Dynamic Performance Considerations in Feedback-Based Optimization and Applications in Power Systems

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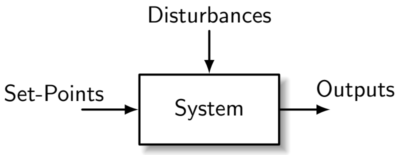
Workshop on Systems Theory of Optimization, Learning, and Control Algorithms, ECC 2026, Reykjavík, Iceland

Offline vs. Feedback-Based Optimization



Goal: Real-time regulation of system to an **optimal constrained equilibrium point**.

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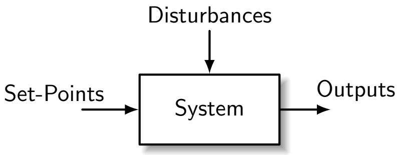

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Example: Optimal dispatch of generation in AC electric power systems, subject to constraints on frequency, voltage, current, power, ...

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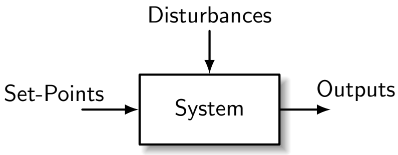

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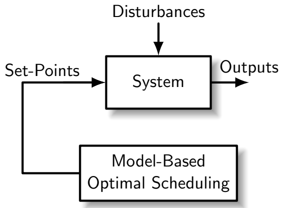
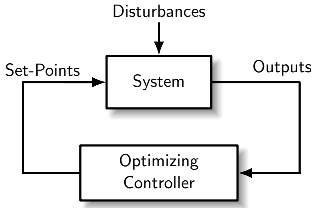
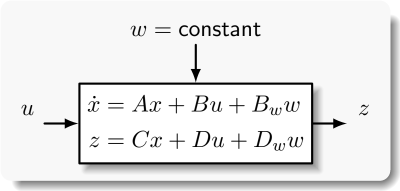


Feedback-Based Optimization



Set-Up: LTI Systems w/ Convex Equilibrium Specifications



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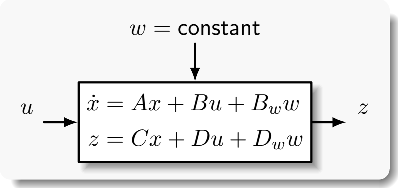

Since A Hurwitz, define

$$G_u \triangleq -CA^{-1}B + D$$
$$G_w \triangleq -CA^{-1}B_w + D_w$$

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$w = \text{constant}$

$$u \longrightarrow \boxed{\begin{aligned} \dot{x} &= Ax + Bu + B_w w \\ z &= Cx + Du + D_w w \end{aligned}} \longrightarrow z$$

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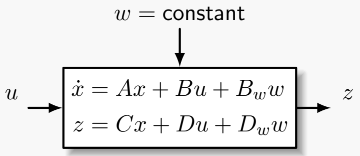

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- $f_0 : \mathcal{U} \to \mathbb{R}$ is C^1, strong. convex
- $g_0 : \mathcal{Z} \to \mathbb{R}$ C^1, convex
- strictly feasible, etc. etc.
- Signals z and $H_z z + H_u u + H_w w$ measurable
- $H_z G_u + H_u$ full row rank

The "Optimization Algorithm" Viewpoint

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2. Obtain controller by "Steady-State Equivalence": $z(t) \approx \bar{z}$

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3. Analyze closed-loop to obtain a stability certificate of form

$$\tau > \tau^\star \triangleq \text{(Function of plant, costs, etc.)}$$

Assessment of "Optimization Algorithm" Viewpoint

Positives:

(i) Only required model info. is DC gain G_u and w unmeasured.
(ii) Unambiguously proven effective in certain applications
(iii) Accessible, conceptually clear, bridges to other communities

Negatives:

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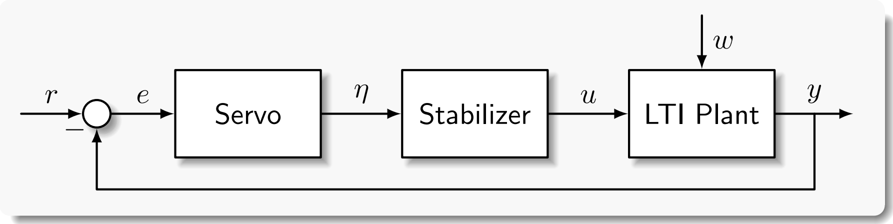
Today: I discuss (1) an alternative paradigm **Optimal Steady-State Control**, and (2) an **Estimator-Enhanced FBO** which removes time-scale separation.

Optimal Steady-State Control

Theory of Tracking and Regulation for LTI Systems



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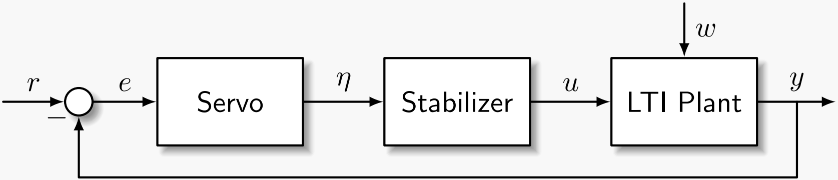

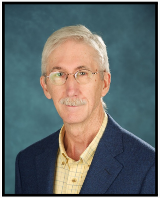

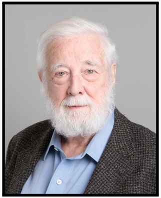

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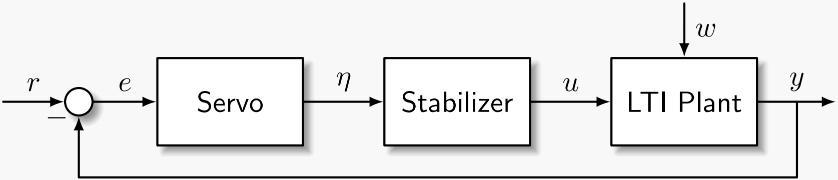

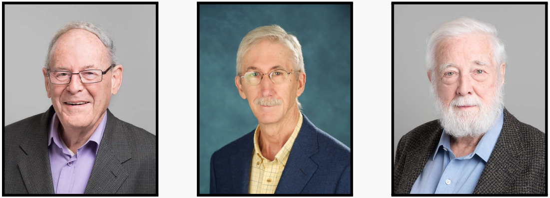

Our goal: incorporate **optimality** and **constraints**.

Optimal Steady-State Control

Optimal Steady-State Control for Linear Time-Invariant Systems

Liam S. P. Lawrence, Zachary E. Nelson, Enrique Mallada, and John W. Simpson-Porco

Linear-Convex Optimal Steady-State Control

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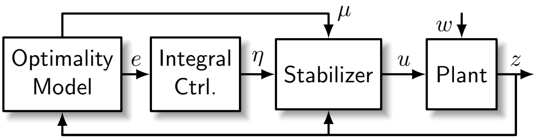

Classical LTI output regulation fused with **optimization**.

