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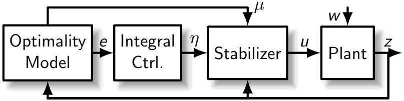

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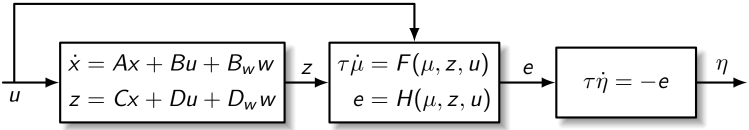
Optimality Model #2

- Parameterize intersection of equality constraints in (z, u):

$$\text{range} \begin{bmatrix} T_z \\ T_u \end{bmatrix} = \text{null} \begin{bmatrix} I_r & -G_u \\ H_z & H_u \end{bmatrix}$$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} T_u^{\mathsf{T}} \nabla f_0(u) + T_z^{\mathsf{T}} \nabla g_0(z) \\ H_z z + H_u u + H_w w \end{bmatrix}.$$

Stabilizers via Low-Gain Integral Control



- We now aim to **stabilize** the cascade.
- We look for simple stabilizers of the form $u = k(\eta, z, \mu)$.
- **Low-gain feedback:** When $\tau \gg 1$, plant is fast compared to controller; replace plant with equilibrium I/O map
- Closed-loop stability determined by **reduced dynamics**

$$\dot{\mu} = F(\mu, G_u u + G_w w, u)$$
$$\dot{\eta} = -H(\mu, G_u u + G_w w, u)$$

JWSP. "*Analysis and Synthesis of Low-Gain Integral Controllers for Nonlinear Systems*". TAC 2021

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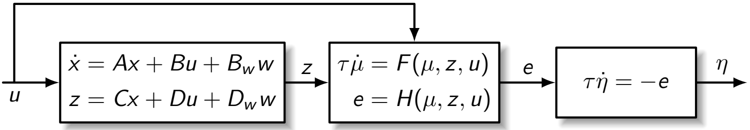

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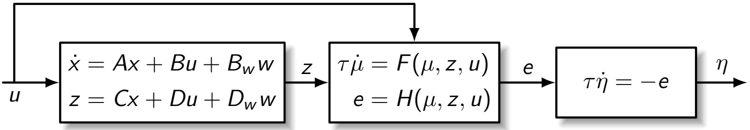

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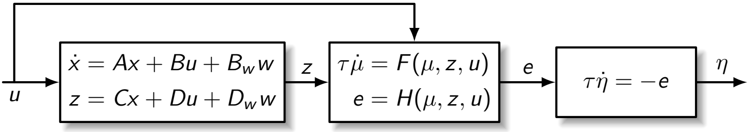

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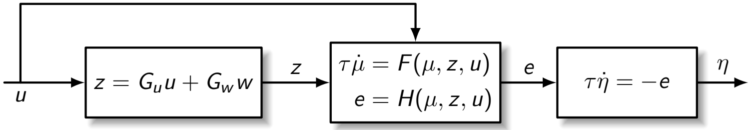

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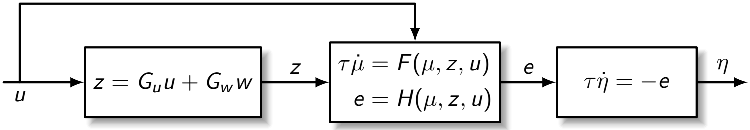

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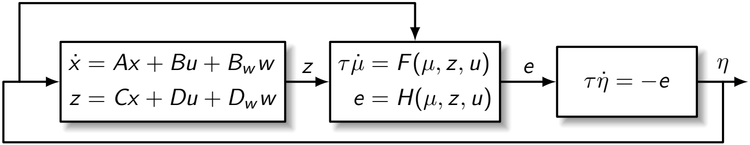
Optimality Model #1: Primal-Dual Stabilizer

$$(PD): \quad \begin{aligned} \tau\dot{\mu} &= H_z z + H_u u + H_w w \\ e &= \nabla f_0(u) + G_u^{\mathsf{T}} \nabla g_0(z) + (H_z G_u + H_u)^{\mathsf{T}} \mu \\ \tau\dot{\eta} &= -e \\ u &= \eta \end{aligned}$$



