The Optimal Steady-State Control Problem

by

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I hereby declare that I have co-authored the content of this thesis. Co-authored chapters and sections are marked by a footnote detailing the relevant publication; I am the sole author of chapters and sections not marked by a footnote. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Many engineering systems — including electrical power networks, chemical processing plants, and communication networks — have a well-defined notion of an "optimal" steady-state operating point. This optimal operating point is often defined mathematically as the solution of a constrained optimization problem that seeks to minimize the monetary cost of distributing electricity, maximize the profit of chemical production, or minimize the communication latency between agents in a network. Optimal steady-state regulation is obviously of crucial importance in such systems.

This thesis is concerned with the *optimal steady-state control problem*, the problem of designing a controller to *continuously* and *automatically* regulate a dynamical system to an optimal operating point that minimizes cost while satisfying equipment constraints and other engineering requirements, even as this optimal operating point changes with time. An optimal steady-state controller must simultaneously solve the optimization problem and force the plant to track its solution.

This thesis makes two primary contributions. The first is a general problem definition and controller architecture for optimal steady-state control for nonlinear systems subject to time-varying exogenous inputs. We leverage *output regulation theory* to define the problem and provide necessary and sufficient conditions on any optimal steady-state controller. Regarding our controller architecture, the typical controller in the output regulation literature consists of two components: an internal model and a stabilizer. Inspired by this division, we propose that a typical optimal steady-state controller should consist of three pieces: an *optimality model*, an internal model, and a stabilizer. We show that our design framework encompasses many existing controllers from the literature.

The second contribution of this thesis is a complete constructive solution to an important special case of optimal steady-state control: the linear-convex case, when the plant is an uncertain linear time-invariant system subject to constant exogenous inputs and the optimization problem is convex. We explore the requirements on the plant and optimization problem that allow for optimal regulation even in the presence of parametric uncertainty, and we explore methods for stabilizer design using tools from *robust control theory*.

We illustrate the linear-convex theory on several examples. We first demonstrate the use of the small-gain theorem for stability analysis when a PI stabilizer is employed; we then show that we can use the solution to the $\mathcal{H}_\infty$ control problem to synthesize a stabilizer when the PI controller fails. Furthermore, we apply our theory to the design of controllers for the optimal frequency regulation problem in power systems and show that our methods recover standard designs from the literature.

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I would also like to acknowledge our collaborators, Dr. Enrique Mallada and Zachary Nelson of Johns Hopkins University. Dr. Mallada's expertise in optimization was of great assistance in this project. Zachary's initial investigations into the problem paved the way for my own work. I say thanks to them both.

Dedication

This thesis is dedicated to my parents, Mark and Lesley, and to my sister, Bronwyn. Thank you for all your love and support during this degree. I further dedicate this thesis to the friends and family who gave me strength when I needed it. You all mean the world to me.

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Chapter 1

Introduction

1.1 Motivation and Literature Review[1]

Control theory is the study of dynamical systems with inputs and outputs. We are free to manipulate the inputs to influence the evolution of the system, and we are able to measure the outputs to acquire information about the system's behaviour. *Feedback* refers to the act of using these measured outputs to make smart decisions about the inputs we apply. Control theorists study the use of feedback to elicit desirable behaviour from a system.

"Desirable" might mean stabilizing the system, guiding measurable outputs to prescribed setpoints, or optimizing state trajectories. Consider the use of feedback control in, for example, an autonomous car driving along a straight stretch of road. The motion of the car is described by a set of differential equations arising from Newton's laws of motion. The steering angle of the front wheels and the speed of rotation of the back wheels serve as inputs that influence how the car moves. We suppose the car is equipped with a speedometer and a camera system that can detect its lateral position and position relative to the car ahead and the car behind — these measurements are the outputs. We would define "desirable behaviour" in this example as the car moving in a straight line at a constant speed, with minimal deviation from the centre of the lane, while maintaining a constant distance from the other cars. The on-board computer system must take the measurements of speed and distance and determine whether the vehicle needs to speed up, slow down, or bank to one side.

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[1]Some of the contents of this section appear in L. S. P. Lawrence, Z. E. Nelson, E. Mallada, and J. W. Simpson-Porco "Optimal Steady-State Control for Linear Time-Invariant Systems", in 2018 IEEE Conference on Decision and Control (CDC), pages 3251–3257.

In many engineering applications, the control goal is to make a system come to rest in a dynamic equilibrium that is, by some definition, *optimal*. Consider an electrical power grid, consisting of a network of power supplies, such as synchronous generators, and power loads, such as residential consumers demanding electricity. The optimal operating point is determined by a constrained optimization problem that seeks to minimize the total cost of power generation, while maintaining supply-demand balance and system stability [32, 70]. The current approach for regulating an electrical power grid to this optimal operating point involves a time-scale separation between the tasks of computing the optimal setpoint and tracking this setpoint using feedback controllers. The optimal generation setpoints are computed offline using demand projections and a model of the network, then the operating points are dispatched as reference commands to local controllers at each generation site [5]. This process is repeated with a fixed update rate: a new optimizer is computed, dispatched, and tracked. If the supply and demand of power changes on a time scale that is slow compared to the update rate, then this method is perfectly acceptable.

If the optimizer changes rapidly, however, as is the case for power networks with a high penetration of renewable energy sources, the conventional approach is inefficient [69]. Profit is reduced as a result of operating in a sub-optimal regime between optimizer updates. In the rapidly-changing optimizer case, then, it would be advantageous to eliminate the time-scale separation by combining the local generator controllers with an online optimization algorithm, so that the optimal operating condition could be tracked in real time. Indeed, this is the direction of much recent research in power system control [2, 18, 19, 21, 22, 31, 38, 46, 47, 51, 52, 54, 66–69, 72, 73, 77, 78, 82].

The same theme of real-time regulation of system variables to optimal values emerges in diverse areas, and much work on controller design to implement online optimization exists in the literature. Fields of application besides the power network control example mentioned already include network congestion management [41, 48, 49, 53, 76], chemical processing [20, 26, 28], wind turbine power capture [8, 58], active flow control and axial flow compressor control for aerospace applications [4, 75], temperature regulation in energy-efficient buildings [30], and beam matching in particle accelerators [65].

The breadth of applications motivates the need for a *general* theory and design procedure for controllers that regulate a system to a maximally efficient operating point defined by an optimization problem, even as the optimizer changes over time due to changing market prices, disturbances to the system dynamics, and operating constraints that depend on external variables. We refer to the problem of designing such a controller as the *optimal steady-state (OSS) control problem.*

Much work on the OSS control problem already exists in the literature under various

names including "real-time optimization" and "online optimization" — see for example, [11, 13, 16, 17, 33, 35, 40, 44, 56, 57, 59, 79]. In the *extremum-seeking control* approach, a harmonic signal is used to perturb an uncertain system, and the gradient of a cost function is then inferred by filtering system measurements; a control signal is applied to drive the gradient to zero [20, 29, 43]. Jokić, Lazar, and van den Bosch propose a *Karush-Kuhn-Tucker (KKT) controller*, employing the necessary and sufficient KKT conditions for optimality as the basis of a nonlinear feedback controller that guides the outputs of a system to an optimizer [39, 40]. Nelson and Mallada consider an optimization problem over system states and apply gradient feedback with a proportional-integral (PI) controller; if the full system state cannot be directly measured, a Luenberger observer is employed [59].

Many of the currently-proposed controllers, however, have limited applicability: some solutions only apply to systems of a special form [13, 79]; some attempt to optimize only the steady-state input [17] or output [11, 35, 40, 57] alone; some apply only to equality-constrained [16, 56] or unconstrained optimization problems [33]; and in all cases, the effects of parametric modelling uncertainty are omitted.

As a result, a number of important questions are raised regarding the existence of solutions and the general architecture of OSS controllers.

(i) Fundamental controller existence theorems are lacking, leaving important questions unanswered: What conditions on the plant and optimization problem are necessary for the OSS control problem to be solvable? What properties must a controller satisfy to solve the OSS control problem?

(ii) Insufficient attention has been paid to understanding when real-time optimization can be performed robustly in the presence of parametric uncertainty.

(iii) The literature lacks a general, unifying architecture for OSS controllers that facilitates controller design and connects to established design methodologies.

This thesis fills these gaps in the optimal steady-state control literature.

1.2 Organization of Thesis

This thesis is organized as follows. In Chapter 2, we give a summary of the background necessary to understand the remainder of the thesis. We then outline the fundamentals of optimal steady-state control, including a formal problem statement, solvability conditions,

and general controller architecture in Chapter 3. We pose our problem in a very general setting for which a solution may be difficult or impossible to determine; in Chapter 4, therefore, we describe a special case for which we formulate a constructive design procedure. We illustrate many of the discussed ideas in Chapter 5, followed by a summary and enumeration of future directions in Chapter 6.

Chapter 2

Background

This chapter summarizes the essential topics required to understand the remainder of the thesis. We survey

   (i) convex optimization,

  (ii) the output regulation problem, and

 (iii) robust control theory.

We assume the reader is familiar with the basics of linear control theory (see [34]), nonlinear systems theory including stability definitions and Lyapunov analysis (see [42]), and linear algebra (see [3]).

Notation

We will do our best to introduce notation in a "just-in-time" manner; however, some notation is so common that it is easier to agree upon its use at the outset.

We denote by $\mathbb{R}$ the set of real numbers. The set $\mathbb{R}^n$ is the set of n-tuples of real numbers. The space of $n \times m$ matrices with real entries is denoted by $\mathbb{R}^{n \times m}$. The matrix I_n is the $n \times n$ identity matrix, $\mathbb{0}_n$ is the n-vector of all zeros, and $\mathbb{0}$ is a matrix of zeros whose size may be inferred from context.

For matrices $A, B \in \mathbb{R}^{n \times n}$, the generalized inequality $A \prec B$ means $B - A$ is positive definite, while $A \preceq B$ means $B - A$ is positive semidefinite.

The notation $f : U \to V$ means that f is a function mapping (a subset of) the set U to (a subset of) the set V. For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, its gradient is $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$.

2.1 Convex Optimization

This section recalls the basics of mathematical optimization, and convex optimization in particular, from [10]. For more on optimization, we also refer the reader to [6, 7, 50, 60].

2.1.1 Optimization Problems

An *optimization problem* is the problem of selecting a "best" decision out of a universe of possible decisions. Formulating this problem in the language of mathematics, we seek to select a vector of *decision variables* $x \in \mathbb{R}^n$ that minimizes a cost as measured by a function $f : \mathbb{R}^n \to \mathbb{R}$ that we call the *objective function*.

We write an optimization problem as follows:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \; f(x) \,. \tag{2.1}$$

An *optimizer* or *minimizer* $x^\star \in \mathbb{R}^n$ satisfies the property that no other vector in $\mathbb{R}^n$ yields a lower value of f, i.e.

$$f(x^\star) \leq f(x) \text{ for all } x \in \mathbb{R}^n \,.$$

We denote the set of minimizers by

$$\underset{x \in \mathbb{R}^n}{\text{argmin}} \; f(x) \,.$$

This set may contain zero, one, or many elements.

The problem (2.1) is called an *unconstrained* problem, since our decision variable may be any element of $\mathbb{R}^n$. Often, not every decision is admissible. We are frequently subject to *constraints* in our choices. We formulate constraints mathematically by imposing that our decision vector lie in some set $\mathcal{C}$ which may be a strict subset of $\mathbb{R}^n$. We call the set $\mathcal{C}$ the *feasible region* and say that $x \in \mathbb{R}^n$ is a *feasible point* if $x \in \mathcal{C}$. We write this *constrained* problem as

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & x \in \mathcal{C} \,, \end{aligned} \tag{2.2}$$

and denote the set of minimizers by

$$\operatorname*{argmin}_{x \in \mathbb{R}^n} \left\{ f(x) \mid x \in \mathcal{C} \right\}.$$

We typically describe the feasible region $\mathcal{C}$ algebraically using a set of functions $g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, \ldots, n_{\mathrm{ic}}\}$, that define *inequality constraints* and $h_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, \ldots, n_{\mathrm{ec}}\}$, that define *equality constraints*, like so:

$$\mathcal{C} = \left\{ x \in \mathbb{R}^n \mid g_i(x) \leq 0 \text{ for all } i \in \{1, \ldots, n_{\mathrm{ic}}\}, h_i(x) = 0 \text{ for all } i \in \{1, \ldots, n_{\mathrm{ec}}\} \right\}.$$

Using this algebraic description, we write the problem (2.2) as

$$\begin{aligned}
\operatorname*{minimize}_{x \in \mathbb{R}^n} \quad & f(x) \\
\text{subject to} \quad & g_i(x) \leq 0, \quad i \in \{1, \ldots, n_{\mathrm{ic}}\} \\
& h_i(x) = 0, \quad i \in \{1, \ldots, n_{\mathrm{ec}}\}.
\end{aligned}$$

2.1.2 Convexity

Certain sets and functions satisfy an important property known as *convexity*. A set $\mathcal{C} \subset \mathbb{R}^n$ is said to be *convex* when the line segment between any two points in $\mathcal{C}$ is also in $\mathcal{C}$. Mathematically, $\mathcal{C}$ is convex if

$$x, y \in \mathcal{C} \text{ and } \lambda \in [0, 1] \text{ imply that } (1 - \lambda)x + \lambda y \in \mathcal{C} \text{ also.} \tag{2.3}$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex when the line segment between any two points on the graph of the function lies above the graph. Mathematically, f is convex when

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y) \text{ for any } x, y \in \mathbb{R}^n \text{ and any } \lambda \in [0, 1]. \tag{2.4}$$

Optimization problems with a convex objective function and a convex feasible region possess structure that permits us to say much more about such problems than generic problems lacking convexity. A convex optimization problem is given by

$$\begin{aligned}
\operatorname*{minimize}_{x \in \mathbb{R}^n} \quad & f(x) \\
\text{subject to} \quad & g_i(x) \leq 0 \quad i \in \{1, \ldots, n_{\mathrm{ic}}\} \\
& Ax = b,
\end{aligned} \tag{2.5}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ define the equality constraints, and the $g_i : \mathbb{R}^n \to \mathbb{R}$ strict inequality, i.e. when

$$f((1 - \lambda)x + \lambda y) < (1 - \lambda)f(x) + \lambda f(y) \text{ for any } x, y \in \mathbb{R}^n \text{ with } x \neq y \text{ and any } \lambda \in [0, 1].$$
(2.6)

We have the following lemma from [10, Section 4.2.1].

Lemma 2.1.1 (Strict Convexity Implies Uniqueness) *If the objective function f in the convex problem* (2.5) *is strictly convex and an optimizer exists, then the optimizer is unique.*

Suppose we are given a convex optimization problem and a feasible point x. To assess whether or not x is optimal, we require a set of necessary and sufficient conditions for optimality. We turn our attention to such conditions next.

2.1.3 The Karush-Kuhn-Tucker Conditions

In an unconstrained, single-variable optimization problem such as

$$\underset{x \in \mathbb{R}}{\text{minimize}} \, f(x) \, ,$$

with $f : \mathbb{R} \to \mathbb{R}$ convex, the point $x^\star \in \mathbb{R}$ satisfies

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x^\star) = 0$$
(2.7)

if and only if $x^\star$ is a minimizer of f. For a convex multivariable constrained optimization problem such as (2.5), the derivative condition (2.7) is generalized to the *Karush-Kuhn-Tucker conditions*. The following theorem holds if a *Slater point* exists for the problem (2.5), a point $\bar{x} \in \mathbb{R}^n$ such that $A\bar{x} = b$ and $g_i(\bar{x}) < 0$ for all $i \in \{1, \ldots, n_{\mathrm{ic}}\}$. If a Slater point exists, then the problem (2.5) is said to satisfy *Slater's constraint qualification*. See [10, Section 5] for more details on the KKT conditions and constraint qualifications.