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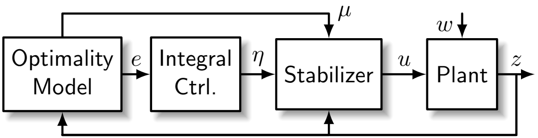

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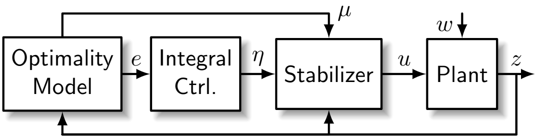

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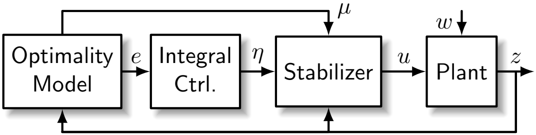

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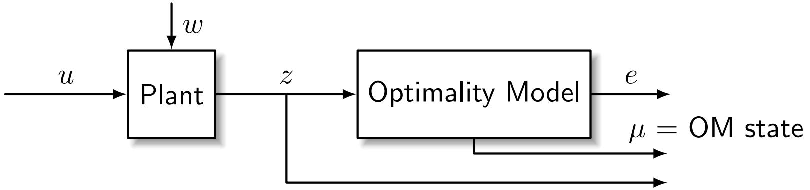
- Integral control forces KKT-like error e to zero
- Stabilizer . . . well, stabilizes . . . and tuned for performance

Optimality Models for OSS Control

An **optimality model** dynamically filters measurements to produce a **proxy error** e quantifying the KKT violation



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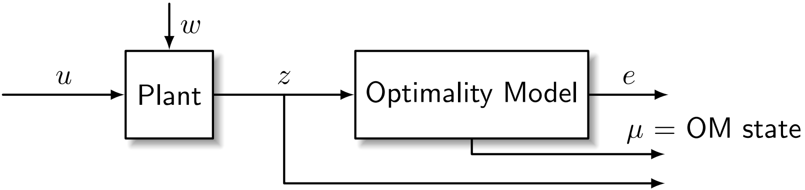

Opt. Model Requirement: if the plant and optimality model are both in equilibrium and $e = \mathbb{0}$, then $(z, u) = (z^\star, u^\star)$.

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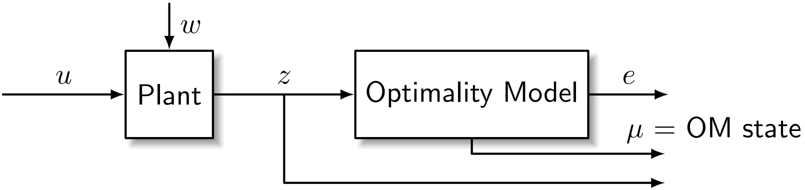

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Interpretation: The loop *incorporates a model of the distance to the optimal solution set of* (EqOpt)

Examples of Optimality Models

$$
\begin{aligned}
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\text{subject to} \quad & \bar{z} = G_u \bar{u} + G_w \bar{w} \\
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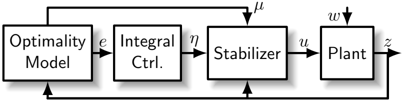

Optimality Model #1

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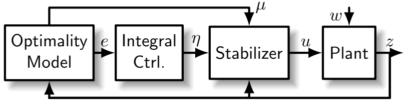

Optimality Models $\approx$ KKT conditions driven by measurements.

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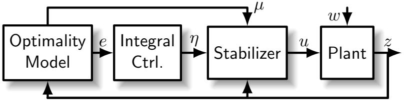

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Optimality Model #1

- Dualize engineering constraint

$$\tau\dot{\mu} = H_z z + H_u u + H_w w$$
$$e = \nabla f_0(u) + G_u^{\mathsf{T}}\nabla g_0(z) + (H_z G_u + H_u)^{\mathsf{T}}\mu$$

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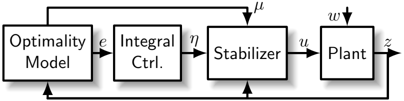

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&\quad + (H_z G_u + H_u)^{\mathsf{T}} \mu
\end{aligned}
$$

Optimality Model #2

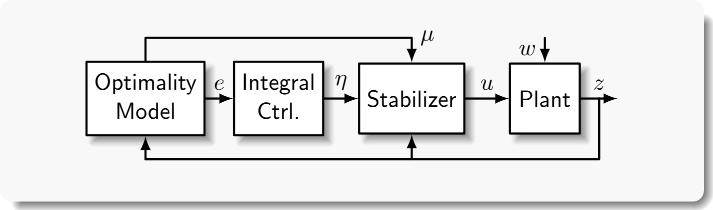
- Parameterize intersection of equality constraints in (z, u):

$$
\text{range} \begin{bmatrix} T_z \\ T_u \end{bmatrix} = \text{null} \begin{bmatrix} I_r & -G_u \\ H_z & H_u \end{bmatrix}
$$

$$
\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} T_u^{\mathsf{T}} \nabla f_0(u) + T_z^{\mathsf{T}} \nabla g_0(z) \\ H_z z + H_u u + H_w w \end{bmatrix}.
$$

From OSS Control to Regulator Problem

Optimality model reduces OSS control to regulator/servomechanism problem



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- **Stabilizer:** Stabilizes the cascade Plant $\rightarrow$ Opt. Model $\rightarrow$ Integrator

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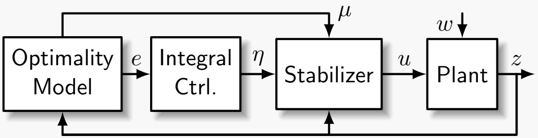

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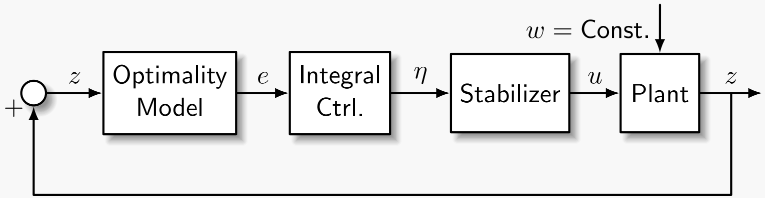
"Theorem" Loop Stable $\implies (u(t), z(t)) \rightarrow (u^\star, z^\star)$

Sanity Check: Existence of Stabilizer for Quadratic Obj.