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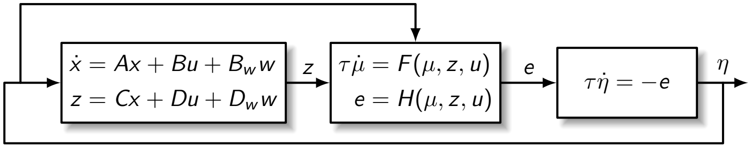

Theorem: If $u \mapsto f_0(u) + g_0(G_u u)$ is strongly convex on $\mathcal{U}$, then there exists $\tau^\star > 0$ such that for all $\tau > \tau^\star$, the controller (PD) solves the OSS control problem.

Optimality Model #1: Inversion-Based Stabilizer

- What if we **directly solve for error-zeroing** u:

$$0 = e = \nabla f_0(u) + G_u^\mathsf{T} \nabla g_0(z) + (H_z G_u + H_u)^\mathsf{T} \mu$$

- When is ∇f_0 invertible? When f_0 is strongly convex and *essentially smooth*, i.e., it blows up at the boundary of $\mathcal{U}$.

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$$(Inv): \quad \begin{aligned} \tau \dot{\mu} &= H_z z + H_u u + H_w w \\ u &= (\nabla f_0)^{-1}(-G_u^{\mathsf{T}} \nabla g_0(z) - (H_z G_u + H_u)^{\mathsf{T}} \mu) \end{aligned}$$

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Optimality Model #2: Two-Loop Stabilizer

Optimality Model:

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} T_u^\mathsf{T} \nabla f_0(u) + T_z^\mathsf{T} \nabla g_0(z) \\ H_z z + H_u u + H_w w \end{bmatrix}$$

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Stabilizer:

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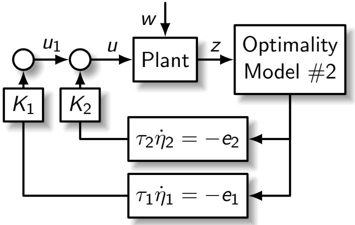

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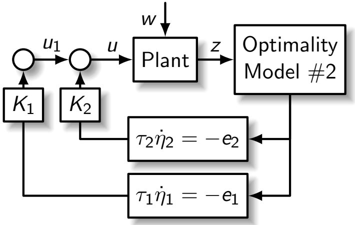

Theorem:

(i) Let $N = H_z G_u + H_u$

(ii) Choose K_2 s.t. $-NK_2$ Hurwitz

(iii) Let $\Pi = I - K_2(NK_2)^{-1}N$

(iv) Choose K_1 s.t. $\Pi K_1 = T_u$

If

$$\xi \mapsto f_0(T_u \xi) + g_0(T_z \xi)$$

is strongly convex, then (TL) solves the OSS control problem for all

$$\tau_1 \gg \tau_2 \gg 0.$$

Optimality Model #2: Robust/Optimal Stabilizer

Optimality Model:

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} T_u^\mathsf{T} \nabla f_0(u) + T_z^\mathsf{T} \nabla g_0(z) \\ H_z z + H_u u + H_w w \end{bmatrix}$$

Stabilizer:

$$(RO): \quad \begin{aligned} \tau_1 \dot{\eta}_1 &= -e_1 \\ \tau_2 \dot{\eta}_2 &= -e_2 \\ u &= K_1 \eta_1 + K_2 \eta_2 \end{aligned}$$

The design of K_1, K_2 can be formulated as a **robust state-feedback design problem** for the **reduced dynamics**.

- ∇f_0 and ∇g_0 are monotone/slope-restricted nonlinearities
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Theorem: An optimal selection of K_1, K_2 can be found by solving an SDP; see paper. Under Lipschitz assumptions, the SDP is always feasible, since (TL) is a feasible point.