Theorem 2.1.2 (Karush-Kuhn-Tucker) *Assume a Slater point exists for the convex optimization problem* (2.5). *The point $x^\star \in \mathbb{R}^n$ is a global minimizer for the problem* (2.5)

if and only if there exist $\mu^\star \in \mathbb{R}^{n_{\mathrm{ec}}}$ *and* $\nu^\star \in \mathbb{R}^{n_{\mathrm{ic}}}$ *such that*

$$\mathbb{0} = \nabla f(x^\star) + A^\mathsf{T}\mu^\star + \sum_{k=1}^{m} \nu_i^\star \nabla g_i(x)$$

$$\nu_i^\star \geq 0\,, \quad i \in \{1, \ldots, n_{\mathrm{ec}}\}$$

$$\nu_i^\star g_i(x^\star) = 0\,, \quad i \in \{1, \ldots, n_{\mathrm{ic}}\}\,. \tag{2.8}$$

$\triangle$

Remark 2.1.3 (Dual Variables) *From the convex problem* (2.5), *one can define a second, closely-related convex problem known as the* dual problem *with decision variables called the* dual variables. *The vectors* $\mu^\star$ *and* $\nu^\star$ *of* (2.8) *are the optimal dual variables for the dual problem. We will not need to examine dual problems at any point in this thesis, but the interested reader is encouraged to consult [10, Chapter 5] for more.* $\triangle$

2.2   The Output Regulation Problem

This section recalls the basic problem setup and results for the global nonlinear output regulation problem from [61]. Our exposition is brief, and we refer the reader to [14, 36, 37, 61, 62] for detailed treatments.

Output regulation is a generalization of integral control to cases where disturbances and/or reference signals are time-varying. Consider a nonlinear plant

$$\dot{x} = f(x, u, w)\,, \qquad x(0) \in X := \mathbb{R}^n$$
$$y_{\mathrm{m}} = h_{\mathrm{m}}(x, u, w)\,, \tag{2.9}$$

where $x \in X$ is the state, $u \in U := \mathbb{R}^m$ is the control input, $y_{\mathrm{m}} \in \mathbb{R}^{p_{\mathrm{m}}}$ is the vector of available measurements, and $w \in \mathbb{R}^{n_w}$ is a set of exogenous inputs which might include disturbances to the plant dynamics, reference signals, or uncertain parameters. The function f is assumed to be locally Lipschitz in x and continuous in u and w, while h_{m} is assumed to be continuous; note for later that y_{m} may contain components of the input u. We define an *error signal* $e \in \mathbb{R}^p$ associated with the plant

$$e = h_e(x, u, w)\,, \tag{2.10}$$

consisting of variables which should be "protected" from the effects of the exogenous inputs and initial conditions. For example, e may be a vector of reference tracking errors, and should be driven to zero asymptotically using feedback control. The function h_e is assumed to be continuous. The class of exogenous inputs of interest is generated by the *exosystem*

$$\dot{w} = s(w), \qquad w(0) \in W \tag{2.11}$$

where s is locally Lipschitz and $W \subset \mathbb{R}^{n_w}$ is an open invariant set for the dynamics (2.11). Note that we can capture the effects of parametric uncertainty in the plant model (2.9) by including such parameters as components of w with static dynamics $\dot{w}_i = 0$. We denote the set of solutions of (2.11) by $\mathcal{I}_s(W)$, the corresponding ω-limit set by $\Omega(W)$, and assume that solutions of the exosystem (2.11) are bounded for all time $t \in \mathbb{R}$.

A general nonlinear feedback controller for (2.9) is given by

$$
\begin{aligned}
\dot{x}_{\mathrm{c}} &= f_{\mathrm{c}}(x_{\mathrm{c}}, y_{\mathrm{m}}), \qquad x_{\mathrm{c}}(0) \in X_{\mathrm{c}} := \mathbb{R}^{n_{\mathrm{c}}} \\
u &= h_{\mathrm{c}}(x_{\mathrm{c}}, y_{\mathrm{m}}),
\end{aligned}
\tag{2.12}
$$

which processes the measurements $y_{\mathrm{m}}(t)$ and produces the control signal $u(t)$ in closed-loop with the plant (2.9). The dynamics of the closed-loop system are described by (2.9) and (2.12), with (2.11) generating the exogenous input, i.e. the closed-loop system is given by the dynamics

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t), w(t)) \\
y_{\mathrm{m}}(t) &= h_{\mathrm{m}}(x(t), u(t), w(t)) \\
\dot{x}_{\mathrm{c}}(t) &= f_{\mathrm{c}}(x_{\mathrm{c}}(t), y_{\mathrm{m}}(t)) \\
u(t) &= h_{\mathrm{c}}(x_{\mathrm{c}}(t), y_{\mathrm{m}}(t)),
\end{aligned}
\tag{2.13}
$$

an exogenous input $w(\cdot)$ satisfying $\dot{w}(t) = s(w(t))$, and a set of initial conditions $(w(0), x(0), x_{\mathrm{c}}(0)) \in W \times X \times X_{\mathrm{c}}$. We will say that the closed-loop system is *well-posed* when the equations (2.13) yield a unique solution for $x(t)$, $y_{\mathrm{m}}(t)$, $x_{\mathrm{c}}(t)$, and $u(t)$, defined for all $t \geq 0$, for all exogenous inputs $w(\cdot) \in \mathcal{I}_s(W)$ and initial conditions $(x(0), x_{\mathrm{c}}(0)) \in X \times X_{\mathrm{c}}$.

The problem of *output regulation* is to design the feedback controller such that the closed-loop system satisfies a generalized stability criterion known as *global uniform convergence with a uniformly bounded steady-state (UBSS)* and such that the error signal $e(t)$ is driven to zero. We recall the definition of this generalized stability property (see [61] for more details) and then define the output regulation problem.

Definition 2.2.1 (Convergence of an Autonomous System) *Consider an autonomous dynamical system with state $z \in \mathbb{R}^d$*

$$\dot{z} = F(z, t). \tag{2.14}$$

We assume F is locally Lipschitz in z and piecewise continuous in t. The autonomous system (2.14) *is said to be* globally uniformly convergent *if there exists a solution $\bar{z}(t)$ such that $\bar{z}(t)$ is defined and bounded for all $t \in \mathbb{R}$ and $\bar{z}(t)$ is globally asymptotically stable for all solutions of* (2.14).

Definition 2.2.2 (Convergence of a System with Inputs) *Consider a dynamical system with state $z \in \mathbb{R}^d$ and input $w \in \mathbb{R}^m$*

$$\dot{z} = F(z, w). \tag{2.15}$$

The function F is assumed to be locally Lipschitz in z and continuous in w. The signal $w(t)$ is assumed to belong to a class of signals $\mathcal{W}$ which are piecewise continuous functions of time defined for all $t \in \mathbb{R}$.

The system (2.15) *is said to be* globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{W}$ *if the following hold:*

(i) *For each $w(\cdot) \in \mathcal{W}$, the system* (2.15) *is globally uniformly convergent;*

(ii) *For every $\rho > 0$ there exists an $R > 0$ such that if $w(\cdot)$ satisfies $\|w(t)\| \leq \rho$ for all $t \in \mathbb{R}$ then the corresponding steady-state solution $\bar{z}_w(t)$ of* (2.15) *with input $w(t)$ satisfies $\|\bar{z}_w(t)\| \leq R$ for all $t \in \mathbb{R}$.*

Problem 2.2.3 (Output Regulation) *For the plant* (2.9), *design, if possible, a dynamic feedback controller of the form* (2.12) *such that the closed-loop system* (2.13) *meets the following criteria:*

(i) well-posedness: *the closed-loop system is well-posed;*

(ii) global convergence: *the closed-loop system is globally uniformly convergent and satisfies the UBSS property for the class of inputs $\mathcal{I}_s(W)$;*

(iii) asymptotic error zeroing: *for every initial condition $(x(0), x_c(0)) \in X \times X_c$ of the closed-loop system and initial condition $w(0) \in W$ of the exosystem, the error signal* (2.10) *asymptotically tends to zero, i.e., $\lim_{t \to \infty} e(t) = \mathbb{0}_p$.*

$\triangle$

The following theorem is a basic necessary condition for the output regulation problem to be solvable [61, Lemma 4.13].

Theorem 2.2.4 (Regulator Equations) *The output regulation problem is solvable only if there exist continuous mappings* $\pi : \Omega(W) \to X$ *and* $\psi : \Omega(W) \to U$ *which satisfy the regulator equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi(w) = f(\pi(w), \psi(w), w)$$
$$\mathbb{0}_p = h_e(\pi(w), \psi(w), w) \tag{2.16}$$

for every solution of the exosystem $w = w(t)$ *satisfying* $w(t) \in \Omega(W)$ *for all* $t \in \mathbb{R}$. $\triangle$

The interpretation of Theorem 2.2.4 is that there must exist a steady-state feedforward control input $u(t) = \psi(w(t))$ with corresponding steady-state state trajectory $x(t) = \pi(w(t))$ such that the error $e = h_e(x, u, w)$ is held identically equal to zero. The set of controllers that solve the output regulation problem is described in the next theorem [61, Theorem 4.16].

Theorem 2.2.5 (Controller Conditions) *The output regulation problem is solved by a controller of the form* (A.4) *if and only if*

(i) *there exists a mapping* $\pi_\mathrm{c} : \Omega(W) \to X_\mathrm{c}$ *such that for some* π *and* ψ *satisfying the regulator equations* (2.16) *the mapping* π_c *satisfies the* generalized internal model principle

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi_\mathrm{c}(w) = f_\mathrm{c}(\pi_\mathrm{c}(w), h_\mathrm{m}(\pi(w), \psi(w), w))$$
$$\psi(w) = h_\mathrm{c}(\pi_\mathrm{c}(w), h_\mathrm{m}(\pi(w), \psi(w), w)) \tag{2.17}$$

for every solution of the exosystem $w = w(t)$ *satisfying* $w(t) \in \Omega(W)$ *for all* $t \in \mathbb{R}$;

(ii) *the closed-loop system corresponding to this controller is globally uniformly convergent with the UBSS property for the class of inputs* $\mathcal{I}_s(W)$. $\triangle$

Theorem 2.2.5 (i) states that that there must exist a steady-state trajectory $x_\mathrm{c}(t) = \pi_\mathrm{c}(w(t))$ of the controller which can produce the error-zeroing steady-state input $u(t) = \psi(w(t))$.

The typical controller solving the output regulation problem consists of two sub-systems: an *internal model* of the exosystem and a *stabilizer*; see [14, Equation (2.20)], [36, Equation (6.14)], [61, Equation (5.13)]. The internal model ensures the controller satisfies the generalized internal model property (2.17), and reduces the output regulation problem to the problem of (robustly) stabilizing the *augmented plant*, consisting of the internal model and the plant in either a series or feedback configuration. The stabilizer ensures that the closed-loop system satisfies the UBSS stability property. The output regulation problem is therefore solved using the following modular design strategy:

(i) design an internal model and construct the augmented plant;

(ii) design a stabilizer to ensure the closed-loop system meets a stability criterion (e.g., global uniform convergence).

Both of these steps are formidable in general, and methods for constructing internal models are described in [36]. The specific case of a static exosystem (constant disturbances and uncertain parameters), however, has the well-known solution of *integral control* when the error e is measurable.

Lemma 2.2.6 (Integral Control) *Suppose the exosystem* (2.11) *is static so that* $\dot{w} = \mathbb{0}_{n_w}$, *and suppose the error e is available for measurement. Consider the controller*

$$\dot{\eta} = e \tag{2.18a}$$
$$\dot{x}_\mathrm{s} = f_\mathrm{s}(x_\mathrm{s}, e, y_\mathrm{m}, \eta) \tag{2.18b}$$
$$u = h_\mathrm{s}(x_\mathrm{s}, e, y_\mathrm{m}, \eta) . \tag{2.18c}$$

If the closed-loop system with plant (2.9) *and controller* (2.18) *is well-posed and has a globally asymptotically stable equilibrium point for every* $w(0) \in W$, *then the regulator equations* (2.16) *are solvable and the controller* (2.18) *solves the output regulation problem.*

Proof: Since the closed-loop system (2.9) and (2.18) has a globally asymptotically stable equilibrium point for every $w(0) \in W$, it follows that there exists a unique solution $(\bar{x}, \bar{u})$ to

$$\mathbb{0}_n = f(\bar{x}, \bar{u}, w)$$
$$\mathbb{0}_p = e = h(\bar{x}, \bar{u}, w) \tag{2.19}$$

for every $w(0) \in W$ since $\dot{x} = \mathbb{0}_n$ and $\dot{\eta} = \mathbb{0}_p$ at an equilibrium point. The equations (2.19), however, are precisely the regulator equations (2.16) for a static exosystem; the maps π

and ψ are the maps from $w(0)$ to the vectors $\bar{x}$ and $\bar{u}$ solving (2.19) respectively. Hence, the regulator equations are solvable.

We now show that items (i) and (ii) of Theorem 2.2.5 are satisfied, proving the claim. Again, the existence of a unique equilibrium point for the closed-loop system implies that there exists a unique solution $(\bar{\eta}, \bar{x}_{\mathrm{s}})$ to

$$\mathbb{0}_{n_{\mathrm{s}}} = f_{\mathrm{s}}(\bar{x}_{\mathrm{s}}, \mathbb{0}_p, \bar{y}_{\mathrm{m}}, \bar{\eta})$$
$$\bar{u} = h_{\mathrm{s}}(\bar{x}_{\mathrm{s}}, \mathbb{0}_p, \bar{y}_{\mathrm{m}}, \bar{\eta})$$

for each $w(0) \in W$. Hence, item (i) of Theorem 2.2.5 is satisfied, with the map π_{c} mapping from $w(0)$ to the vector $(\bar{\eta}, \bar{x}_{\mathrm{s}})$. Furthermore, in the case of a static exosystem, global uniform convergence with the UBSS property is equivalent to the existence of a globally asymptotically stable equilibrium point for each $w(0) \in W$. Item (ii) of Theorem 2.2.5 is therefore satisfied by assumption. $\qquad\square$

The first component of the controller (2.18a) is the internal model (in this case, a pure integrator), while the second component (2.18b) and (2.18c) is the stabilizer.

2.3   Robust Control

The subfield of control known as *robust control* is concerned with questions of stability and performance for feedback systems in the presence of *nonlinearity*, *uncertainty*, or otherwise "troublesome" components that prevent us from applying traditional linear systems theory. See [25, 83] for detailed treatments of robust control.

In a robust control setting, we are interested in feedback connections of the form depicted in Figure 2.1. We have a *nominal system G*, which we assume to be linear and time-invariant, and a *troublesome system Δ*. We assume that we have some coarse description of the input-output behaviour of Δ, but otherwise the exact manner by which Δ maps input signals to output signals is unknown to us. We ask whether we can certify stability of the closed-loop system despite our limited knowledge of Δ.

In this thesis, the troublesome component Δ is related to the gradient of a convex function. This section on robust control will detail how to design a controller that guarantees closed-loop stability when such a nonlinearity is present in the feedback path.

We will first explore the idea of a "system" as a map between spaces of input and output signals. This perspective on systems leads naturally to the idea of "gain." Our
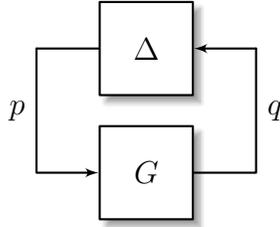
Figure 2.1: The standard robust stability problem: an LTI system G is connected in feedback with a "troublesome" component Δ.

main analysis tool, the small-gain theorem, gives a sufficient condition for stability of the feedback interconnection of two systems based on the systems' respective gains. We then show how to apply the small-gain theorem to our particular application (when Δ is the gradient of a convex function). Finally, we give an overview of our main synthesis tool, $\mathcal{H}_\infty$ controller design, which allows us to compute a controller that enforces the condition of the small-gain theorem.

2.3.1 The Signal Space $\mathscr{L}_2$

We will view systems as maps between vector spaces we call *signal spaces*. We will only be interested in the signal space $\mathscr{L}_2^n[0, \infty)$, the set of vector-valued functions square-integrable on the positive real line. For brevity of notation, we shall drop the time range $[0, \infty)$ from $\mathscr{L}_2^n[0, \infty)$. We also drop the superscript n if it can be inferred from context, and simply refer to the space $\mathscr{L}_2$.

The function $f : \mathbb{R}_+ \to \mathbb{R}^n$ is a member of $\mathscr{L}_2$ when

$$\int_0^\infty f(t)^\mathsf{T} f(t) \, \mathrm{d}t < \infty \,.$$

We interpret the integral in the above equation as a measure of the "energy" contained in the signal f (consider the case when f is a current signal, for instance). Then f being a member of $\mathscr{L}_2$ means that f is a signal carrying finite energy. This energy integral defines a *norm* on the vector space $\mathscr{L}_2$ by

$$\|f\| := \int_0^\infty f(t)^\mathsf{T} f(t) \, \mathrm{d}t \,.$$

Unstable systems do not necessarily produce outputs of finite energy when the inputs are of finite energy. We must therefore broaden the signal space $\mathscr{L}_2$ to accommodate such

systems. We define the *extended* space $\mathscr{L}_{2e}$, which contains all signals of finite energy over finite time intervals. Specifically, $f \in \mathscr{L}_{2e}$ if

$$\int_0^T f(t)^{\mathsf{T}} f(t) \, \mathrm{d}t < \infty \text{ for all } T \geq 0 \,.$$

The space $\mathscr{L}_{2e}$ contains such signals as e^t and $\sin(t)$, in addition to all the signals in $\mathscr{L}_2$ of finite energy.