- **Convex quadratic objective** $\begin{bmatrix} z \\ u \end{bmatrix}^{\mathsf{T}} Q \begin{bmatrix} z \\ u \end{bmatrix} + c^{\mathsf{T}} \begin{bmatrix} z \\ u \end{bmatrix}$

Plant→OM→Integrator **cascade is stabilizable/detectable** $\Longleftrightarrow$

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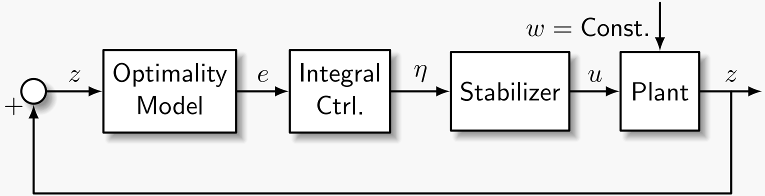

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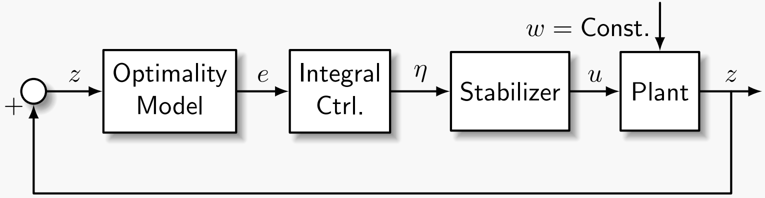

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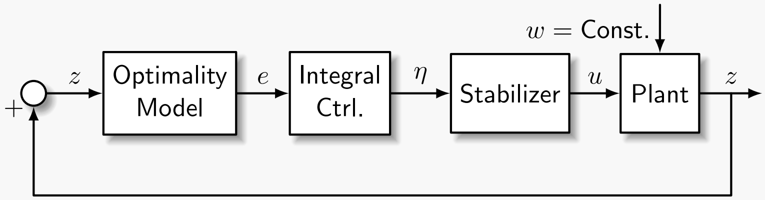

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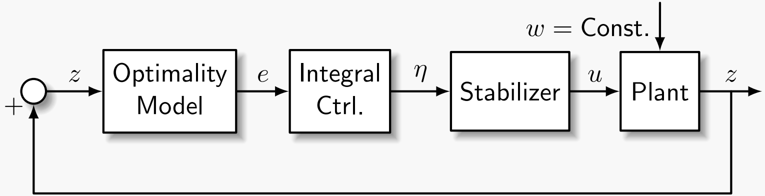
(iii) constraint matrix $H_z G_u + H_u$ full row rank;

Sanity Check: Existence of Stabilizer for Quadratic Obj.



- **Convex quadratic objective** $\begin{bmatrix} z \\ u \end{bmatrix}^\mathsf{T} Q \begin{bmatrix} z \\ u \end{bmatrix} + c^\mathsf{T} \begin{bmatrix} z \\ u \end{bmatrix}$

Plant→OM→Integrator **cascade is stabilizable/detectable** $\iff$

(i) plant stabilizable/detectable;

(ii) optimization problem has a unique solution;

(iii) constraint matrix $H_z G_u + H_u$ full row rank;

(iv) some full rank conditions on optimality model.

Low-Gain Stabilizer Designs

Problem: Quadratic objectives are very restrictive, and do not permit, e.g., log barrier functions, quadratic penalty functions.
Optimality model introduces nonlinearity.

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Low-Gain Stabilizers for Linear-Convex Optimal Steady-State Control

John W. Simpson-Porco

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Low-Gain Stabilizers for Linear-Convex Optimal Steady-State Control

John W. Simpson-Porco

- Several constructive stabilizer designs given
- Provides streamlined treatment of optimality models
- Based on stability results from JWSP TAC 2020

Academic Example: 30 states, 7 inputs, 5 outputs

- **Objectives:** (z_1, z_2) step tracking, min. control, constraints

$$\begin{aligned} \text{minimize} \quad & \left[\sum_{k=1}^{m} \tfrac{1}{2}\bar{u}_k^2 + \gamma \mathsf{B}(\bar{u}_k)\right] + c\,\mathsf{P}(\bar{z}) \\ \text{subject to} \quad & \bar{z} = G_u \bar{u} + G_w w \\ & 0 = z_i - r_i, \qquad i \in \{1, 2\} \end{aligned}$$

$$\mathsf{B}(u_k) = \text{log barrier fcn.}$$
$$\mathsf{P}(\bar{z}_3, \bar{z}_4, \bar{z}_5) = \text{penalty fcn.}$$

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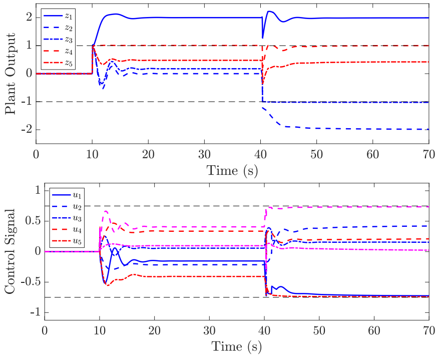

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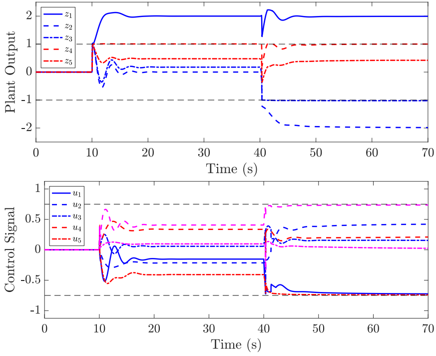
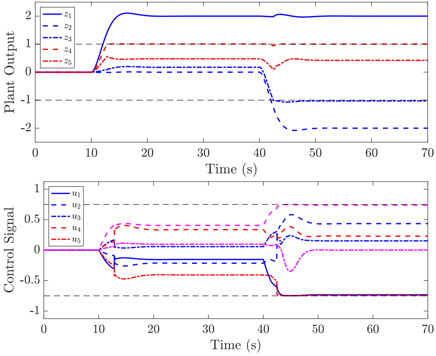


Optimal/Robust Stabilizer



Designing for Performance in the OSS Control Framework

A Generalized Plant Perspective on Linear-Convex Feedback Optimization

Fabian Jakob, Andrea Iannelli

***Abstract*—Feedback optimization is a control approach for driving a dynamical system to the solution of an optimization problem by interconnecting the plant with an algorithm. Existing stability guarantees typically rely on timescale separation, enforced by conservative gain bounds that limit transient performance and require a pre-stabilized plant. This paper revisits the robust control perspective on feedback optimization. We formulate the plant-optimizer interconnection as a generalized plant, where the cost gradients are characterized by Zames–Falb Integral Quadratic Constraints. Classical timescale-separation bounds are recovered as a special case of static multipliers, with dynamic multipliers yielding substantially tighter stability margins. The formulation also enables IQC based synthesis of dynamic output feedback controllers that jointly stabilize the plant and optimize transient performance, with possible model uncertainty absorbed into an uncertainty channel. For constrained problems, the framework extends to dynamic controllers that generalize projected gradient flows. Numerical examples illustrate the benefits and flexibility of the proposed approach.**

***Index Terms*—Feedback optimization, integral quadratic constraints, robust control design.**

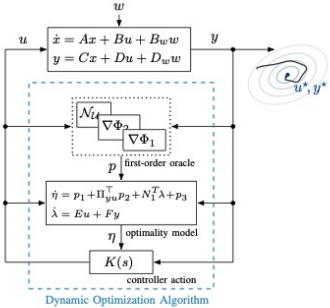


Fig. 1: Feedback optimization of LTI plants driven by structured dynamic algorithms.