Simulation: 30 states, 7 inputs, 5 outputs

- **Objectives:** (z_1, z_2) step tracking, min. control, constraints

$$\begin{aligned} \text{minimize} \quad & \left[\sum_{k=1}^{m} \tfrac{1}{2}\bar{u}_k^2 + \gamma \mathsf{B}(\bar{u}_k)\right] + c\,\mathsf{P}(\bar{z}) \\ \text{subject to} \quad & \bar{z} = G_u \bar{u} + G_w w \\ & 0 = z_i - r_i, \qquad i \in \{1, 2\} \end{aligned}$$

$$\mathsf{B}(u_k) = \text{log barrier fcn.}$$
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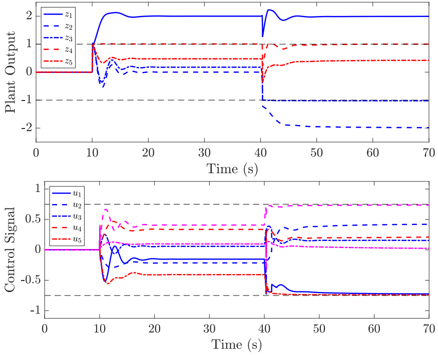
Inversion Stabilizer $\tau = 2$



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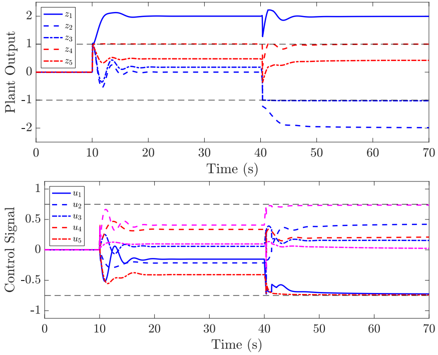
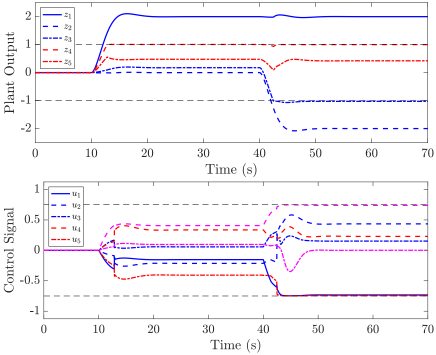
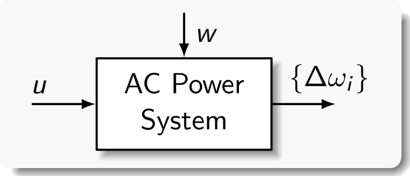


Optimal/Robust Stabilizer



Application: Secondary Frequency Control in Bulk Grid



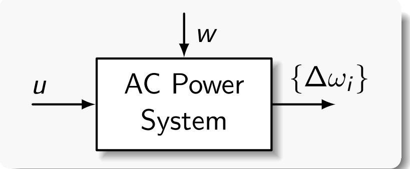
$$
\begin{aligned}
\underset{u}{\text{minimize}} \quad & \sum\nolimits_{i=1}^{n} C_i(\bar{u}_i) \\
\text{subject to} \quad & \Delta\boldsymbol{\omega} = \tfrac{1}{\beta}\mathbb{1}_n(\mathbb{1}_n^{\mathsf{T}}\bar{u} + w) \\
& \Delta\omega_n = 0
\end{aligned}
$$

- **Note:** the way one encodes the frequency regulation constraint can lead to different controller architectures; this is just one possible choice.
- $G_u = \frac{1}{\beta}\mathbb{1}_n\mathbb{1}_n^{\mathsf{T}}$, $G_w = \frac{1}{\beta}\mathbb{1}_n$