2.3.2   Systems and $\mathscr{L}_2$ Gain

We consider systems as maps between $\mathscr{L}_{2e}$ spaces. A system G with n_p inputs and n_q outputs is taken to be a mapping $G : \mathscr{L}_{2e}^{n_p} \to \mathscr{L}_{2e}^{n_q}$, i.e. G takes input signals $p \in \mathscr{L}_{2e}^{n_p}$ and produces output signals $q \in \mathscr{L}_{2e}^{n_q}$. We denote this relationship by $q = G(p)$.

We will say that G is $\mathscr{L}_2$ *stable* if $p \in \mathscr{L}_2 \implies q \in \mathscr{L}_2$. That is, G is stable if an input of finite energy results in an output of finite energy. Furthermore, G is said to be $\mathscr{L}_2$ *stable with finite gain* if the greatest multiplicative increase in the output energy as compared to the input energy is finite, i.e. if the quantity

$$\sup \left\{ \frac{\|Gp\|}{\|p\|} \,\middle|\, p \in \mathscr{L}_2, p \neq 0 \right\}$$

is finite. We define the $\mathscr{L}_2$ *gain* $\|G\|_{\mathscr{L}_2}$ of a norm-bounded system G as

$$\|G\|_{\mathscr{L}_2} := \sup \left\{ \frac{\|Gp\|}{\|p\|} \,\middle|\, p \in \mathscr{L}_2, p \neq 0 \right\} \,.$$

The system $G : \mathscr{L}_{2e}^{n_p} \to \mathscr{L}_{2e}^{n_q}$ is *linear* if

$$G(\alpha p_1 + \beta p_2) = \alpha G(p_1) + \beta G(p_2)$$

for all $\alpha, \beta \in \mathbb{R}$ and $p_1, p_2 \in \mathscr{L}_{2e}^{n_p}$.

The system $G : \mathscr{L}_{2e}^{n_p} \to \mathscr{L}_{2e}^{n_q}$ is *time-invariant* if, for every input $p \in \mathscr{L}_{2e}^{n_p}$ producing output $q \in \mathscr{L}_{2e}^{n_q}$, the input $p(t-\tau)h(t-\tau)$ produces output $q(t-\tau)h(t-\tau)$ for any $\tau \geq 0$, where $h(\cdot)$ is the unit step function.

A linear, time-invariant system $G : \mathscr{L}_{2e}^{n_p} \to \mathscr{L}_{2e}^{n_q}$ admits a state-space representation

$$\begin{aligned} \dot{x} &= Ax + Bp, \quad x(0) = \mathbb{0}_n \\ q &= Cx + Dp \end{aligned} \tag{2.20}$$

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for some matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times m}$. We use the notation

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

to mean that the system G is linear and time-invariant, with a state-space representation given by the matrices A, B, C, and D. Linear, time-invariant systems are stable if their corresponding dynamics matrix A is *Hurwitz*, i.e. has all of its eigenvalues in the open left-half plane.

We can completely characterize the $\mathscr{L}_2$ gain of a stable LTI system via the system's state-space representation. This is summarized by the following lemma, whose proof may be found in [37, Theorem 3.1].

Lemma 2.3.1 ($\mathscr{L}_2$ Gain for LTI Systems) *For the LTI system*

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] ,$$

with A Hurwitz, the following are equivalent:

(i) *The $\mathscr{L}_2$ gain of G is less than γ;*

(ii) *There exists a positive definite symmetric matrix P such that*

$$\begin{bmatrix} A^\mathsf{T} P + P A & P B \\ B^\mathsf{T} P & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C & D \end{bmatrix}^\mathsf{T} \begin{bmatrix} C & D \end{bmatrix} \prec \mathbb{0} .$$

$\triangle$

Remark 2.3.2 (Linear Matrix Inequalities) *The inequality of item (ii) is known as a linear matrix inequality (LMI), and the question of the existence of a matrix P satisfying the inequality can be answered through* semidefinite programming *(SDP). For more on LMIs and SDPs, see [9, 27, 74].* $\triangle$

2.3.3   Sector-Bounded and Slope-Restricted Functions

In this thesis, the trouble-making component Δ will be a *static, sector-bounded nonlinearity.*

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ defines a system called a *static nonlinearity* $\Delta : \mathscr{L}_{2e}^n \to \mathscr{L}_{2e}^n$ with input q and output p through the simple relation

$$p(t) = \Delta(q(t)) = f(q(t)) \,.$$

A function f (equivalently, the static nonlinearity Δ) satisfying $f(0) = 0$ is said to be *in the sector* $[\alpha, \beta]$ with $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, when

$$(f(x) - \alpha x)^{\mathsf{T}} (\beta x - f(x)) \geq 0 \tag{2.21}$$

for every $x \in \mathbb{R}^n$. In the one-dimensional case ($n = 1$), we can interpret the condition (2.21) as saying that the graph of the function f lies between the straight lines αx and βx. As a shorthand notation, we will use $\Delta \in \sec[\alpha, \beta]$ to mean that Δ is a static nonlinearity in the sector $[\alpha, \beta]$.

A stronger condition than a sector bound is a *slope restriction.* A function $h : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *slope restricted to* $[\alpha, \beta]$ with $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$ when

$$\begin{bmatrix} x - y \\ h(x) - h(y) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} -2\alpha\beta I & (\alpha + \beta)I \\ (\alpha + \beta)I & -2I \end{bmatrix} \begin{bmatrix} x - y \\ h(x) - h(y) \end{bmatrix} \geq 0 \tag{2.22}$$

for all $x, y \in \mathbb{R}^n$. Note that if $h : \mathbb{R}^n \to \mathbb{R}^n$ is slope restricted to $[\alpha, \beta]$ then the function $\psi : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\psi(x) := h(x + \bar{x}) - h(\bar{x})$ is in the sector $[\alpha, \beta]$ for any $\bar{x} \in \mathbb{R}^n$.

The gradient of a convex function is slope restricted under certain conditions. A convex function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be *strongly convex with parameter* κ, or κ-*strongly convex*, if

$$g((1 - \lambda)x + \lambda y) \leq (1 - \lambda)g(x) + \lambda g(y) - \frac{1}{2}\kappa\lambda(1 - \lambda)\|x - y\|^2 \,,$$

where κ is a positive real number. Strong convexity is a stronger condition than strict convexity.

A function $h : \mathbb{R}^n \to \mathbb{R}^m$ is said to be (globally) *Lipschitz continuous with parameter* L, or L-*Lipschitz*, if

$$\|h(x) - h(y)\| \leq L\|x - y\|$$

for all $x, y \in \mathbb{R}^n$, where L is a positive real number.

We have the following lemma that relates the previous two definitions to a slope restriction on convex function gradients

Lemma 2.3.3 (Sector-Bounded Gradient) *If $g : \mathbb{R}^n \to \mathbb{R}$ is κ-strongly convex and $\nabla g : \mathbb{R}^n \to \mathbb{R}^n$ is L-Lipschitz then ∇g is slope restricted to $[\kappa, L]$.*

Proof: See [45, Section 3.3]. $\qquad\square$

The sector $[-1, 1]$ deserves special mention because we will focus on a related stability theorem shortly. If a static nonlinearity Δ is in the sector $[-1, 1]$, the $\mathscr{L}_2$ gain of Δ is less than one. We therefore call the sector $[-1, 1]$ the *small-gain sector*.

In the next section, we present a stability theorem for the interconnection of an LTI system and a static nonlinearity in the small-gain sector. We subsequently show how to extend these results to more general sectors.

2.3.4   The Small-Gain Theorem

The small-gain theorem is a stability condition for the interconnection of two (not necessarily linear and time-invariant) systems. We consider a special case of the small gain theorem for the interconnection of an LTI system and a static nonlinearity in the small gain sector.

Consider the feedback interconnection of the LTI system

$$
G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]
$$

with A Hurwitz and a static nonlinearity $\Delta \in \sec[-1, 1]$. The closed-loop system equations are

$$
\begin{aligned}
\dot{x} &= Ax + Bp \\
q &= Cx + Dp \\
p &= \Delta(q) \, .
\end{aligned}
\tag{2.23}
$$

Theorem 2.3.4 (Small Gain) *The origin $x = \mathbb{0}$ is globally asymptotically stable for the dynamics (2.23) if $\|G\|_{\mathscr{L}_2} < 1$.*

Proof: Since $\|G\|_{\mathscr{L}_2} < 1$, by Lemma 2.3.1 there exists a positive definite symmetric matrix P such that

$$
\begin{bmatrix} A^\mathsf{T} P + PA & PB \\ B^\mathsf{T} P & -I \end{bmatrix} + \begin{bmatrix} C & D \end{bmatrix}^\mathsf{T} \begin{bmatrix} C & D \end{bmatrix} \prec \mathbb{0} \, .
\tag{2.24}
$$

We claim that for any P satisfying (2.24),

$$V(x) = x^\mathsf{T} P x$$

is a Lyapunov function for the closed-loop system. Note that V is a positive definite function satisfying $V(\mathbb{0}) = 0$. It remains to show that $\dot{V}$ is a negative definite function.

If there exists a positive definite matrix P such that (2.24) holds, then there exists a $\varepsilon > 0$ such that

$$\begin{bmatrix} A^\mathsf{T} P + PA & PB \\ B^\mathsf{T} P & -I \end{bmatrix} + \begin{bmatrix} C & D \end{bmatrix}^\mathsf{T} \begin{bmatrix} C & D \end{bmatrix} \preceq -\varepsilon I \,. \tag{2.25}$$

Multiply (2.25) on the left and right by any $\operatorname{col}(x, p) \in \mathbb{R}^n \times \mathbb{R}^{n_p}$ to obtain

$$\begin{bmatrix} x \\ p \end{bmatrix}^\mathsf{T} \begin{bmatrix} A^\mathsf{T} P + PA & PB \\ B^\mathsf{T} P & -I \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} x \\ p \end{bmatrix}^\mathsf{T} \begin{bmatrix} C & D \end{bmatrix}^\mathsf{T} \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \leq -\varepsilon(\|x\|^2 + \|p\|^2)$$

$$x^\mathsf{T} A^\mathsf{T} P x + x^\mathsf{T} PAx + x^\mathsf{T} PBp + p^\mathsf{T} B^\mathsf{T} P x - p^\mathsf{T} p + (Cx + Dp)^\mathsf{T}(Cx + Dp) \leq -\varepsilon(\|x\|^2 + \|p\|^2)$$

$$(Ax + Bp)^\mathsf{T} P x + x^\mathsf{T} P(Ax + Bp) - p^\mathsf{T} p + (Cx + Dp)^\mathsf{T}(Cx + Dp) \leq -\varepsilon(\|x\|^2 + \|p\|^2) \,.$$

Using the system equations for G

$$\dot{x} = Ax + Bp$$
$$q = Cx + Dp \,,$$

we find

$$\dot{x}^\mathsf{T} P x + x^\mathsf{T} P \dot{x} - p^\mathsf{T} p + q^\mathsf{T} q \leq -\varepsilon(\|x\|^2 + \|p\|^2) \,. \tag{2.26}$$

We use the fact that

$$\dot{x}^\mathsf{T} P x + x^\mathsf{T} P \dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(x^\mathsf{T} P x\right) = \frac{\mathrm{d}}{\mathrm{d}t} V(x)$$

to rewrite (2.26) as

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x) - p^\mathsf{T} p + q^\mathsf{T} q \leq -\varepsilon(\|x\|^2 + \|p\|^2) \,. \tag{2.27}$$

Now, because $\Delta \in \sec[-1, 1]$, we have

$$(\Delta(q) + q)^\mathsf{T}(q - \Delta(q)) \geq 0$$

for all inputs q. In other words, p and q satisfy

$$(p + q)^\mathsf{T}(q - p) = -p^\mathsf{T} p + p^\mathsf{T} q - q^\mathsf{T} p + q^\mathsf{T} q = -p^\mathsf{T} p + q^\mathsf{T} q \geq 0 \,,$$

or

$$p^\mathsf{T} p - q^\mathsf{T} q \leq 0 \,.$$

Therefore, (2.27) implies

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x) \leq -\varepsilon \|x\|^2 \underbrace{-\varepsilon \|p\|^2}_{\leq 0} + \underbrace{p^\mathsf{T} p - q^\mathsf{T} q}_{\leq 0} \leq -\varepsilon \|x\|^2 \,.$$

Hence $\dot{V}$ is a negative definite function. The origin $x = \mathbb{0}$ is therefore globally asymptotically stable for the dynamics (2.23). $\qquad\square$

2.3.5   Loop Transformations

We aren't always handed a feedback system with Δ in the small-gain sector. As we saw in Section 2.3.3, κ-strongly convex functions with L-Lipschitz gradients are in the sector $[\kappa, L]$. Thankfully, the feedback interconnection of a static nonlinearity in a general sector $[\alpha, \beta]$ (with $\alpha, \beta \in \mathbb{R}$ and $\alpha \leq \beta$) and an LTI system may be converted to the form required by Theorem 2.3.4 using what is known as a *loop transformation*.

The idea is to perform manipulations on the feedback loop to transform the interconnection of the system G and nonlinearity $\Delta \in \sec[\alpha, \beta]$ to the interconnection of a new system G' and new nonlinearity $\Delta' \in \sec[-1, 1]$ in such a way that the resulting closed-loop system is equivalent to the original one. The loop transformation is depicted in Figure 2.2 and consists of the following steps:

(i) apply a negative feedforward of $\frac{\beta + \alpha}{2}$ to Δ and cancel this with a positive feedback of $\frac{\beta + \alpha}{2}$ to G;

(ii) place a gain of $\frac{2}{\beta - \alpha}$ at the output of the feedforward summing junction and cancel this with a gain of $\frac{\beta + \alpha}{2}$ at the input to the feedback summing junction.

Suppose G has the state-space representation

$$\dot{x} = Ax + Bp$$
$$q = Cx + Dp \,.$$
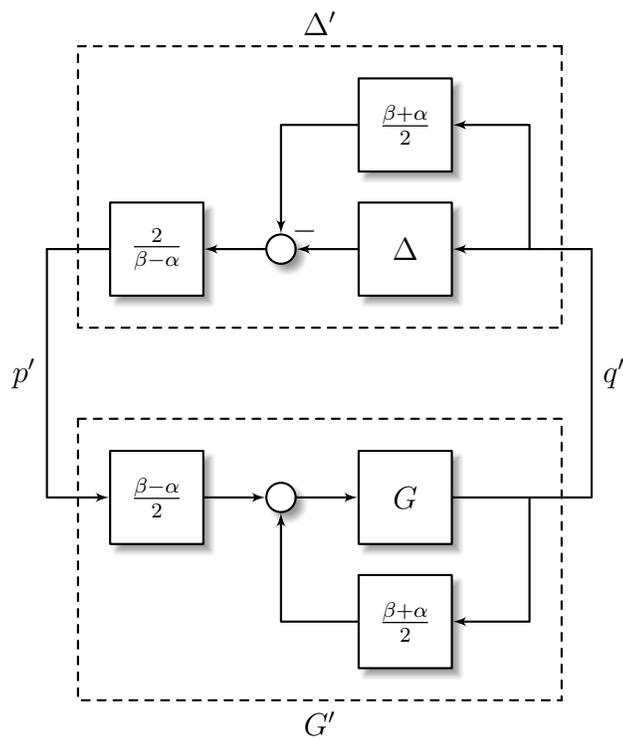
Figure 2.2: A loop transformation converts the nonlinearity $\Delta \in \sec[\alpha, \beta]$ to $\Delta' \in \sec[-1, 1]$, the small-gain sector. The LTI system G becomes the LTI system G'.