Estimator-Enhanced Feedback-Based Optimization

"But I *really* <u>*really*</u> *like* my algorithm . . ."

Recall: Three-step FBO paradigm

(i)	**Algorithm:**	$\tau\dot{\bar{u}} = -\bar{u} + \mathrm{Proj}_{\mathcal{U}}\left(\bar{u} - \alpha\Phi(\bar{u}, \bar{z})\right)$
(ii)	**Steady-State Equivalence:**	$\tau\dot{u} = -u + \mathrm{Proj}_{\mathcal{U}}\left(u - \alpha\Phi(u, z)\right)$
(iii)	**Analysis:**	C.L.S. Stable for $\tau > \tau^\star$

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Idea: Instead insert **estimate** $\hat{\bar{z}}$ of $\bar{z}$

$$\hat{\bar{z}} = G_u u + G_w \hat{w}, \qquad \hat{w} = \text{Estimate of } w$$

Estimator-Enhanced FBO

- Disturbance model $\dot{w} = 0$, full-order Luenberger estimator

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{bmatrix} = \begin{bmatrix} A & B_w \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} (\hat{z} - z)$$

$$\hat{z} = C\hat{x} + Du + D_w\hat{w}$$

- Feasible iff $G_w = -CA^{-1}B_w + D_w$ has full column rank

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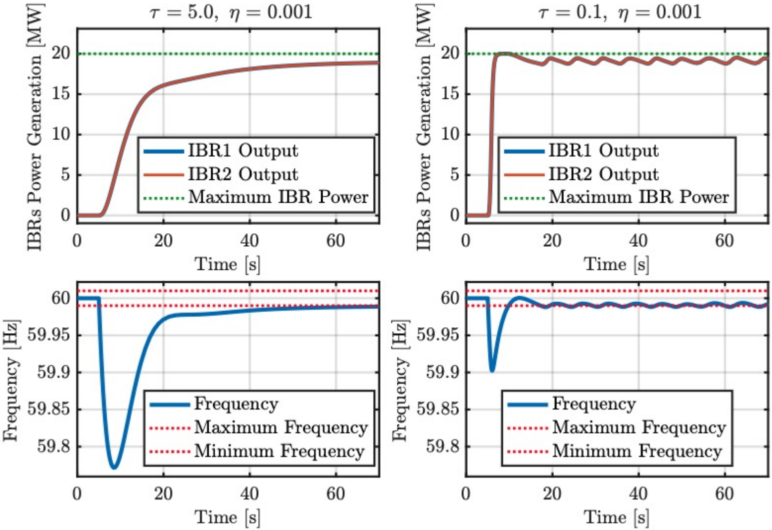
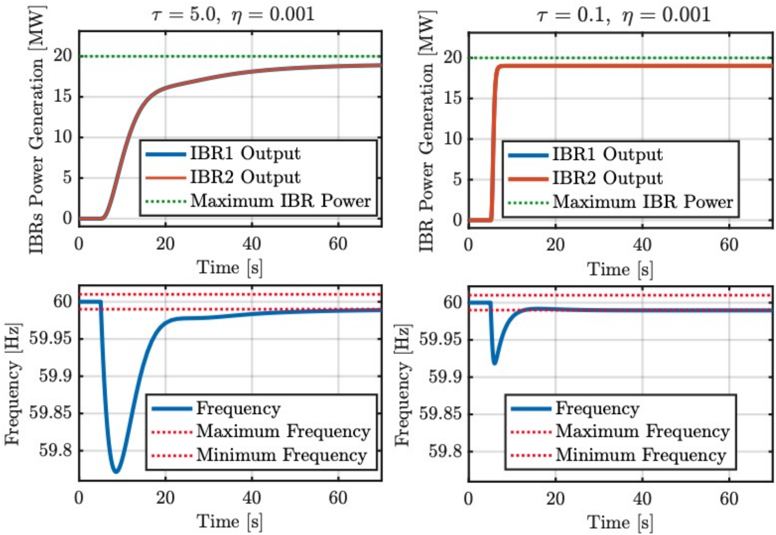
Application: Fast Frequency Control using IBRs