Optimality Model #1

Optimality Model #2

Application: Secondary Frequency Control in Bulk Grid



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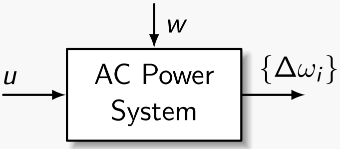

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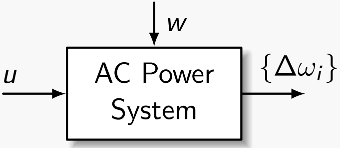
Optimality Model #1

$$
\begin{aligned}
\tau\dot{\mu} &= -\Delta\omega_n \\
e_i &= \nabla C_i(u_i) - \mu
\end{aligned}
$$

Optimality Model #2

Application: Secondary Frequency Control in Bulk Grid



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Optimality Model #1

$$\begin{aligned} \tau\dot{\mu} &= -\Delta\omega_n \\ e_i &= \nabla C_i(u_i) - \mu \end{aligned}$$

Optimality Model #2

$$\text{null}\begin{bmatrix} I_n & -\frac{1}{\beta}\mathbb{1}_n\mathbb{1}_n^{\mathsf{T}} \\ \mathrm{e}_n^{\mathsf{T}} & 0 \end{bmatrix} = \text{range}\begin{bmatrix} 0 \\ L^{\mathsf{T}} \end{bmatrix}$$

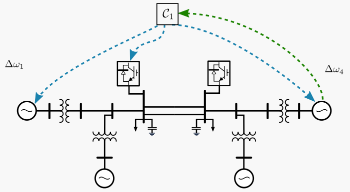
$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} L\nabla C(u) \\ \Delta\omega_n \end{bmatrix}.$$

Application: Secondary Frequency Control in Bulk Grid

Both designs provably stable for large τ

1. Inversion-based controller

$$\tau \dot{\mu} = -\Delta\omega_n$$
$$u_i = (\nabla C_i)^{-1}(\mu)$$



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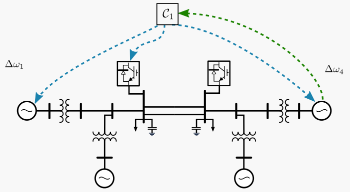
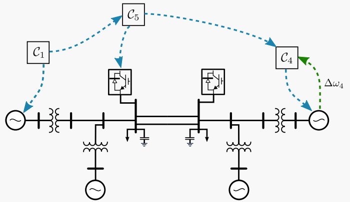


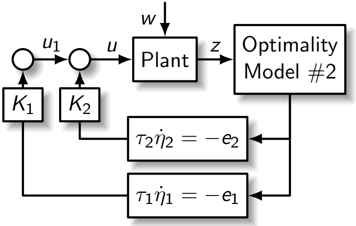
2. Two-loop controller

$$\tau_1 \dot{\eta}_i = -\sum_{j=1}^{n} a_{ij}(\nabla C_i(u_i) - \nabla C_j(u_j))$$
$$\tau_2 \dot{\eta}_n = -\Delta\omega_n$$
$$u_i = \eta_i$$



Conclusions

1. Optimal steady-state (OSS) control is a feedback optimization method
2. Constructive design techniques based on low-gain integral control
3. Only model information required is plant DC gain



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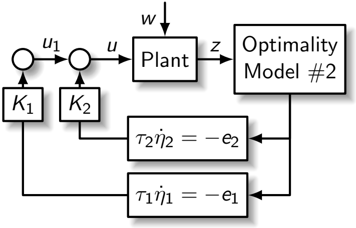
Open directions

1. Nonlinear systems
2. Robustness of the optimization
3. Time-varying disturbances



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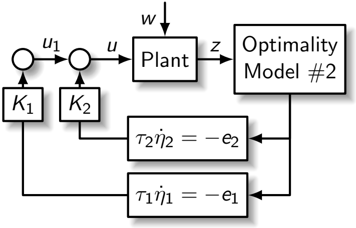

Note: Open PhD position at University of Toronto for Fall 2023, focusing on data-driven control and estimation for energy systems!

Questions

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