The uncertainty input p' of the loop-transformed system G' is related to the old input p and output q by

$$p = \frac{\beta + \alpha}{2} q + \frac{\beta - \alpha}{2} p'$$
$$= \frac{\beta + \alpha}{2} (Cx + Dp) + \frac{\beta - \alpha}{2} p',$$

which can be solved to obtain

$$p = \frac{\beta + \alpha}{2} \left(I - \frac{\beta + \alpha}{2} D \right)^{-1} Cx + \frac{\beta - \alpha}{2} \left(I - \frac{\beta + \alpha}{2} D \right)^{-1} p',$$

assuming the matrix $I - \frac{\beta+\alpha}{2} D$ is invertible. The system G' therefore has the state-space representation

$$\dot{x} = \left(A + \frac{\beta + \alpha}{2} \left(I - \frac{\beta + \alpha}{2} D \right)^{-1} C \right) x + \frac{\beta - \alpha}{2} Bp'$$

$$q' = \left(C + \frac{\beta + \alpha}{2} D \left(I - \frac{\beta + \alpha}{2} D \right)^{-1} C \right) x + \frac{\beta - \alpha}{2} D \left(I - \frac{\beta + \alpha}{2} D \right)^{-1} p'.$$

The next lemma follows trivially from the fact that a loop transformation is nothing more than a change-of-variables.

Proposition 2.3.5 (Loop-Transformed Equivalence) *The origin $x = \mathbb{0}$ is globally asymptotically stable for the dynamics*

$$\dot{x} = Ax + Bp$$
$$q = Cx + Dp$$
$$p = \Delta(q)$$

with $\Delta \in \sec[\alpha, \beta]$ if and only if the origin $x = \mathbb{0}$ is globally asymptotically stable for the loop-transformed dynamics

$$\dot{x} = \left(A + \frac{\beta + \alpha}{2} \left(I - \frac{\beta + \alpha}{2} D \right)^{-1} C \right) x + \frac{\beta - \alpha}{2} Bp'$$

$$q' = \left(C + \frac{\beta + \alpha}{2} D \left(I - \frac{\beta + \alpha}{2} D \right)^{-1} C \right) x + \frac{\beta - \alpha}{2} D \left(I - \frac{\beta + \alpha}{2} D \right)^{-1} p'$$

$$p' = \Delta'(q')$$

with $\Delta' \in \sec[-1, 1]$. $\triangle$

Proposition 2.3.5 tells us that we may apply the small-gain theorem to the loop-transformed system (i.e. check that the $\mathscr{L}_2$ gain of G' is less than one) to verify stability of the original feedback interconnection.

2.3.6 $\mathcal{H}_\infty$ Controller Synthesis

The small-gain theorem is an *analysis* theorem: we are simply given an LTI system G and asked whether it is stable when placed in feedback with a troublesome block $\Delta \in \sec[-1, 1]$. However, suppose G is a system with two channels: an *uncertainty channel* $p \to q$ with uncertainty input p and uncertainty output q (through which G is connected to Δ) and a *control channel* $u \to y$ with control inputs u and measured outputs y, as shown in Figure 2.3a. We might ask whether it is possible to design a feedback controller K for G that enforces the conditions of the small-gain theorem through the $p \to q$ channel. Attaching a nonlinearity $\Delta \in \sec[-1, 1]$ to the $p \to q$ channel would then result in a stable closed-loop system. The problem of designing a controller for the $u \to y$ channel to minimize the $\mathscr{L}_2$ gain of the $p \to q$ channel is a *synthesis* problem known as the $\mathcal{H}_\infty$ *control problem*[1], which we now describe.

We define the linear, time-invariant *augmented system* G with two inputs, p (uncertainty input) and u (control input), and two outputs, q (uncertainty output) and y (measured output).

$$\dot{x} = Ax + B_p p + B_u u$$
$$q = C_q x + D_{qp} p + D_{qu} u$$
$$y = C_y x + D_{yp} p\,.$$

or, in transfer function notation,

$$G = \left[\begin{array}{c|cc} A & B_p & B_u \\ \hline C_q & D_{qp} & D_{qu} \\ C_y & D_{yp} & \mathbb{0} \end{array} \right]. \tag{2.28}$$

Suppose we close the loop around the control input/measured output channel with a controller K given by

$$K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right]. \tag{2.29}$$

---

[1]The symbol $\mathcal{H}_\infty$, pronounced "aitch-infinity," refers to the $\mathcal{H}_\infty$ norm of a transfer function, which is equal to the corresponding system's $\mathscr{L}_2$ gain. See [25, Chapter 3] for more on $\mathcal{H}_\infty$ norms.

Figure 2.3: Diagram of the $\mathcal{H}_\infty$ control problem. Figure (a) shows the LTI system G with uncertainty channel $p \to q$ and control channel $u \to y$, through which we attach controller K. Figure (b) shows the same system after the loop through the control channel is closed, resulting in the system $G \star K$ with input p and output q.

This defines a new system with input p and output q which we denote by $G \star K$:

$$
G \star K = \left[\begin{array}{cc|c}
A + B_u D_K C_y & B_u C_K & B_p + B_u D_K D_{yp} \\
B_K C_y & A_K & B_K D_{yp} \\
\hline
C_q + D_{qu} D_K C_q & D_{qu} C_K & D_{qp} + D_{qu} D_K D_{yu}
\end{array} \right]. \tag{2.30}
$$

Closing the control loop is depicted in Figure 2.3b. We now state the $\mathcal{H}_\infty$ control problem.

Problem 2.3.6 ($\mathcal{H}_\infty$ Control) *For the LTI system* (2.28), *design a controller of the form* (2.29) *such that for the closed-loop system* (2.30), *the dynamics matrix*

$$
\begin{bmatrix}
A + B_u D_K C_y & B_u C_K \\
B_K C_y & A_K
\end{bmatrix}
$$

is Hurwitz and the $\mathscr{L}_2$ gain $\|G \star K\|_{\mathscr{L}_2}$ is minimized. $\triangle$

Two common approaches exist to solve the $\mathcal{H}_\infty$ control problem. The first approach is based on coupled Riccati equations, while the second formulates the problem in a linear matrix inequality (LMI) framework. The details of each approach are substantial and we will not discuss them here; we refer the reader to [23,24] for the Riccati equation approach and to [63,64] for the LMI approach. We note that the controller K obtained from each of these approaches is of the same order as the controlled plant. That is, if $A \in \mathbb{R}^{n \times n}$, then $A_K \in \mathbb{R}^{n \times n}$ also. The $\mathcal{H}_\infty$ synthesis procedure is therefore said to produce a controller of "full order."

2.3.7   Stabilizing Controller Design Procedure

We now have all the ingredients necessary to formulate a design procedure for a controller K that stabilizes the feedback interconnection of an LTI system G and a sector-bounded nonlinearity Δ, as depicted in Figure 2.4a. We will employ these procedures in our design of optimal steady-state controllers.

The first, and simpler, option is to fix a controller structure for K with tunable parameters using engineering judgment. We then close the control loop to obtain the feedback interconnection of $G \star K$ and Δ. After performing a loop transformation to shift Δ from the sector $[\alpha, \beta]$ to the small-gain sector $[-1, 1]$, we perform a grid search over the tunable parameters, applying Theorem 2.3.4 to find a combination that guarantees stability.

The second, more sophisticated, option is to use the $\mathcal{H}_\infty$ controller synthesis procedure of the preceding section as follows.

(i) Perform a loop transformation through the $p \to q$ channel to convert the upper interconnection of G and $\Delta \in \sec[\alpha, \beta]$ to the interconnection of G' and $\Delta' \in \sec[-1, 1]$ as described in Section 2.3.5.

(ii) Synthesize a controller K for the $u' \to y'$ channel to minimize the $\mathcal{L}_2$ gain of the $p' \to q'$ channel using the methods of Section 2.3.6.

(iii) Verify that the minimum achieved $\mathcal{L}_2$ gain of the $p' \to q'$ channel is less than one. If so, the origin of the original closed-loop system is globally asymptotically stable by Theorem 2.3.4 and Proposition 2.3.5.

These steps are illustrated in Figure 2.4.

The designer should attempt the first controller design procedure before employing the second. The former has the advantage of simplicity; it is possible that a low-order controller, such as a lead-lag or pure output feedback controller, is sufficient to stabilize the closed-loop system. The design procedure using $\mathcal{H}_\infty$ synthesis results in a high-order, fully dynamic controller, which will generally be more complex to implement. However, the second design procedure may yield a stabilizing controller when the first method fails in solving a difficult problems, such as when the LTI plant is unstable.
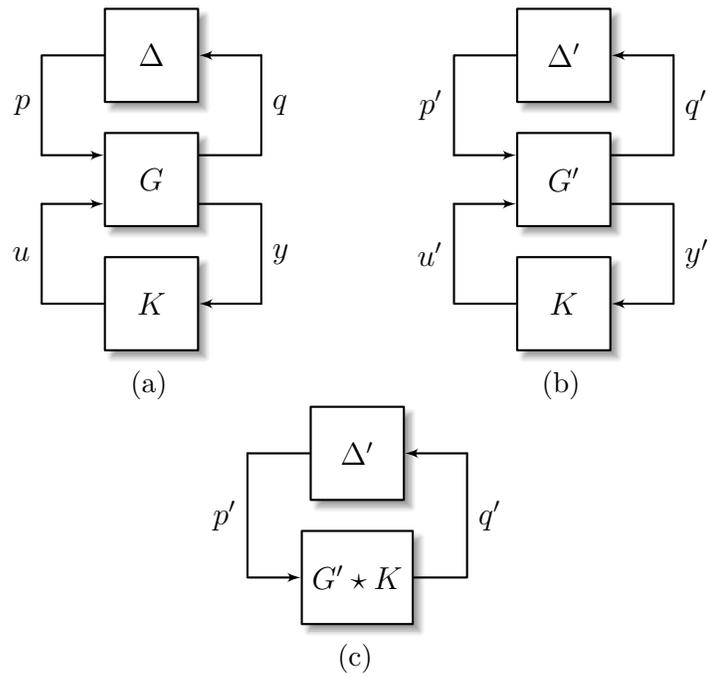
Figure 2.4: Diagram of the design procedure to stabilize the feedback interconnection of an LTI system G and sector-bounded nonlinearity $\Delta \in \sec[\alpha, \beta]$ using feedback controller K. The original system is depicted in (i), the loop-transformed system in (ii), and the system with closed control channel in (iii).

Chapter 3

Fundamentals

The reader should now be sufficiently prepared to understand the fundamentals of the optimal steady-state control problem. In this chapter, we formulate the basic problem statement, present necessary and sufficient conditions for solvability of the OSS control problem, and detail a general architecture for OSS controllers.

3.1 Problem Statement

Consider the general nonlinear plant (2.9) of the output regulation problem. Suppose that instead of trying to asymptotically zero an error signal, our objective is to design a feedback controller of the form (A.4) so that a specified subset of the control inputs and plant states are asymptotically driven to a cost-minimizing steady-state, determined by the solution of a constrained optimization problem. This objective should be achieved despite parametric uncertainty in the plant, and in the presence of unknown exogenous disturbances.

Formally, define the *optimization output* $y \in Y := \mathbb{R}^p$ as

$$y = h_{\mathrm{r}}(x, u, w) \,, \tag{3.1}$$

where h_{r} is a continuous function. The cost-minimizing steady-state is determined by an optimization problem with the optimization output y as the decision variable. Specifically, consider the following nonlinear optimization problem parameterized by $w \in W$ with

decision variable $y \in Y$,

$$\underset{y \in Y}{\text{minimize}} \quad g(y; w) \tag{3.2a}$$

$$\text{subject to} \quad l_i(y, w) = 0\,, \qquad\qquad i \in \{1, \ldots, n_{\text{ec}}\} \tag{3.2b}$$

$$\qquad\qquad k_j(y, w) \leq 0\,, \qquad\qquad j \in \{1, \ldots, n_{\text{ic}}\}\,. \tag{3.2c}$$

The *cost function* is $g : Y \times W \to \mathbb{R}$. The constraints (3.2b) and (3.2c) represent $n_{\text{ec}} \geq 0$ *engineering equality constraints* and $n_{\text{ic}} \geq 0$ *engineering inequality constraints* which should be satisfied in the desired steady-state. The steady-state optimization problem (3.2) is flexible enough to encompasses many situations of interest. To wit, the components of w included in the cost function (3.2a) could represent uncertain parameters, such as changing market prices. The engineering equality constraints (3.2b) might represent required setpoint tracking or balance conditions. The engineering inequality constraints (3.2c) can be used to ensure that states and inputs do not exceed their maximum continuous operation ratings.

For each $w \in W$, let

$$y^\star(w) \coloneqq \underset{y \in Y}{\text{argmin}} \left\{ g(y; w) \mid (3.2b) \text{ and } (3.2c) \text{ hold} \right\} \tag{3.3}$$

denote the *optimal solution function* of the problem (3.2). In general, the optimal solution function is set-valued, and its value for a particular w may be empty. Going forward we will assume that the optimization problem (3.2) has a unique minimizer for each $w \in W$, and hence $y^\star$ is a single-valued map. We further assume that $y^\star : W \to Y$ is continuous — this assumption is essential for the application of output regulation results to the optimal steady-state control problem.

Assumption 3.1.1 (Properties of $y^\star$) *The optimal solution function $y^\star : W \to Y$ defined in (3.3) is single-valued and continuous on W.* △

The goal in optimal steady-state control is to asymptotically guide the optimization output $y(t)$ to the time-varying optimizer $y^\star(w(t))$, where $w(t)$ is generated by the exosystem (2.11). We define the optimal steady-state control problem in the language of the output regulation problem, Problem 2.2.3, with error signal $e \coloneqq y - y^\star(w)$.

Problem 3.1.2 (Optimal Steady-State Control) *Design, if possible, a dynamic feedback controller of the form (A.4) for the nonlinear plant (2.9) such that the closed-loop system meets the following criteria:*

(i) well-posedness: *the closed-loop system is well-posed;*

(ii) global convergence: *the closed-loop system is globally uniformly convergent and satisfies the UBSS property for the class of inputs $\mathcal{I}_s(W)$;*

(iii) asymptotic optimality: *For every initial condition $(x(0), x_c(0)) \in X \times X_c$ of the closed-loop system and initial condition $w(0) \in W$ of the exosystem, the optimization output y is asymptotically brought into agreement with the optimizer*

$$\lim_{t \to \infty} (h_r(x(t), u(t), w(t)) - y^\star(w(t)) = \mathbb{0}_p \,.$$

$\triangle$

An important feature that distinguishes the OSS control problem from the standard output regulation problem is the absence of knowledge of the error signal: the optimal solution function $y^\star$ is generally unknown, and the time-varying optimizer $y^\star(w(t))$ depends on the unknown exogenous disturbance w. Technically, the statement of the output regulation problem does not require the error signal e to be measurable, but standard controller designs make such an assumption — again, see [14, Equation (2.20)], [36, Equation (6.14)], [61, Equation (5.13)]. This creates a new set of challenges for optimal steady-state control beyond the substantial challenges already present in the output regulation problem.

3.2   Solvability Conditions

Before outlining a general design strategy for the OSS control problem, we present solvability theorems that follow immediately from output regulation results. The optimal steady-state control problem is defined as the output regulation problem with continuous error signal $e = h_e(x, u, w) := h_r(x, u, w) - y^\star(w)$. The results of Section 2.2 therefore apply; in particular, we have the following necessary condition for solvability which follows from Theorem 2.2.4.

Theorem 3.2.1 (OSS Regulator Equations) *The OSS control problem is solvable only if there exist continuous mappings $\pi : \Omega(W) \to X$ and $\psi : \Omega(W) \to U$ that satisfy the OSS regulator equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi(w) = f(\pi(w), \psi(w), w)$$
$$y^\star(w) = h_r(\pi(w), \psi(w), w)$$

(3.4)

for every solution of the exosystem $w = w(t)$ satisfying $w(t) \in \Omega(W)$ for all $t \in \mathbb{R}$. $\triangle$

Theorem 3.2.1 for the optimal steady-state control problem admits a similar interpretation to Theorem 2.2.4 for the output regulation problem: for every exogenous input signal $w(t) \in \Omega(W)$, there must exist a control input $u(t) = \psi(w(t))$ that produces the state trajectory $x(t) = \pi(w(t))$ in the plant such that the optimization output is optimal, i.e., $y(t) = y^\star(w(t))$. An alternative interpretation is that (3.4) expresses *compatibility* between the set of all time-varying optimizers and the set of possible steady-state behaviours of the plant.

The necessary and sufficient conditions for a controller to solve the optimal steady-state control problem follow from Theorem 2.2.5.