(a) Stable FBO with large τ (b) Unstable FBO with small τ

Application: Fast Frequency Control using IBRs



(a) Stable FBO with large τ (b) EE-FBO based on full model

Full Talk: Thursday AM "Optimization" 11:40am, Árna 1

Removing Time-Scale Separation in Feedback-Based Optimization via Estimators

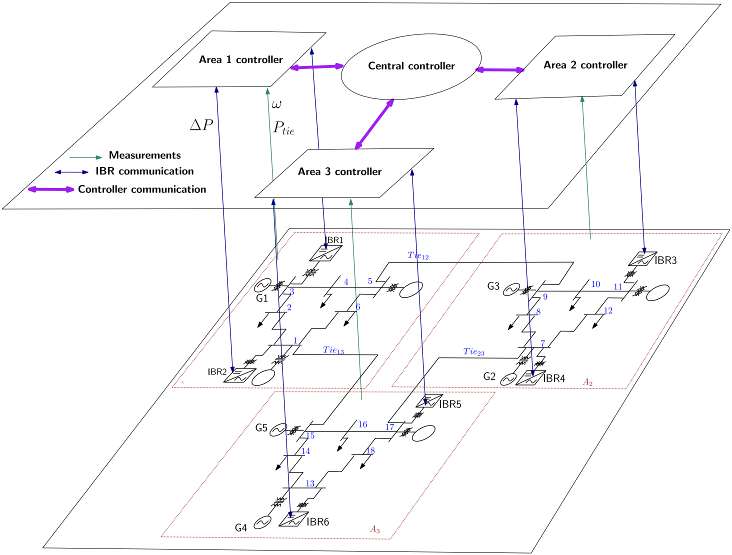
Niloufar Yousefi, John W. Simpson-Porco

Department of Electrical and Computer Engineering

- Main stability result
- Extension to partial plant model information

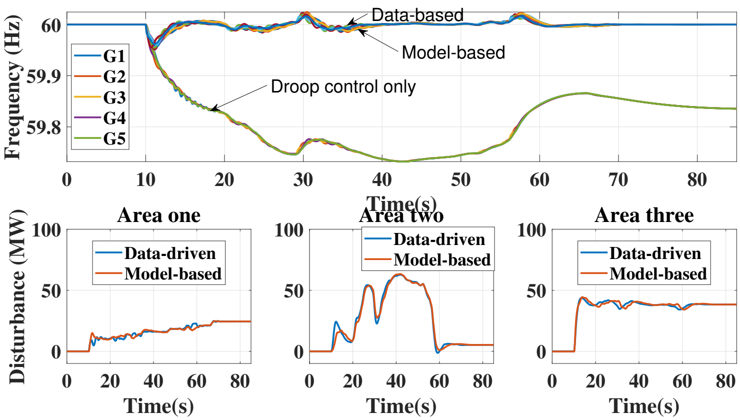
Power System Applications

Fast Frequency Control using IBRs

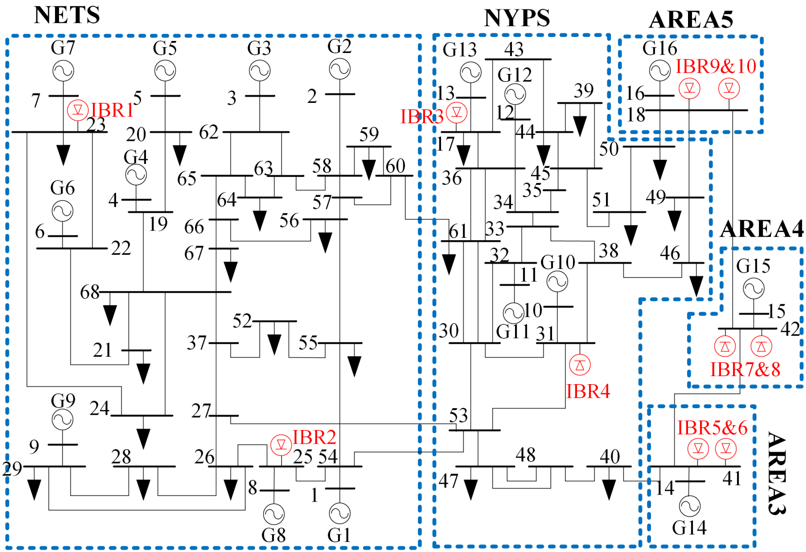


Scenario: Compensating for Renewables

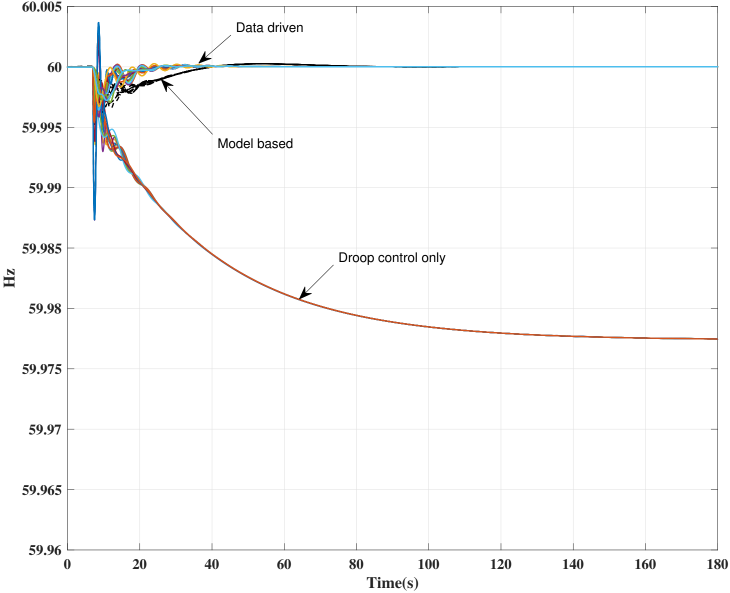
- Uncontrolled wind and solar ($\approx$ 300 MW) integrated into system
- Major unexpected decreases in RES across grid ($\approx$ 125MW total peak)



Five-Area 68 Bus Test System

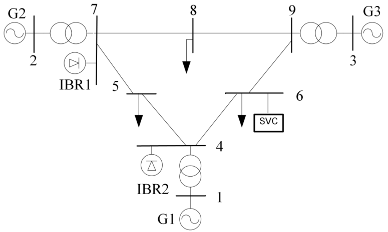


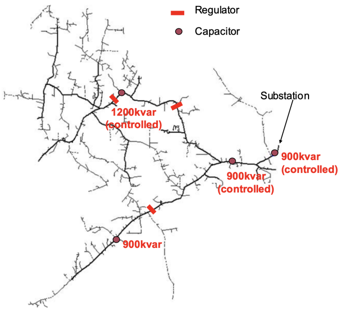
Scenario: 300MW Load Change in NYPS Area



The Need for Transmission-Distribution Coordination

- Only very large storage and RES facilities are transmission-connected
- Most RES/DERs are connected at the **distribution level**





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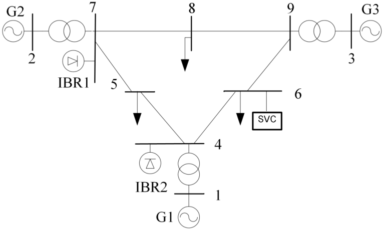
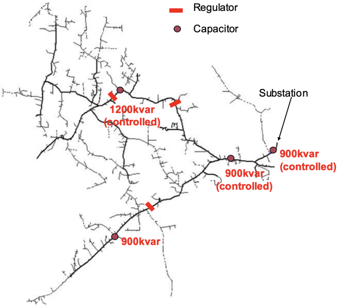




Can we **hierarchically coordinate** resources in the distribution system to **collectively respond** like a fast transmission-connected resource?

Multi-Area Hierarchical DN Control: Objectives

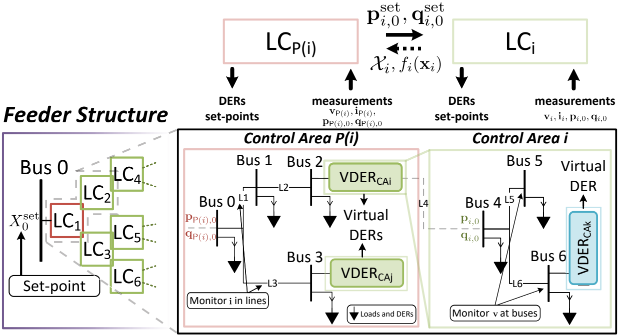
(i) Respond **fast** to power change requests by TSO

(ii) **Area-based** (e.g., aggregator) control using only local measurements

(iii) Maintain voltage and current constraints in DN

(iv) Grid and DER models kept **private** to each area controller

(v) Design must be **agnostic** to specifics of DERs



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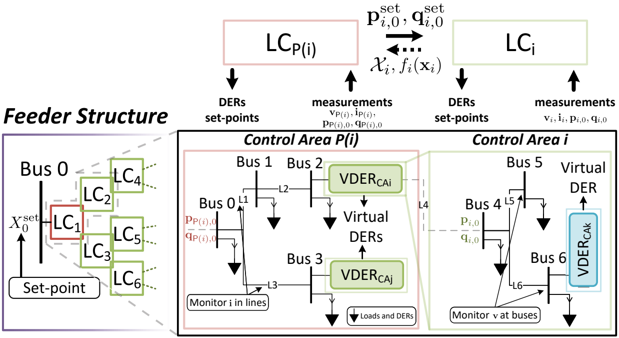