Theorem 3.2.2 (OSS Controller Conditions) *The OSS control problem is solved by the controller* (A.4) *if and only if*

(i) *there exists a mapping $\pi_c : \Omega(W) \to X_c$ such that for some $\pi : \Omega(W) \to X$ and $\psi : \Omega(W) \to U$ satisfying the OSS regulator equations* (3.4) *the mapping π_c satisfies the* generalized internal model principle

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\pi_c(w) &= f_c(\pi_c(w), h_m(\pi(w), \psi(w), w)) \\
\psi(w) &= h_c(\pi_c(w), h_m(\pi(w), \psi(w), w))
\end{aligned}
\tag{3.5}
$$

for every solution of the exosystem $w = w(t)$ satisfying $w(t) \in \Omega(W)$ for all $t \in \mathbb{R}$;

(ii) *the closed-loop system corresponding to this controller is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(W)$.* $\triangle$

3.3 Design Architecture

We will now describe a framework for the design of optimal steady-state controllers that encompasses and generalizes many designs present in the literature. As we have already stated, the optimal steady-state control problem is more difficult to solve than the output regulation problem because the optimizer set is unknown and therefore the regulation error is not measurable. However, if we could produce a measurable proxy for the optimality error, then we could mirror the design techniques from the output regulation literature.

Recall that in the design of a controller to solve the output regulation problem, employing an internal model of the exosystem reduces the problem of output regulation to a problem

of robust stabilization. Inspired by this approach, we will introduce the idea of an *optimality model* to reduce the problem of optimal steady-state control to an output regulation problem *with a measurable error signal.*

An optimality model is a filter applied to the measured output of the plant that, when in steady-state, produces an output ϵ which is a proxy for the optimality error $e = y - y^\star(w)$. To make this idea precise, consider a filter with state $\xi \in \Xi := \mathbb{R}^{n_\xi}$, inputs y_m (plant measurements) and $v \in \mathcal{V} := \mathbb{R}^{n_v}$ (auxiliary stabilizing input), and output ϵ described by

$$
\begin{aligned}
\dot{\xi} &= \varphi(\xi, y_\mathrm{m}, v) \\
\epsilon &= h_\epsilon(\xi, y_\mathrm{m}, v) \,,
\end{aligned}
\tag{3.6}
$$

and define the *filtered plant* as the original plant (2.9) in cascade with the filter (3.6):

$$
\begin{array}{ll}
\dot{x} = f(x, u, w) & \dot{\xi} = \varphi(\xi, y_\mathrm{m}, v) \\
y_\mathrm{m} = h_\mathrm{m}(x, u, w) & \epsilon = h_\epsilon(\xi, y_\mathrm{m}, v) \\
& y_\mathrm{f} = \mathrm{col}(\xi, y_\mathrm{m}, \epsilon) \,.
\end{array}
\tag{3.7}
$$

The state of the filtered plant is $\mathrm{col}(x, \xi)$, the control inputs are u and v, the error output is ϵ, and the measured output is $y_\mathrm{f} = \mathrm{col}(\xi, y_\mathrm{m}, \epsilon)$ consisting of the filter state, original plant measurements, and error output. The purpose of the additional input v will become clear shortly. Consider now the steady-state behaviours of the filtered plant that would lead to our error proxy signal ϵ being identically zero. In other words, consider the solutions $(\pi, \pi_\xi, \psi_u, \psi_v) : \Omega(W) \to X \times \Xi \times U \times \mathcal{V}$ of the regulator equations for the filtered plant:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\pi(w) &= f(\pi(w), \psi_u(w), w) \,, \\
\frac{\mathrm{d}}{\mathrm{d}t}\pi_\xi(w) &= \varphi(\pi_\xi(w), h_\mathrm{m}(\pi(w), \psi_u(w), w), \psi_v(w)) \,, \\
\mathbb{0} &= h_\epsilon(\pi_\xi(w), h_\mathrm{m}(\pi(w), \psi_u(w), w), \psi_v(w)) \,.
\end{aligned}
\tag{3.8}
$$

This leads us to the definition of an optimality model.

Definition 3.3.1 (Optimality Model) *The filter* (3.6) *is said to be an* optimality model *(for the OSS control problem, Problem 3.1.2) if the following implication holds: if the quadruple $(\pi, \pi_\xi, \psi_u, \psi_v)$ is a solution of the regulator equations* (3.8) *for the filtered plant* (3.7), *then the pair (π, ψ_u) satisfies the OSS regulator equations* (3.4). $\triangle$

An optimality model encodes sufficient conditions for optimality during steady-state operation with the plant. For instance, given a convex optimization problem where strong duality holds, the optimality model might encode the KKT conditions when it is in dynamic steady-state with the plant; we explore this case further in Section 4.2. Just as knowledge of an internal model can be used to reduce the output regulation problem to a robust stabilization problem, an optimality model can be used to reduce the optimal steady-state control problem to an output regulation problem with measurable error, as the following theorem shows.

Theorem 3.3.2 (Reduction of OSS to Output Regulation) *Suppose that the filter* (3.6) *is an optimality model for the OSS control problem (Problem 3.1.2), and consider the filtered plant* (3.7). *If the controller*

$$
\begin{aligned}
\dot{x}_\mathrm{c} &= f_\mathrm{c}(x_\mathrm{c}, \xi, y_\mathrm{m}, \epsilon) \\
u &= h_\mathrm{c}^u(x_\mathrm{c}, \xi, y_\mathrm{m}, \epsilon) \\
v &= h_\mathrm{c}^v(x_\mathrm{c}, \xi, y_\mathrm{m}, \epsilon)
\end{aligned}
\tag{3.9}
$$

solves the output regulation problem for the filtered plant (3.7) *with error signal ϵ, then the controller*

$$
\begin{aligned}
\dot{\xi} &= \varphi(\xi, y_\mathrm{m}, v) \\
\dot{x}_\mathrm{c} &= f_\mathrm{c}(x_\mathrm{c}, \xi, y_\mathrm{m}, h_\epsilon(\xi, y_\mathrm{m}, v)) \\
u &= h_\mathrm{c}^u(x_\mathrm{c}, \xi, y_\mathrm{m}, h_\epsilon(\xi, y_\mathrm{m}, v)) \\
v &= h_\mathrm{c}^v(x_\mathrm{c}, \xi, y_\mathrm{m}, h_\epsilon(\xi, y_\mathrm{m}, v))
\end{aligned}
\tag{3.10}
$$

solves the optimal steady-state control problem.

Proof: Suppose the controller (3.9) solves the output regulation problem for the filtered plant (3.7). One consequence of Theorem 2.2.5 is that there must exist a solution $(\pi, \pi_\xi, \psi_u, \psi_v)$ to the regulator equations for the filtered plant (3.8). Define $y_\mathrm{m}(w) \coloneqq h_\mathrm{m}(\pi(w), \psi_u(w), w)$ to be the steady-state plant measurements and $y_\mathrm{f}(w) \coloneqq \mathrm{col}(\pi_\xi(w), y_\mathrm{m}(w), \mathbb{0})$ to be the steady-state output of the filtered plant — note that $\epsilon = \mathbb{0}$ since $(\pi, \pi_\xi, \psi_u, \psi_v)$ solve the regulator equations for the filtered plant. Also by Theorem 2.2.5, we can conclude there exists a mapping π_c such that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\pi_\mathrm{c}(w) &= f_\mathrm{c}(\pi_\mathrm{c}(w), y_\mathrm{f}(w)) \\
\psi_u(w) &= h_\mathrm{c}^u(\pi_\mathrm{c}(w), y_\mathrm{f}(w)) \\
\psi_v(w) &= h_\mathrm{c}^v(\pi_\mathrm{c}(w), y_\mathrm{f}(w))
\end{aligned}
$$

and the closed-loop system corresponding to this controller is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(W)$.

Define the OSS controller with state $\mathrm{col}(\xi, x_\mathrm{c})$ and dynamics given by (3.10). Since the filter is an optimality model and $(\pi, \pi_\xi, \psi_u, \psi_v)$ solve the regulator equations for the filtered plant, it follows from Definition 3.3.1 that the pair (π, ψ_u) solve the OSS regulator equations (3.4). Therefore, by Theorem 3.2.2, there exist mappings π_ξ, π_c such that for the solution (π, ψ_u) of the OSS regulator equations, the OSS controller satisfies

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\pi_\xi(w) &= \varphi(\pi_\xi(w), y_\mathrm{m}(w), h_\mathrm{c}^v(\pi_\mathrm{c}(w), y_\mathrm{f}(w))) \\
\frac{\mathrm{d}}{\mathrm{d}t}\pi_\mathrm{c}(w) &= f_\mathrm{c}(\pi_\mathrm{c}(w), y_\mathrm{f}(w)) \\
\psi_u(w) &= h_\mathrm{c}^u(\pi_\mathrm{c}(w), y_\mathrm{f}(w))
\end{aligned}
$$

and the closed-loop system corresponding to this controller is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(W)$. Employing Theorem 2.2.5 in the other direction, we conclude the OSS controller solves the optimal steady-state control problem. $\square$

Based on Theorem 3.3.2, we obtain the following modular design strategy for solving the OSS control problem:

(i) design an optimality model and construct the filtered plant;

(ii) design a controller that solves the output regulation problem for the filtered plant.

A controller solving the OSS control problem will therefore typically consist of three cascaded subsystems: an optimality model, an internal model of the exosystem, and a stabilizer. See Figure 3.1 for a diagram of this proposed scheme. The purpose of the auxiliary input v to the optimality model can now be made clear: it provides additional inputs for stabilization of the closed-loop system.
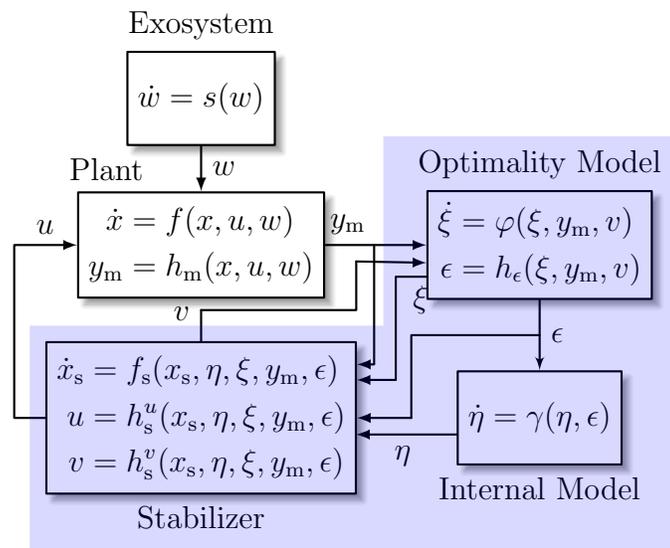
Figure 3.1: A general architecture for OSS controllers. The plant and optimality model are placed in cascade to form the filtered plant. The internal model and stabilizer solve the output regulation problem for the filtered plant with error signal ϵ. The overall controller is contained in the shaded blue region.

Chapter 4

The Linear-Convex Case [1]

For an arbitrary nonlinear plant, exosystem, and optimization problem, the OSS control problem is likely intractable. The remainder of this paper focuses in detail on the important case of a LTI plant with constant parametric uncertainty, a static exosystem, and a convex steady-state optimization problem. We call this case *linear-convex optimal steady-state control*, and leverage the results of Chapter 3 to provide a complete solution to this problem. In Section 4.1, we consider the fundamentals of linear-convex OSS control, before presenting a constructive controller design procedure in Section 4.2.

4.1 Linear-Convex Fundamentals

4.1.1 Uncertain LTI Plant

We specialize the nonlinear plant (2.9) and the output to be optimized (3.1) to the case of linear time-invariant dynamics with structured parametric uncertainty in the system matrices:

$$
\begin{aligned}
\dot{x} &= A(\delta)x + B(\delta)u + B_w(\delta)w, \qquad x(0) \in X \\
y &= C(\delta)x + D(\delta)u + Q(\delta)w, \\
y_{\mathrm{m}} &= C_{\mathrm{m}}(\delta)x + D_{\mathrm{m}}(\delta)u + Q_{\mathrm{m}}(\delta)w.
\end{aligned}
\tag{4.1}
$$

---

[1]The contents of this chapter will be incorporated into a publication: L. S. P. Lawrence, J. W. Simpson-Porco, and E. Mallada, "The Linear-Convex Optimal Steady-State Control Problem," to be submitted to IEEE Transactions on Automatic Control.

Note the difference in notation from Chapter 3, as we have split the exogenous input into two components, w and δ. The vector $w \in \mathbb{R}^{n_w}$ models exogenous disturbances and setpoints, while $\delta \in \boldsymbol{\delta} \subset \mathbb{R}^{n_\delta}$ is a vector that characterizes structured parametric model uncertainty and belongs to a set $\boldsymbol{\delta}$ containing the origin ($\delta = \mathbb{0}$ is the nominal model). All matrices are assumed to be continuous functions of $\delta \in \boldsymbol{\delta}$. We assume the corresponding exosystems are static:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} w \\ \delta \end{bmatrix} = \begin{bmatrix} \mathbb{0}_{n_w} \\ \mathbb{0}_{n_\delta} \end{bmatrix}, \quad w(0) \in \mathbb{R}^{n_w}, \quad \delta(0) \in \boldsymbol{\delta}, \tag{4.2}$$

which yields constant signals w and δ.

4.1.2 Robust Steady-State Optimization Problem

Recall from Theorem 3.2.1 that a necessary condition for solvability of the OSS control problem is that the OSS regulator equations (3.4) have a solution; that is, there must exist a steady-state operation of the plant that yields the optimal output. We can either *assume* that the OSS regulator equations have a solution, or we can constrain our optimization problem to *guarantee* that solutions exist. We opt for the latter strategy by embedding the steady-state operation constraint into the steady-state optimization problem.

Consider the equilibrium outputs $\bar{y}$ that can be generated from (4.1) by an equilibrium state $\bar{x}$ and input $\bar{u}$:

$$\begin{aligned} \mathbb{0}_n &= A(\delta)\bar{x} + B(\delta)\bar{u} + B_w(\delta)w \\ \bar{y} &= C(\delta)\bar{x} + D(\delta)\bar{u} + Q(\delta)w. \end{aligned} \tag{4.3}$$

We define the set-valued mapping $\overline{Y} : W \times \boldsymbol{\delta} \rightrightarrows Y$ so that $\overline{Y}(w, \delta)$ is the set of all such achievable equilibrium optimization outputs $\bar{y}$ for fixed values of w and δ:

$$\begin{aligned} \overline{Y}(w, \delta) &\coloneqq \{\bar{y} \in Y \mid \text{there exists an } (\bar{x}, \bar{u}) \text{ such that} \\ &\qquad (\bar{x}, \bar{u}, \bar{y}) \text{ satisfy } (4.3)\}. \end{aligned} \tag{4.4}$$

For each (w, δ), the set $\overline{Y}(w, \delta)$ is an affine subset of Y, which we assume is nonempty.[2] We shall include $\bar{y} \in \overline{Y}(w, \delta)$ as a constraint of the steady-state optimization problem to ensure compatibility between the optimizers and steady-state operation of the plant, thereby ensuring solvability of the OSS regulator equations (Theorem 3.2.1).

---

[2]Equivalently, we assume that $\mathbf{range}\, B_w(\delta) \subseteq \mathbf{range} \begin{bmatrix} A(\delta) & B(\delta) \end{bmatrix}$ for all $\delta \in \boldsymbol{\delta}$.

The cost-minimizing equilibrium point is determined by the convex optimization problem

$$\underset{y \in Y}{\text{minimize}} \quad g(y; w) \tag{4.5a}$$

$$\text{subject to} \quad y \in \overline{Y}(w, \delta) \tag{4.5b}$$

$$Hy = Lw \tag{4.5c}$$

$$k_i(y, w) \leq 0, \quad i \in \{1, \ldots, n_{\text{ic}}\} \tag{4.5d}$$

in which $g : Y \times W \to \mathbb{R}$ is assumed to be a continuous function of all of its arguments, and differentiable and convex in y for each w. The constraint (4.5b) is the steady-state constraint just discussed. The constraints (4.5c) and (4.5d) represent n_{ec} engineering equality constraints and n_{ic} engineering inequality constraints which should be satisfied in the desired steady-state. To ensure the optimization problem is convex, the engineering equality constraints must be linear and the functions $k_i : Y \times W \to \mathbb{R}$ of the engineering inequality constraints must be convex in y for each w. The matrices H, L and functions k_i are part of the design specifications, and are therefore not subject to parametric uncertainty.

Proceeding from (3.3), as before $y^\star : W \times \boldsymbol{\delta} \to Y$ is the optimal solution function of (4.5), and we assume $y^\star$ satisfies Assumption 3.1.1.[3] If the objective function g is strictly convex, then we may simply assume that an optimizer exists for each (w, δ); uniqueness of the optimizer then follows from Lemma 2.1.1. We further assume that a strictly feasible point exists for the optimization problem (4.5).

Assumption 4.1.1 (Constraints Strictly Feasible) *There exists a point $\tilde{y} \in Y$ that satisfies $\tilde{y} \in \overline{Y}(w, \delta)$, $H\tilde{y} = Lw$, and $k_i(\tilde{y}, w) < 0$ for all $i \in \{1, \ldots, n_{\text{ic}}\}$.*

The existence of a strictly feasible point ensures *Slater's constraint qualification* holds, and therefore guarantees that the *Karush-Kuhn-Tucker (KKT) conditions* are necessary and sufficient for optimality [10, Sections 5.2.3 and 5.5.3].

The optimality condition associated with the constraint $y \in \overline{Y}(w, \delta)$ involves the unique subspace associated with the affine set $\overline{Y}(w, \delta)$. Recall that any affine set in Euclidean space can be written as the sum of a (unique) subspace and a (non-unique) "offset" vector.

---

[3]Unlike in the general case, we can verify that Assumption 3.1.1 holds under certain conditions for convex problems like (4.5). Continuity of $y^\star$ follows from properties of feasible region, while single-valuedness of $y^\star(w, \delta)$ follows if $g(\cdot, w)$ is strictly convex for each w. If $g(\cdot; w)$ is not strictly convex, one may perturb the objective function to enforce strict convexity without changing the optimal value significantly. See Appendix B for more details.

It follows from the definition of $\overline{Y}$ that for each (w, δ), there exists an offset vector $\mathsf{y}(w, \delta)$ and a *unique* subspace $V(\delta)$ such that

$$\overline{Y}(w, \delta) = \mathsf{y}(w, \delta) + V(\delta) . \tag{4.6}$$

For each (w, δ), the optimal solution $y^\star$ is characterized as the unique vector such that there exists $\mu^\star \in \mathbb{R}^{n_{\mathrm{ec}}}$ and $\nu^\star \in \mathbb{R}^{n_{\mathrm{ic}}}$ such that $(y^\star, \mu^\star, \nu^\star)$ satisfy the *KKT conditions*

$$\nabla g(y^\star, w) + H^\mathsf{T} \mu^\star + \sum_{i=1}^{n_{\mathrm{ic}}} \nu_i^\star \nabla k_i(y, w) \perp V(\delta) \tag{4.7a}$$

$$0 = \nu_i^\star k_i(y^\star, w) , \quad \nu_i^\star \geq 0 , \quad i \in \{1, \ldots, n_{\mathrm{ic}}\} \tag{4.7b}$$

along with the primal feasibility conditions (4.5b)–(4.5d). The gradient condition (4.7a) is written in a non-standard manner in terms of the subspace $V(\delta)$, which can be interpreted as the subspace of first-order feasible variations for the affine constraint $y \in \overline{Y}(w, \delta)$ — see [6, Section 3.1] for details.

We draw the reader's attention to a second, equivalent, way to write the gradient condition (4.7a). There exists a $(y^\star, \mu^\star, \nu^\star)$ satisfying (4.7) if and only if there exists a $(y^\star, \nu^\star)$ satisfying

$$\nabla g(y^\star, w) + \sum_{i=1}^{n_{\mathrm{ic}}} \nu_i^\star \nabla k_i(y^\star, w) \perp (V(\delta) \cap \mathbf{null}\, H) \tag{4.8a}$$

$$0 = \nu_i^\star k_i(y^\star, w) , \quad \nu_i^\star \geq 0 , \quad i \in \{1, \ldots, n_{\mathrm{ic}}\} \tag{4.8b}$$

Going forward, we will make use of both formulations (4.7) and (4.8) when appropriate.

Remark 4.1.2 (Comments on Linear-Convex OSS Formulation) *The assumption that H, L in (4.5c) are free of parametric uncertainty can be relaxed without much difficulty. One could relax the assumption of differentiability of g by using subgradients or proximal operator methods, as in [16]; we do not pursue this here.* △

4.1.3 Linear-Convex OSS Regulator Equations

Recall that solvability of the OSS regulator equations (3.4) is necessary for the solvability of the OSS control problem. The inclusion of the equilibrium constraint (4.5b) in the optimization problem ensures this necessary condition is met.

Proposition 4.1.3 (Solvability of OSS Regulator Equations) *For the linear-convex OSS control problem with plant* (4.1), *exosystem* (4.2), *and convex optimization problem* (4.5), *there exist functions* $\pi : W \times \boldsymbol{\delta} \to X$ *and* $\psi : W \times \boldsymbol{\delta} \to U$ *satisfying the OSS regulator equations* (3.4). $\triangle$

Proof: We consider whether there exist functions π and ψ satisfying (3.4) for the LTI dynamics (4.1) with exosystem (4.2). That is, we consider solutions to

$$
\begin{aligned}
\mathbb{0}_n &= A(\delta)\pi(w,\delta) + B(\delta)\psi(w,\delta) + B_w(\delta)w \\
y^\star(w,\delta) &= C(\delta)\pi(w,\delta) + D(\delta)\psi(w,\delta) + Q(\delta)w \, .
\end{aligned}
\tag{4.9}
$$

Since, by the constraints of the optimization problem, $y^\star(w,\delta) \in \overline{Y}(w,\delta)$, it follows from (4.3) and (4.4) that the mappings π and ψ exist. $\square$

Remark 4.1.4 (Necessity of Steady-State Constraints) *Failing to include the steady-state constraints* (4.5b) *in the optimization problem* (4.5) *can result in an instance of the OSS control problem in which* $y^\star(w,\delta) \notin \overline{Y}(w,\delta)$ *for some* (w,δ). *That is, the optimizer of* (4.5) *might be* inconsistent *with steady-state operation of the plant* (4.1) *for some* (w,δ). *In this case, the OSS regulator equations* (3.4) *will fail to have globally defined solutions, and the OSS control problem will be insolvable.* $\triangle$

4.2 Constructive Solutions

This section presents constructive solutions to the linear-convex OSS control problem outlined in Section 4.1.

4.2.1 Robust Subspaces

Following the design strategy outlined in Section 3.3, we must construct an optimality model for the linear-convex OSS control problem, and then design a controller solving the output regulation problem for the series interconnection of the LTI plant and the optimality model (Theorem 3.3.2). A major roadblock to constructing optimality models is the presence of parametric uncertainty, and we therefore devote significant effort in this subsection to studying it. Indeed, consider the KKT conditions (4.7) or (4.8), and notice that both
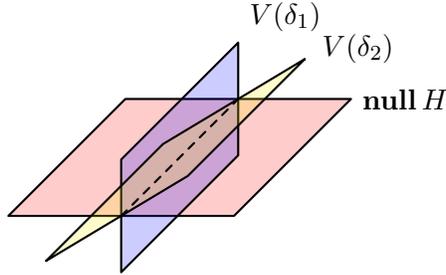
Figure 4.1: An illustration of the robust feasible subspace property. For two distinct values of uncertainty δ_1 and δ_2, the subspaces $V(\delta_1)$ and $V(\delta_2)$ are different. However, the intersected subspaces $V(\delta_1) \cap \mathbf{null}\,H$ and $V(\delta_2) \cap \mathbf{null}\,H$, shown by the dashed line, are equal.

involve the subspace $V(\delta)$, which depends on the uncertain parameters δ. It is therefore impossible for our controller to incorporate the gradient condition within an optimality model without knowledge of δ *unless $V(\delta)$ or $V(\delta) \cap \mathbf{null}\,H$ is, in fact, independent of δ.* While we cannot expect such a result to hold for an arbitrary uncertain LTI plant, this uncertainty-independence always holds when $V(\delta) = \mathbb{R}^p$ for all δ. What's more, this property *may* hold even when $V(\delta)$ is a strict subset of $\mathbb{R}^p$ for all δ, provided that the manner in which the uncertainty enters our model possesses some structure.[4]

We now explore the means by which one can verify the robustness (i.e., uncertainty independence) of these subspaces to parametric uncertainty, and some of the consequences. Our first definition makes precise the notion of $V(\delta) \cap \mathbf{null}\,H$ being independent of δ.

Definition 4.2.1 (Robust Feasible Subspace (RFS)) *Let $V(\delta)$ be the unique subspace associated with $\overline{Y}(w, \delta)$ as in (4.6). The optimization problem (4.5) is said to satisfy the robust feasible subspace (RFS) property if there exists a fixed $l \in \mathbb{N}$ and a fixed set of vectors $\{v_1, v_2, \ldots, v_l\} \subset \mathbb{R}^p$ such that $V(\delta) \cap \mathbf{null}\,H = \mathrm{span}(v_1, v_2, \ldots, v_l)$ for all $\delta \in \boldsymbol{\delta}$.* $\triangle$

The robust feasible subspace property is illustrated in Figure 4.1. In this example, one can visualize the subspace $V(\delta)$ rotating along the dashed-line axis as δ changes in value. So long as the subspaces $V(\delta)$ and $\mathbf{null}\,H$ are never equal, there exists a fixed basis, independent of δ, for the subspace $V(\delta) \cap \mathbf{null}\,H$. In this case, the basis consists of one vector in the direction of the dashed line.

A sufficient condition for $V(\delta) \cap \mathbf{null}\,H$ to be independent of δ is that $V(\delta)$ is itself independent of δ; this leads to our second definition.

---

[4]We show that this special structure exists in, for example, power system models in Section 5.2.

Definition 4.2.2 (Robust Output Subspace (ROS)) *The uncertain LTI plant* (4.1) *is said to satisfy the* robust output subspace (ROS) property *if there exists a fixed* $l \in \mathbb{N}$ *and a fixed set of vectors* $\{v_1, v_2, \ldots, v_l\} \subset \mathbb{R}^p$ *such that* $V(\delta) = \mathrm{span}(v_1, v_2, \ldots, v_l)$ *for all* $\delta \in \boldsymbol{\delta}$. △

We next discuss how to verify the RFS and ROS properties algebraically through the construction of certain matrices whose column vectors form a basis of the subspace $V(\delta)$ or $V(\delta) \cap \mathbf{null}\, H$. These matrices will play a key role in defining optimality models for the linear-convex OSS control problem. We first present the construction of a matrix whose range is $V(\delta)$, which will be useful for assessing both the RFS and ROS properties.

Lemma 4.2.3 (Construction of $V(\delta)$) *Let* $N_{AB}(\delta)$ *be any matrix such that* $\mathbf{range}\, N_{AB}(\delta) = \mathbf{null} \begin{bmatrix} A(\delta) & B(\delta) \end{bmatrix}$. *Then the columns of the matrix*

$$R(\delta) := \begin{bmatrix} C(\delta) & D(\delta) \end{bmatrix} N_{AB}(\delta) \tag{4.10}$$

span the subspace $V(\delta)$, *and hence* $V(\delta) = \mathbf{range}\, R(\delta)$.

Proof: We can view the affine space $\overline{Y}(w, \delta)$ from (4.4) as being constructed by a two-step process. In the first step, we examine the steady-state solutions $(\bar{x}, \bar{u})$ to

$$A(\delta)\bar{x} + B(\delta)\bar{u} + B_w(\delta)w = \mathbb{0}_n. \tag{4.11}$$

In the second step, we compute each corresponding output as $\bar{y} = C(\delta)\bar{x} + D(\delta)\bar{u} + Q(\delta)w$ and place $\bar{y}$ into $\overline{Y}$. Let the set of solutions to the linear equations (4.11) be denoted $\mathsf{L} \subset \mathbb{R}^n \times \mathbb{R}^m$. The set L is affine, and therefore can be written as the sum of a subspace and an offset vector. Fix a particular solution $(\tilde{x}, \tilde{u})$ to (4.11), and note that

$$\mathsf{L} = (\tilde{x}, \tilde{u}) + \mathbf{null} \begin{bmatrix} A(\delta) & B(\delta) \end{bmatrix}.$$

The set $\overline{Y}$ can then be written as

$$\begin{aligned}
\overline{Y}(w, \delta) &= \{ C(\delta)\bar{x} + D(\delta)\bar{u} + Q(\delta)w \mid (\bar{x}, \bar{u}) \in \mathsf{L} \} \\
&= \underbrace{C(\delta)\tilde{x} + D(\delta)\tilde{u} + Q(\delta)w}_{\mathsf{y}(w,\delta)} + \\
&\quad \underbrace{\begin{bmatrix} C(\delta) & D(\delta) \end{bmatrix} \left(\mathbf{null} \begin{bmatrix} A(\delta) & B(\delta) \end{bmatrix} \right)}_{V(\delta)}.
\end{aligned}$$

It follows that the construction described in the statement of the lemma indeed yields the subspace $V(\delta)$. $\qquad\square$

We now consider how to verify when the robust feasible subspace property holds. For $\delta \in \boldsymbol{\delta}$, let $R_\perp(\delta)$ be a matrix satisfying $\mathbf{null}\, R_\perp(\delta) = V(\delta)$.[5] We then have

$$\begin{aligned}
V(\delta) \cap \mathbf{null}\, H &= \mathbf{null}\, R_\perp(\delta) \cap \mathbf{null}\, H \\
&= \mathbf{null} \begin{bmatrix} R_\perp(\delta) \\ H \end{bmatrix}.
\end{aligned} \qquad (4.12)$$

If the null space of the last line of (4.12) has a basis independent of δ, then the robust feasible subspace property holds. These observations lead us to the following result.

Proposition 4.2.4 (Algebraic Characterization of RFS Property) *Let $R_\perp(\delta)$ be any matrix satisfying* $\mathbf{null}\, R_\perp(\delta) = V(\delta)$ *for all δ. The optimization problem* (4.5) *satisfies the robust feasible subspace property if and only if there exists a fixed matrix T_0 such that*

$$\mathbf{range}\, T_0 = \mathbf{null} \begin{bmatrix} R_\perp(\delta) \\ H \end{bmatrix} \qquad (4.13)$$

for all $\delta \in \boldsymbol{\delta}$. $\qquad\triangle$

Proof: First suppose the optimization problem (4.5) satisfies the RFS property. Then there exists an $l \in \mathbb{N}$ and a set of vectors $\{v_1, v_2, \ldots, v_l\} \subset \mathbb{R}^p$ such that $V(\delta) \cap \mathbf{null}\, H = \mathrm{span}(v_1, v_2, \ldots, v_l)$ for all δ. It follows that $T_0 := \begin{bmatrix} v_1 & v_2 & \ldots & v_l \end{bmatrix}$ satisfies

$$\mathbf{range}\, T_0 = V(\delta) \cap \mathbf{null}\, H = \mathbf{null} \begin{bmatrix} R_\perp(\delta) \\ H \end{bmatrix}$$

for all δ. Conversely, suppose there exists a matrix T_0 satisfying (4.13) for all δ. Then $V(\delta) \cap \mathbf{null}\, H = \mathbf{range}\, T_0$ for all δ, hence the column vectors of T_0 span $V(\delta) \cap \mathbf{null}\, H$ for all δ, and therefore the problem (4.5) satisfies the ROS property. $\qquad\square$

---

[5] One can either construct $R_\perp(\delta)$ from $R(\delta)$ by requiring that $R_\perp(\delta)R(\delta) = \mathbb{0}$ and $\begin{bmatrix} R(\delta) & R_\perp^\mathsf{T}(\delta) \end{bmatrix}$ is full rank, or one can use a more direct procedure. First, construct a matrix $\Gamma(\delta)$ such that

$$\mathbf{range}\, \Gamma(\delta) = \mathbf{null} \begin{bmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta) \end{bmatrix}^\mathsf{T}, \qquad \text{for all } \delta \in \boldsymbol{\delta}.$$

Then, partition $\Gamma(\delta)^\mathsf{T}$ as $\Gamma(\delta)^\mathsf{T} = \begin{bmatrix} X(\delta) & Z(\delta) \end{bmatrix}$ where $X(\delta)$ has n columns and $Z(\delta)$ has p columns. One can show that $V(\delta) = \mathbf{null}\, Z(\delta)$ and therefore one may use $R_\perp(\delta) := Z(\delta)$ in Proposition 4.2.4.

We now have an algebraic characterization of the ROS property — the existence of a matrix T_0 satisfying (4.13). We can make an analogous statement for the robust output subspace property. The proof of the following proposition is essentially identical to the proof of Proposition 4.2.4.