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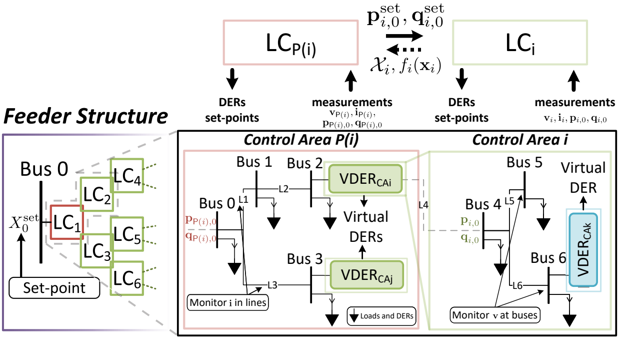

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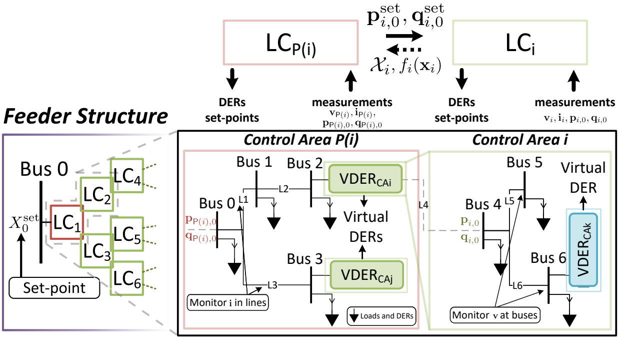

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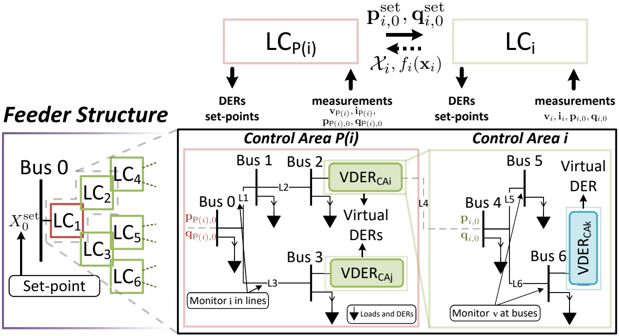

Area (Nonlinear) Controller Design

Farhat *et al. A Multi-Area Architecture for Real-Time Feedback-Optimization ...*, IEEE TSG, 2024

Control Area i Design:

- $x_j = (\mathrm{p}_j, \mathrm{q}_j)$ are DER **set-points**
- *Child areas* $\mathsf{C}(i)$ abstracted as **virtual DERs**
- $f_{ij}(x_j) =$ control cost of DER j
- $\mathbf{p}_{i,0}, \mathbf{q}_{i,0} =$ power export to the *Parent Area* $\mathsf{P}(i)$

$$\begin{aligned}
\underset{\mathbf{x}_i \in \boldsymbol{\mathcal{X}}_i}{\text{minimize}} \quad & \sum_{j \in \mathcal{D}_i} f_{ij}(x_j) \\
\text{subject to} \quad & \mathbb{1}^\mathsf{T} \mathbf{p}_{i,0}(\mathbf{x}_i) = \mathrm{p}_{i,0}^{\text{set}}(\mathbf{x}_{\mathsf{P}(i)}) \\
& \mathbb{1}^\mathsf{T} \mathbf{q}_{i,0}(\mathbf{x}_i) = \mathrm{q}_{i,0}^{\text{set}}(\mathbf{x}_{\mathsf{P}(i)}) \\
& \underline{\mathbf{v}}_i \leq \mathbf{v}_i(\mathbf{x}_i) \leq \overline{\mathbf{v}}_i \\
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\end{aligned}
$$

Algorithm 1: LC Controller for ith CA

At each sampling time

[Step 1]: Receive set-points from LC of parent area $\mathsf{P}(i)$

$$\mathrm{p}^{\text{set}}_{i,0}(\mathbf{x}_{\mathsf{P}(i)}) = T^{\mathrm{p}}_i \mathbf{x}_{\mathsf{P}(i)}, \quad \mathrm{q}^{\text{set}}_{i,0}(\mathbf{x}_{\mathsf{P}(i)}) = T^{\mathrm{q}}_i \mathbf{x}_{\mathsf{P}(i)}$$

[Step 2]: Collect local measurements $\mathbf{p}_{i,0}, \mathbf{q}_{i,0}, \mathbf{v}_i, \mathbf{i}_i, \underline{\mathbf{x}}_i, \overline{\mathbf{x}}_i$

[Step 3]: LC performs the updates

$$
\begin{aligned}
\lambda_i^+ &= \mathcal{P}_{\geq 0}\left(\lambda_i + \alpha_{\lambda_i}\left(\mathbb{1}^\mathsf{T}\mathbf{p}_{i,0} - \mathrm{p}^{\text{set}}_{i,0} - E_{i_p} - r_{\lambda_i}\lambda_i\right)\right) \\
\mu_i^+ &= \mathcal{P}_{\geq 0}\left(\mu_i + \alpha_{\mu_i}\left(\mathrm{p}^{\text{set}}_{i,0} - \mathbb{1}^\mathsf{T}\mathbf{p}_{i,0} - E_{i_p} - r_{\mu_i}\mu_i\right)\right) \\
\eta_i^+ &= \mathcal{P}_{\geq 0}\left(\eta_i + \alpha_{\eta_i}\left(\mathbb{1}^\mathsf{T}\mathbf{q}_{i,0} - \mathrm{q}^{\text{set}}_{i,0} - E_{i_q} - r_{\eta_i}\eta_i\right)\right) \\
\psi_i^+ &= \mathcal{P}_{\geq 0}\left(\psi_i + \alpha_{\psi_i}\left(\mathrm{q}^{\text{set}}_{i,0} - \mathbb{1}^\mathsf{T}\mathbf{q}_{i,0} - E_{i_q} - r_{\psi_i}\psi_i\right)\right) \\
\boldsymbol{\gamma}_i^+ &= \mathcal{P}_{\geq 0}\left(\boldsymbol{\gamma}_i + \alpha_{\gamma_i}\left(\mathbf{v}_i - \overline{\mathbf{v}}_i - r_{\gamma_i}\boldsymbol{\gamma}_i\right)\right) \\
\boldsymbol{\nu}_i^+ &= \mathcal{P}_{\geq 0}\left(\boldsymbol{\nu}_i + \alpha_{\nu_i}\left(\underline{\mathbf{v}}_i - \mathbf{v}_i - r_{\nu_i}\boldsymbol{\nu}_i\right)\right) \\
\boldsymbol{\zeta}_i^+ &= \mathcal{P}_{\geq 0}\left(\boldsymbol{\zeta}_i + \alpha_{\zeta_i}\left(\mathbf{i}_i - \bar{\mathbf{i}}_i - r_{\zeta_i}\boldsymbol{\zeta}_i\right)\right)
\end{aligned}
$$