Proposition 4.2.5 (Algebraic Characterization of ROS Property) *Let $R(\delta)$ be the matrix defined in Lemma 4.2.3. The LTI plant (4.1) satisfies the robust output subspace property if and only if there exists a fixed matrix R_0 such that $\mathbf{range}\, R_0 = \mathbf{range}\, R(\delta)$ for all $\delta \in \boldsymbol{\delta}$.*

The next result gives a strong sufficient condition for the ROS property.

Proposition 4.2.6 (Robust Full Rank Implies ROS) *The LTI system (4.1) satisfies the ROS property if*

$$\mathbf{rank} \begin{bmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta) \end{bmatrix} = n + p, \qquad \text{for all } \delta \in \boldsymbol{\delta}. \tag{4.14}$$

Moreover,

(i) *the requirements of Proposition 4.2.5 are satisfied by the matrix $R_0 := I_p$;*

(ii) *the requirements of Proposition 4.2.4 are satisfied by any matrix T_0 such that $\mathbf{range}\, T_0 = \mathbf{null}\, H$.*

Proof: If (4.14) holds, then $\overline{Y}(w, \delta) = \mathbb{R}^p$ for all (w, δ), and therefore $V(\delta) = \mathbb{R}^p$ for all δ. The standard basis vectors $\{e_1, e_2, \ldots, e_p\} \subset \mathbb{R}^p$ therefore satisfy $V(\delta) = \mathrm{span}(e_1, e_2, \ldots, e_p)$ for all δ, and hence the plant satisfies the ROS property. Since $\mathbf{range}\, R(\delta) = V(\delta) = \mathbb{R}^p$ for all $\delta \in \boldsymbol{\delta}$, one may indeed take $R_0 = I_p$ in Proposition 4.2.5. Finally, considering Proposition 4.2.4, the only valid choice of the matrix $R_\perp(\delta)$ such that $\mathbf{null}\, R_\perp(\delta) = V(\delta)$ for all $\delta \in \boldsymbol{\delta}$ is $R_\perp(\delta) := \mathbb{0}$. Hence the matrix T_0 can be selected as any matrix satisfying $\mathbf{range}\, T_0 = \mathbf{null} \begin{bmatrix} \mathbb{0} \\ H \end{bmatrix} = \mathbf{null}\, H.$ $\qquad\qquad\square$

Note that (4.14) can hold only when the number of outputs to be optimized is less than or equal to the number of control inputs. The rank condition (4.14) of Proposition 4.2.6 — sometimes referred to as the *non-resonance condition* [37, Lemma 4.1] — is a
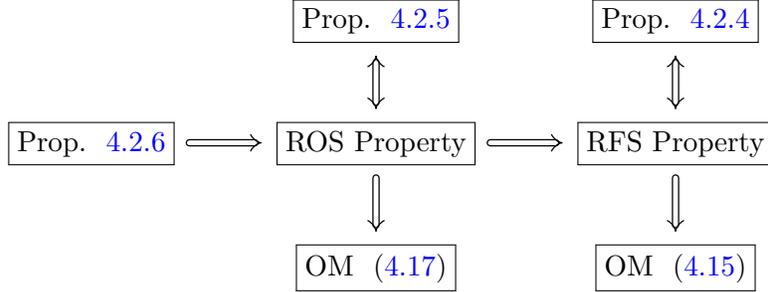
Figure 4.2: Relationships between robust subspace results and optimality models.

standard assumption of the linear output regulation problem with constant disturbances. We emphasize that (4.14) is only a sufficient condition for the ROS property, which itself is merely sufficient for the RFS property. The relationships between these conditions, and the optimality models of the following section, are summarized in Figure 4.2.

Remark 4.2.7 (Enforcing the RFS Property) *Two potential remedies exist if the RFS property fails to hold for an instance of OSS control. First, the designer can consider adding additional engineering equality constraints (rows of H) to ensure the subspace* **null** $\begin{bmatrix} R_\perp(\delta) \\ H \end{bmatrix}$ *of Proposition 4.2.4 is independent of δ. This can be accomplished by observing the manner in which the uncertain parameters δ enter the matrix $R_\perp(\delta)$ and adding rows of H accordingly. Second, the designer can consider* changing *the selection of system variables to be optimized, i.e. changing the definition of y, to modify the matrices $C(\delta)$ and $D(\delta)$ and hence the matrix $R(\delta)$ of Lemma 4.2.3. This modification might consist of* reducing *the number of optimization outputs p to a value less than or equal to the number of control inputs m, to make it possible for the full-rank condition of Proposition 4.2.6, and therefore the ROS property, to hold.* △

4.2.2 Optimality Models for Linear-Convex OSS Control

We now consider the construction of optimality models for linear-convex OSS control. The options available to us depend on which of the two previously-defined subspace robustness properties hold. For simplicity, we omit the auxiliary stabilizing input v from consideration in the remainder of our discussion.

We assume the constraint violations $Hy - Lw$ and $k_i(y, w)$, and objective function gradient, $\nabla g(y; w)$, are available for feedback, in that they are either directly measurable

or can be calculated using measurements. Incorporating the inequality constraints and associated dual variable conditions relies on the following lemma, which is straightforward to prove by checking every sign combination.

Lemma 4.2.8 *For real numbers a and b, the pair (a, b) satisfies $a = \max(a + b, 0)$ if and only if $a \geq 0$, $b \leq 0$, and $ab = 0$.* $\triangle$

It follows that, for each $i \in \{1, \ldots, n_{\mathrm{ic}}\}$, the conditions $\nu_i \geq 0$, $k_i(y, w) \leq 0$, and $\nu_i k_i(y, w) = 0$ are equivalent to $\nu_i = \max(\nu_i + k_i(y, w), 0)$. In compact notation, we write $\nu = \mathbf{max}(\nu + k(y, w), \mathbb{0})$, where $\mathbf{max}$ evaluates max elementwise and $k(y, w) := \mathrm{col}(k_1(y, w), \ldots, k_{n_{\mathrm{ic}}}(y, w))$.

Proposition 4.2.9 (Robust Feasible Subspace Optimality Model) *Suppose the optimization problem (4.5) satisfies the robust feasible subspace property, and let T_0 be any matrix satisfying the statement of Proposition 4.2.4. Then*

$$
\begin{aligned}
\dot{\nu} &= \mathbf{max}(\nu + k(y, w), \mathbb{0}) - \nu \\
\epsilon &= \begin{bmatrix} Hy - Lw \\ T_0^{\mathsf{T}} \left(\nabla g(y; w) + \sum_{i=1}^{n_{\mathrm{ic}}} \nu_i \nabla k_i(y, w) \right) \end{bmatrix}
\end{aligned}
\tag{4.15}
$$

is an optimality model for the linear-convex OSS control problem.

Proof: For each (w, δ), consider the solutions $(\bar{x}, \bar{\mu}, \bar{\nu}, \bar{u})$ of the regulator equations for the LTI plant (4.1) and optimality model (4.15) in series:

$$
\begin{align}
\mathbb{0} &= A(\delta)\bar{x} + B(\delta)\bar{u} + B_w(\delta)w \tag{4.16a} \\
\bar{y} &= C(\delta)\bar{x} + D(\delta)\bar{u} + Q(\delta)w \tag{4.16b} \\
\mathbb{0} &= \mathbf{max}(\bar{\nu} + k(\bar{y}, w), \mathbb{0}) - \bar{\nu} \tag{4.16c} \\
\mathbb{0} &= H\bar{y} - Lw \tag{4.16d} \\
\mathbb{0} &= T_0^{\mathsf{T}} \left(\nabla g(\bar{y}, w) + \sum_{i=1}^{n_{\mathrm{ic}}} \bar{\nu}_i \nabla k_i(\bar{y}, w) \right). \tag{4.16e}
\end{align}
$$

All time derivatives on the left-hand side of the regulator equations (4.16) are zero since the exosystem is static. We show that the regulator equations are equivalent to the KKT conditions (4.8). The first two equations (4.16a) and (4.16b) imply $\bar{y} \in \overline{Y}(w, \delta)$. The equation (4.16d) is the engineering equality constraint. The engineering inequality constraints

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and remaining KKT conditions (4.8) are encoded by (4.16c) and (4.16e). Since the KKT conditions are sufficient for optimality, the following implication holds for all (w, δ): if $(\bar{x}, \bar{\mu}, \bar{\nu}, \bar{u})$ satisfy the regulator equations (4.16), then $(\bar{x}, \bar{u})$ satisfy the linear-convex OSS regulator equations (4.9). The filter (4.15) satisfies the criterion of Definition 3.3.1, and is therefore an optimality model. $\qquad\square$

The robust feasible subspace optimality model (RFS-OM) may be employed whenever the RFS property holds. If, furthermore, the ROS property holds, then we have a second option: the robust output subspace optimality model (ROS-OM).

Proposition 4.2.10 (Robust Output Subspace Optimality Model) *Suppose the plant* (4.1) *satisfies the robust output subspace property, and let R_0 be any matrix satisfying the statement of Proposition 4.2.4. Then*

$$
\begin{aligned}
\dot{\mu} &= Hy - Lw \\
\dot{\nu} &= \mathbf{max}(\nu + k(y, w), \mathbb{0}) - \nu \\
\epsilon &= R_0^{\mathsf{T}} \left(\nabla g(y; w) + H^{\mathsf{T}} \mu + \sum\nolimits_{i=1}^{n_{\mathrm{ic}}} \nu_i \nabla k_i(y, w) \right)
\end{aligned}
\tag{4.17}
$$

is an optimality model for the linear-convex OSS control problem.

Proof: The proof is almost identical to the proof of Proposition 4.2.9, except that we compare the gradient condition to (4.7a) instead of (4.8a). $\qquad\square$

If the robust output subspace property holds, then we are free to employ either (4.15) or (4.17) as our optimality model.[6] In each optimality model, the choice of the matrix T_0 or R_0 provides a great deal of design flexibility. When combined with different controller design options solving the output regulation problem for the filtered plant, this gives a huge variety of design options for synthesizing OSS controllers.

4.2.3 Linear-Convex OSS Controller

Suppose that we have constructed an optimality model for our linear-convex OSS control problem, perhaps using one of the two optimality models of the previous section. We

---

[6]In fact, even more variations are possible by considering other equivalent formulations of the KKT conditions and developing appropriate robust subspace notions; for brevity we omit the details here.

represent the optimality model generically by

$$\dot{\xi} = \varphi(\xi, y_{\mathrm{m}})$$
$$\epsilon = h_\epsilon(\xi, y_{\mathrm{m}}) \,.$$

We need to construct a controller that solves the output regulation problem for the filtered plant comprising the LTI plant and optimality model in series:

$$\dot{x} = A(\delta)x + B(\delta)u + B_w(\delta)w$$
$$y_{\mathrm{m}} = C_{\mathrm{m}}(\delta)x + D_{\mathrm{m}}(\delta)u + Q_{\mathrm{m}}(\delta)w$$
$$\dot{\xi} = \varphi(\xi, y_{\mathrm{m}})$$
$$\epsilon = h_\epsilon(\xi, y_{\mathrm{m}})$$

(4.18)

Observe that the filtered plant is a nonlinear system subject to constant exogenous inputs with a measurable error ϵ — Lemma 2.2.6 immediately yields the solution. We must employ an integrator and stabilizer to solve the output regulation problem for the filtered plant (4.18):

$$\dot{\eta} = \epsilon$$
$$\dot{x}_{\mathrm{s}} = f_{\mathrm{s}}(x_{\mathrm{s}}, y_{\mathrm{m}}, \xi, \eta, \epsilon)$$
$$u = h_{\mathrm{s}}(x_{\mathrm{s}}, y_{\mathrm{m}}, \xi, \eta, \epsilon) \,.$$

(4.19)

If the closed-loop system is well-posed and stable, the linear-convex OSS control problem is solved.

Proposition 4.2.11 (Linear-Convex OSS Controller) *Let (φ, h_ϵ) be an optimality model for the linear-convex OSS control problem. If the stabilizer $(f_{\mathrm{s}}, h_{\mathrm{s}})$ is designed such that the closed-loop system of the filtered plant (4.18) and controller (4.19) in feedback is well-posed and has a globally asymptotically stable equilibrium point for all (w, δ), then the controller*

$$\dot{\xi} = \varphi(\xi, y_{\mathrm{m}})$$
$$\dot{\eta} = h_\epsilon(\xi, y_{\mathrm{m}})$$
$$\dot{x}_{\mathrm{s}} = f_{\mathrm{s}}(x_{\mathrm{s}}, y_{\mathrm{m}}, \xi, \eta, h_\epsilon(\xi, y_{\mathrm{m}}))$$
$$u = h_{\mathrm{s}}(x_{\mathrm{s}}, y_{\mathrm{m}}, \xi, \eta, h_\epsilon(\xi, y_{\mathrm{m}}))$$

(4.20)

solves the linear-convex optimal steady-state control problem. △

4.2.4 Stabilizer Design

Using an optimality model and a bank of integrators, Proposition 4.2.11 tells us that we have reduced the linear-convex OSS control problem to a stabilization problem. If we restrict our attention to linear time-invariant stabilizers and if we employ the optimality model of Proposition 4.2.9 or Proposition 4.2.10, the only nonlinearity in the closed-loop system is the static, slope-restricted nonlinearity $\nabla g(\cdot, w)$. The map $y \mapsto \nabla g(y; w)$ is slope-restricted to $[\kappa, L]$ if $y \mapsto g(y; w)$ is κ-strongly convex and $y \mapsto \nabla g(y; w)$ is L-Lipschitz continuous.

After centering the closed-loop system equations about an equilibrium point, the only nonlinearity present is in the sector $[\kappa, L]$. As a result, the controller design procedures of Section 2.3.7 apply. We can either propose a simple stabilizer structure, such as a PI controller, and search for stabilizing gains using the small-gain theorem; otherwise, we can synthesize a full-order dynamic controller using $\mathcal{H}_\infty$ synthesis techniques. We illustrate these procedures on concrete examples in the next chapter.

Chapter 5

Illustrative Examples

In this section, we consider several illustrative examples of the concepts discussed in the previous portions of the thesis. We first examine two academic examples that demonstrate the application of the small-gain theorem and $\mathcal{H}_\infty$ controller synthesis to the construction of the stabilizer. Subsequently, we apply the OSS control framework to power system models. We show that we are able to recover several standard controller designs from the literature to solve the optimal frequency regulation problem.

5.1   Stabilization Examples[1]

We demonstrate the construction of the stabilizer assuming the simplest possible form for the OSS control problem. We assume no parametric uncertainty and no engineering constraints. This allows us to illustrate the fundamental tools for stabilizer design with a minimum of clutter.

We suppose we are interested in regulating the output y of the LTI system

$$\dot{x} = Ax + Bu + B_w w$$
$$y = Cx + Du$$
$$y_{\mathrm{m}} = y \,.$$

---

[1]Some of the contents of this section appear in L. S. P. Lawrence, Z. E. Nelson, E. Mallada, and J. W. Simpson-Porco "Optimal Steady-State Control for Linear Time-Invariant Systems", in 2018 IEEE Conference on Decision and Control (CDC), pages 3251–3257.

to the solution of the equilibrium-constrained problem

$$\begin{aligned}
\underset{y \in Y}{\text{minimize}} \quad & g(y; w) \\
\text{subject to} \quad & y \in \overline{Y}(w) \,,
\end{aligned} \tag{5.1}$$

where $g : Y \times W \to \mathbb{R}$ is κ-strongly convex and $\nabla g(y; w)$ is L-Lipschitz in y for each w. Since we assume no parametric uncertainty, the equilibrium output map $\overline{Y}$ depends only upon w, and not (w, δ).

5.1.1 Analysis of Proportional-Integral Stabilizer via the Small-Gain Theorem

Suppose we wish to keep the design of the stabilizer as simple as possible, and use a *proportional-integral (PI)* controller structure. The plant satisfies the robust output subspace property since no parametric uncertainty is present. We may therefore apply the ROS optimality model

$$\epsilon = R_0^{\mathsf{T}} \nabla g(y; w) \,,$$

with the matrix R_0 constructed as in Lemma 4.2.3.

The closed-loop system equations are

$$\begin{aligned}
\dot{x} &= Ax + Bu + B_w w & \text{(5.2a)} \\
y &= Cx + Du & \text{(5.2b)} \\
\dot{\eta} &= \epsilon = R_0^{\mathsf{T}} \nabla g(y; w) & \text{(5.2c)} \\
u &= -K_P \epsilon - K_I \eta \,. & \text{(5.2d)}
\end{aligned}$$

The closed-loop system (5.2) possesses a unique equilibrium point under mild assumptions.