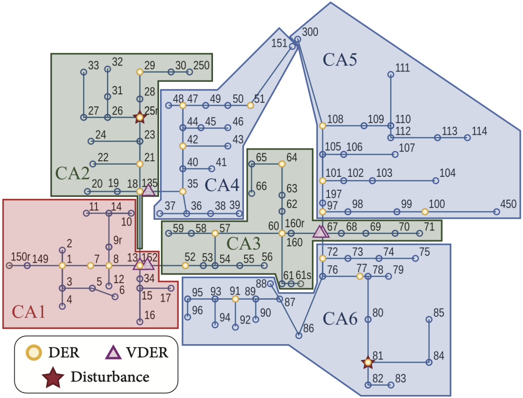
[Step 4]: LC updates (V)DER set-points

$$\mathbf{x}_i^+ = \underset{\mathbf{x}_i \in \boldsymbol{\mathcal{X}}_i}{\arg\min}\, L^{\mathrm{r}}_i(\mathbf{x}_i, \mathbf{d}_i^+; \mathbf{x}_{\mathsf{P}(i)})$$

[Step 5]: Transmit set-points to LCs of each child area

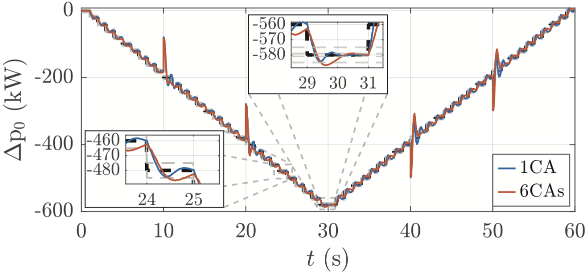
$$\mathrm{p}^{\text{set}}_{j,0}(\mathbf{x}_i^+) = T^{\mathrm{p}}_j \mathbf{x}_i^+, \quad \mathrm{q}^{\text{set}}_{j,0}(\mathbf{x}_i^+) = T^{\mathrm{q}}_j \mathbf{x}_i^+, \qquad j \in \mathsf{C}(i).$$

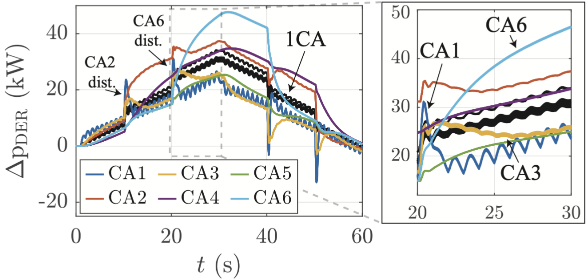
Example: 123 Bus Feeder



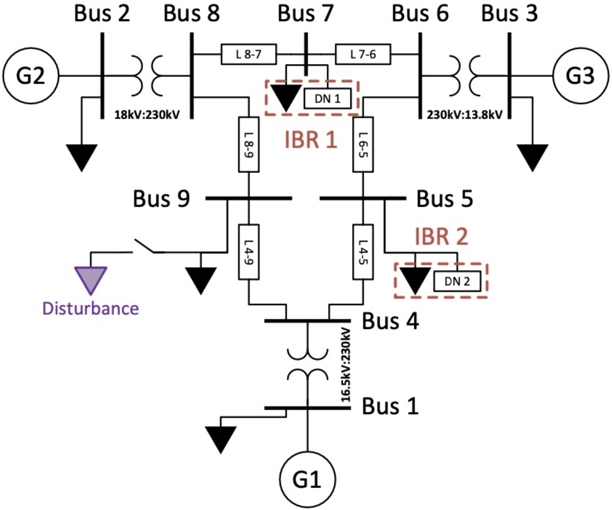
- 6 Control Areas
- 17 DERs
- Track **ramp signal** from TSO

Example: 123 Bus Feeder

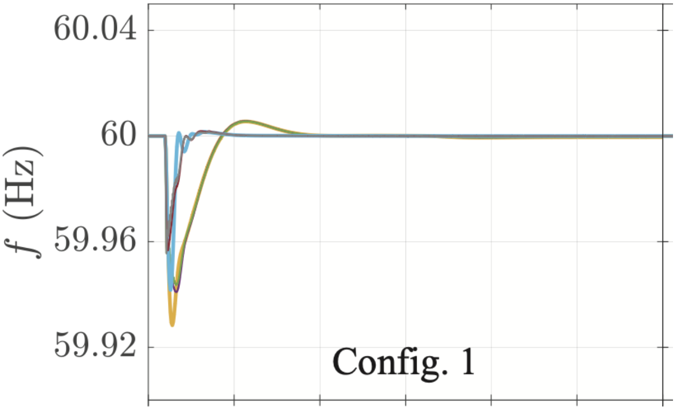




Example: Fast Frequency Control



Example: Fast Frequency Control



Conclusions

- Opt. Alg. FBO is simple but has major limitations
- **OSS Control:** a flexible and general control-theoretic framework
- **EE-FBO:** enhances dynamics of your favourite Opt. Alg. FBO

Opportunities:

- Trade-offs between performance and model info / data quality
- Layering and hierarchical control
- Further validation in applications where dynamics matter

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Conclusions

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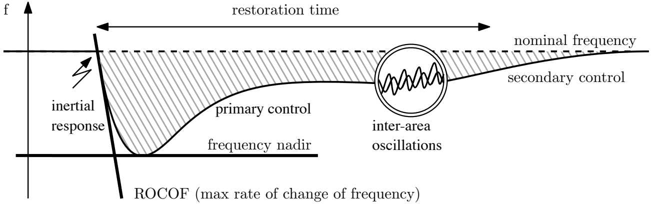
Related Papers

- L. S. P. Lawrence, J. W. Simpson-Porco, and E. Mallada, "Linear-convex optimal steady-state control," *IEEE Transactions on Automatic Control*, vol. 66, no. 11, pp. 5377–5384, Nov. 2021.
- L. S. P. Lawrence, Z. E. Nelson, E. Mallada, and J. W. Simpson-Porco, "Optimal steady-state control for linear time-invariant systems," in *IEEE Conf. on Decision and Control*, Miami Beach, FL, USA, Dec. 2018, pp. 3251–3257.
- J. W. Simpson-Porco, "Analysis and synthesis of low-gain integral controllers for nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 66, no. 9, pp. 4148–4159, Sep. 2021.
- ——, "Low-gain stabilizers for linear-convex optimal steady-state control," in *IEEE Conf. on Decision and Control*, Cancún, Mexico, Dec. 2022, pp. 2552–2559.
- N. Yousefi and J. W. Simpson-Porco, "Removing time-scale separation in feedback-based optimization via estimators," in *European Control Conference*, 2026, to appear.
- I. Farhat, E. Ekomwenrenren, J. W. Simpson-Porco, E. Farantatos, M. Patel, and A. Haddadi, "A multi-area architecture for real-time feedback-based optimization of distribution grids," *IEEE Transactions on Smart Grid*, vol. 16, no. 2, pp. 1448–1461, Mar. 2025.

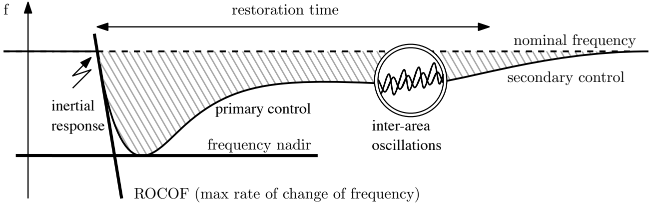
appendix

Application: Secondary Frequency Control in Bulk Grid



- **Goal**: efficiently regulate $\Delta\omega \to 0$

Application: Secondary Frequency Control in Bulk Grid



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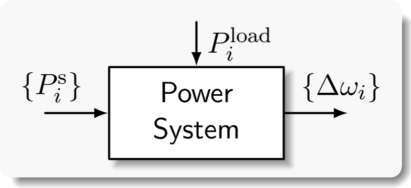
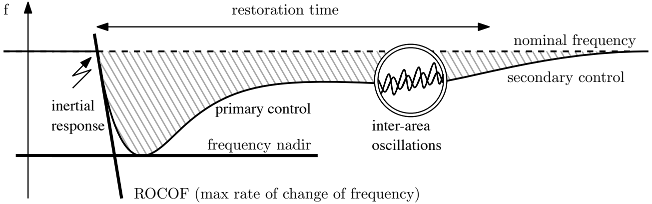


Application: Secondary Frequency Control in Bulk Grid



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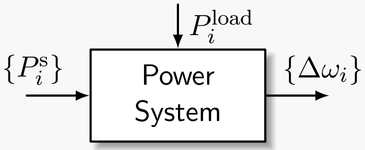
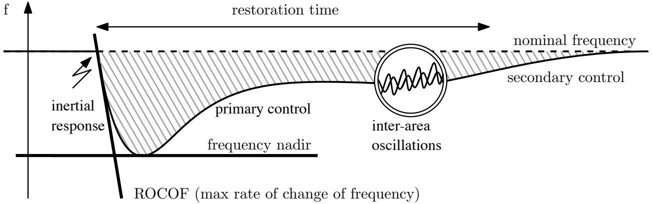


$$\begin{aligned} \underset{P_i^{\text{s}} \in \{\text{limits}\}}{\text{minimize}} \quad & \sum\nolimits_{i=1}^{n} C_i(P_i^{\text{s}}) \\ \text{subject to} \quad & \Delta\omega_i = 0 \\ & \text{(System)} \end{aligned}$$

Application: Secondary Frequency Control in Bulk Grid



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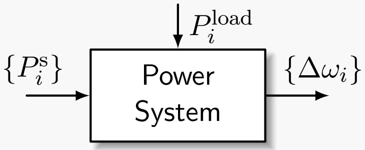

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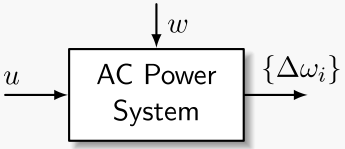
Want: Fast resource-allocating control loops (*architecture?*)

Application: Secondary Frequency Control in Bulk Grid



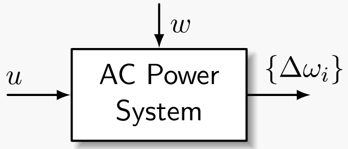
$$\begin{array}{ll} \underset{u}{\text{minimize}} & \sum_{i=1}^{n} C_i(\bar{u}_i) \\ \text{subject to} & \Delta\boldsymbol{\omega} = \frac{1}{\beta}\mathbb{1}_n(\mathbb{1}_n^{\mathsf{T}}\bar{u} + w) \\ & \Delta\omega_n = 0 \end{array}$$

- **Note:** the way one encodes the frequency regulation constraint can lead to different controller architectures; this is just one possible choice.
- $G_u = \frac{1}{\beta}\mathbb{1}_n\mathbb{1}_n^{\mathsf{T}}$, $G_w = \frac{1}{\beta}\mathbb{1}_n$

Optimality Model #1

Optimality Model #2

Application: Secondary Frequency Control in Bulk Grid



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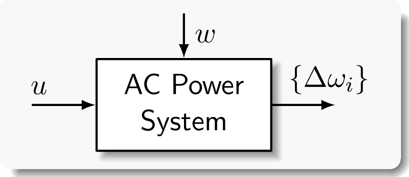

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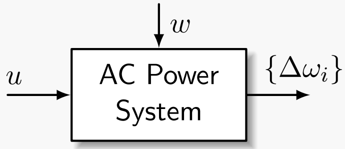
Optimality Model #1

$$\begin{aligned} \tau\dot{\mu} &= -\Delta\omega_n \\ e_i &= \nabla C_i(u_i) - \mu \end{aligned}$$

Optimality Model #2

Application: Secondary Frequency Control in Bulk Grid



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Optimality Model #2

$$\text{null}\begin{bmatrix} I_n & -\frac{1}{\beta}\mathbb{1}_n\mathbb{1}_n^{\mathsf{T}} \\ \mathsf{e}_n^{\mathsf{T}} & 0 \end{bmatrix} = \text{range}\begin{bmatrix} 0 \\ L^{\mathsf{T}} \end{bmatrix}$$

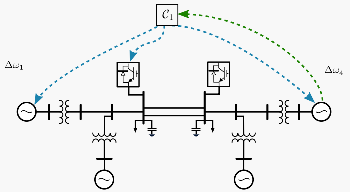
$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} L\nabla C(u) \\ \Delta\omega_n \end{bmatrix}.$$

Application: Secondary Frequency Control in Bulk Grid

Both designs provably stable for large τ

1. Inversion-based controller

$$\tau\dot{\mu} = -\Delta\omega_n$$
$$u_i = (\nabla C_i)^{-1}(\mu)$$



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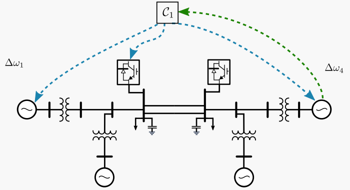

2. Two-loop controller

$$\tau_1 \dot{\eta}_i = -\sum_{j=1}^{n} a_{ij}(\nabla C_i(u_i) - \nabla C_j(u_j))$$
$$\tau_2 \dot{\eta}_n = -\Delta\omega_n$$
$$u_i = \eta_i$$