Proposition 5.1.1 (Unique Equilibrium Point) *Suppose $y \mapsto g(y; w)$ is strictly convex,*

$$\mathbf{rank} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = n + m \,, \tag{5.3}$$

and K_I is invertible. Then for each $w \in \mathbb{R}^{n_w}$ the closed-loop system (5.2) has a unique equilibrium point.

Proof: The equilibria $(\bar{x}, \bar{\eta})$ of (5.2) satisfy

$$\mathbb{0} = A\bar{x} + B\bar{u} + B_w w \tag{5.4a}$$

$$\bar{y} = C\bar{x} + D\bar{u} \tag{5.4b}$$

$$\mathbb{0} = R_0^\mathsf{T} \nabla g(\bar{y}; w) \tag{5.4c}$$

$$\bar{u} = -K_I \bar{\eta}. \tag{5.4d}$$

Since $R_0^\mathsf{T} \nabla g(y; w)$ is an optimality model, (5.4a)–(5.4c) imply that $\bar{y}$ is optimal. Strict convexity of $y \mapsto g(y; w)$ then implies this $\bar{y}$ is unique. By the rank assumption (5.3), the state-input pair $(\bar{x}, \bar{u})$ is unique. Finally, invertibility of K_I implies $\bar{\eta}$. $\qquad\square$

Assuming the requirements of Proposition 5.1.1 hold, we denote the unique equilibrium values by $x^\star, u^\star, y^\star$, and so on. We centre the system equations about this equilibrium point by making the change-of-variables $\tilde{x} := x - x^\star$, $\tilde{u} = u - u^\star$, $\tilde{y} = y - y^\star$, etc. and extract the nonlinearity ∇g to obtain a feedback system in the standard form of the robust control problem with the troublesome block $\Delta \in \sec[\kappa, L]$ (given by $\Delta(y) := \nabla g(y + y^\star; w) - \nabla g(y^\star; w)$) and LTI system G given by

$$G = \left[\begin{array}{cc|c} A & -K_I & -BK_P R_0^\mathsf{T} \\ \mathbb{0} & \mathbb{0} & R_0^\mathsf{T} \\ \hline C & -DK_I & -DK_P R_0^\mathsf{T} \end{array} \right].$$

We may then apply Theorem 2.3.4 and Proposition 2.3.5 to assess the stability of the closed-loop system.

We take as the plant matrices

$$A := \begin{bmatrix} -1 & -4 & -1 & 3 \\ 1 & -4 & -1 & -3 \\ -1 & 4 & -1 & -9 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \qquad\qquad B := \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

$$C := \begin{bmatrix} 1 & -1 & 0 & -4 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad\qquad D := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The matrix B_w is irrelevant for this example. The matrix A has four stable eigenvalues at $\{-2, -2 + 2i, -2 - 2i, -4\}$. The state matrices satisfy

$$\mathbf{rank} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = n + m.$$

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Notice that the optimization output y comprises two measurements from the state of the system as well as the control input. The matrix R_0 as constructed in Lemma 4.2.3 is given by

$$R_0^\mathsf{T} := \begin{bmatrix} -0.7474 & -1.1626 & -0.6644 \end{bmatrix} .$$

We suppose we are interested in assessing stability for *any* objective function g which is κ-strongly convex with $\kappa := 1/9$ and whose gradient ∇g is L-Lipschitz continuous with $L := 1$.

Using `CVX` in `MATLAB` to verify the $\mathscr{L}_2$ gain of the loop-transformed system G' (as detailed in Section 2.3.7), we find the closed-loop system to be stable for all 100 gain combinations $(K_P, K_I) \in \{0.2, 0.4, \ldots, 1.8, 2\}^2$. (Note that K_I is invertible for each of these gain combinations.) That is, employing a PI controller as the stabilizer with any of these gain combinations results in a controller (5.2c) and (5.2d) that solves the OSS control problem.

5.1.2 $\mathcal{H}_\infty$ Synthesis of Full-Order Dynamic Controller

A simple PI controller will not always work to stabilize the closed-loop system. Suppose we are interested in OSS control for the *unstable* plant

$$A := \begin{bmatrix} -1 & -4 & -1 & 3 \\ 1 & -4 & -1 & -3 \\ -1 & 4 & -1 & -9 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad B := \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

$$C := \begin{bmatrix} 1 & -1 & 0 & -4 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad D := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The matrix A has eigenvalues $\{-2, -2+2i, -2-2i, 1\}$, three stable and one unstable. We assume the objective function is strongly convex with parameter $\kappa := 1$ and has a Lipschitz continuous gradient with parameter $L := 2$.

Using a PI controller as the stabilizer, we attempted stability verification using the methods of Section 2.3.7 for the 49 gain combinations $(K_P, K_I) \in \{10^{-3}, 10^{-2}, \ldots, 10^2, 10^3\}^2$. The LMI solver failed in each case, and simulations further suggest the PI controller is incapable of stabilizing the closed-loop system.

By contrast, we can employ the $\mathcal{H}_\infty$ synthesis procedure of Section 2.3.7 to design a full-order dynamic stabilizer. Using the `hinfsyn` function in `MATLAB`, we were able to
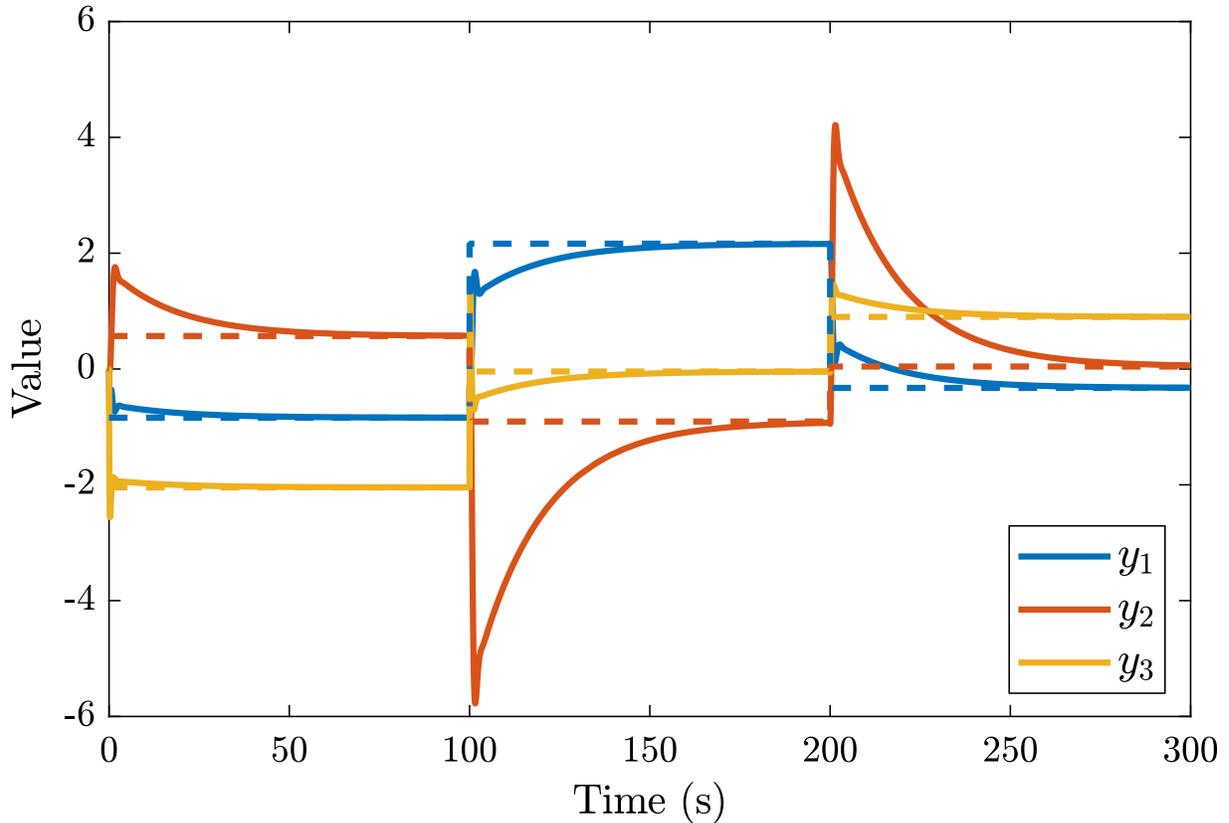
Figure 5.1: The optimization output variables (y_1, y_2, y_3) plotted as a function of time. The optimizer for each variable over each time interval is shown as a dashed line.

synthesize a functioning dynamic stabilizer whose behaviour is demonstrated in Figure 5.1 for the objective function $g(y; w) = y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{2}y_3^2$ and piecewise constant disturbance $w(t)$ given by

$$w(t) := \begin{cases} \begin{bmatrix} -1 & 3 & 1 & 2 \end{bmatrix}^\mathsf{T} & \text{for } 0 \leq t < 100 \\ \begin{bmatrix} 2 & -3 & 0 & 0 \end{bmatrix}^\mathsf{T} & \text{for } 100 \leq t < 200 \\ \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^\mathsf{T} & \text{for } 200 \leq t < 300 \,. \end{cases}$$

In summary, $\mathcal{H}_\infty$ synthesis can produce a stabilizer when a simple structure such as a PI controller fails; however, the resulting dynamic controller is of high order. In general, one should begin with simpler stabilizer designs and move to progressively more complex ones as need dictates.

5.2 Optimal Frequency Regulation in Power Systems[2]

This final section illustrates the application of our theory to a power system control problem. Our main objective is to work through the constructions presented in Section 4.2, and to simultaneously illustrate the many sources of design flexibility within our proposed framework. In particular, we will show that several centralized and distributed frequency controllers proposed in the literature are recoverable as special cases of our framework.

The dynamics of synchronous generators in a connected AC power network with n buses and n_t transmission lines is modelled in a reduced-network framework by the *swing equations*. The vectors of angular frequency (deviations from nominal) $\omega \in \mathbb{R}^n$ and real power flows $p \in \mathbb{R}^{n_t}$ along the transmission lines obey the dynamic equations

$$
\begin{aligned}
M(\delta)\dot{\omega} &= P^\star - D(\delta)\omega - \mathcal{A}p + u \\
\dot{p} &= \mathcal{B}(\delta)\mathcal{A}^{\mathsf{T}}\omega \,,
\end{aligned}
\tag{5.5}
$$

in which $M(\delta) \succ \mathbb{0}$ is the (diagonal) inertia matrix, $D(\delta) \succ \mathbb{0}$ is the (diagonal) damping matrix, $\mathcal{A} \in \{0, 1, -1\}^{n \times n_t}$ is the signed node-edge incidence matrix of the network, $\mathcal{B}(\delta) \succ \mathbb{0}$ is the diagonal matrix of transmission line susceptances, $P^\star \in \mathbb{R}^n$ is the vector of uncontrolled power injections (generation minus demand) at the buses, and $u \in \mathbb{R}^n$ is the controllable reserve power produced by the generator. The diagonal elements of the inertia, damping, and branch susceptance matrices are uncertain but positive; for example, they could be known within some bounds. See [81, Section VII] for a first-principles derivation of this model.

The incidence matrix satisfies $\mathbf{null}\,\mathcal{A}^{\mathsf{T}} = \mathrm{span}(\mathbb{1}_n)$, and strictly for simplicity we assume that the network is acyclic, in which case $n_t = n - 1$ and $\mathbf{null}\,\mathcal{A} = \{\mathbb{0}\}$.

We consider the *optimal frequency regulation problem* (OFRP), wherein we minimize the total cost $\sum_i J_i(u_i)$ of reserve power production in the system subject to network-wide balancing of supply and demand. We will consider two equivalent formulations, yielding two different OSS control problems. In both formulations, we take steady-state operation of the plant (5.5) as an implicit constraint, as in the definition of the optimization problem (4.5) for linear-convex OSS control.

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[2]The contents of this section will be incorporated into a publication: L. S. P. Lawrence, J. W. Simpson-Porco, and E. Mallada, "The Linear-Convex Optimal Steady-State Control Problem," to be submitted to IEEE Transactions on Automatic Control.

5.2.1 Economic Dispatch Formulation of OFRP

The first formulation of the optimization problem requires balance between power supply and demand

$$\underset{u}{\text{minimize}} \quad J(u) := \sum_{i=1}^{n} J_i(u_i)$$
$$\text{subject to} \quad \mathbb{1}_n^\mathsf{T} u = -\mathbb{1}_n^\mathsf{T} P^\star . \tag{5.6}$$

With state vector $x := \text{col}(\omega, p)$, the dynamics (5.5) can be put into the standard LTI form (4.1) with matrices

$$A(\delta) := \begin{bmatrix} -M(\delta)^{-1} D(\delta) & -M(\delta)^{-1}\mathcal{A} \\ \mathcal{B}(\delta)\mathcal{A}^\mathsf{T} & \mathbb{0} \end{bmatrix}$$
$$B(\delta) := \begin{bmatrix} M(\delta)^{-1} \\ \mathbb{0} \end{bmatrix} \quad B_w(\delta) := \begin{bmatrix} M(\delta)^{-1} \\ \mathbb{0} \end{bmatrix} .$$

Based on (5.6), we define the output to be optimized as $y := u$. Therefore

$$C = \begin{bmatrix} \mathbb{0} & \mathbb{0} \end{bmatrix} , \quad D = I_n .$$

We assume the measured output y_m consists of the inputs u and the exogenous disturbance term $\mathbb{1}_n^\mathsf{T} P^\star$, so that $y_\mathrm{m} = \text{col}(u, \mathbb{1}_n^\mathsf{T} P^\star)$. As a consequence, the constraint violation $\mathbb{1}_n^\mathsf{T} u + \mathbb{1}_n^\mathsf{T} P^\star$ is measurable.

We begin by determining whether this OSS control problem satisfies the robust feasible subspace property of Definition 4.2.1. We first check whether the robust output subspace property (Definition 4.2.2) holds by constructing the matrix $R(\delta)$ as outlined in Lemma 4.2.3. We construct a matrix $N_{AB}(\delta)$ satisfying $\mathbf{range}\, N_{AB}(\delta) = \mathbf{null} \begin{bmatrix} A(\delta) & B(\delta) \end{bmatrix}$ by examining

$$\mathbf{null} \begin{bmatrix} -M(\delta)^{-1} D(\delta) & -M(\delta)^{-1}\mathcal{A} & M(\delta)^{-1} \\ \mathcal{B}(\delta)\mathcal{A}^\mathsf{T} & \mathbb{0} & \mathbb{0} \end{bmatrix}$$

One may verify that choosing

$$N_{AB}(\delta) := \begin{bmatrix} \mathbb{1}_n & \mathbb{0} \\ \mathbb{0} & I_n \\ D(\delta)\mathbb{1}_n & \mathcal{A} \end{bmatrix} \tag{5.7}$$

yields the required property. We find that the matrix $R(\delta) = \begin{bmatrix} C & D \end{bmatrix} N_{AB}(\delta)$ is given by

$$R(\delta) = \begin{bmatrix} D(\delta)\mathbb{1}_n & \mathcal{A} \end{bmatrix} .$$

Because $\mathbf{range}\,\mathcal{A} = (\mathbf{null}\,\mathcal{A}^{\mathsf{T}})^{\perp} = (\mathrm{span}(\mathbb{1}_n))^{\perp}$ and $\mathbb{1}_n^{\mathsf{T}}D(\delta)\mathbb{1}_n > 0$ for all δ, the matrix $R(\delta)$ is full column rank for all δ. Therefore, we may choose $R_0 := I_n$ as a matrix satisfying $\mathbf{range}\,R_0 = \mathbf{range}\,R(\delta)$ for all δ. The robust output subspace property therefore holds by Proposition 4.2.5.[3]

Because the plant satisfies the robust output subspace property, we have access to the optimality models for both the ROS property (Proposition 4.2.10) and the RFS property (Proposition 4.2.9). The ROS optimality model (4.17) reduces to

$$\dot{\mu} = \mathbb{1}_n^{\mathsf{T}}u + \mathbb{1}_n^{\mathsf{T}}P^{\star}, \qquad \epsilon = \nabla J(u) + \mu\mathbb{1}_n\,,$$

and applying Proposition 4.2.11, two possible OSS controllers corresponding to different stabilizer choices are

$$\begin{aligned} \dot{\mu} &= \mathbb{1}_n^{\mathsf{T}}u + \mathbb{1}_n^{\mathsf{T}}P^{\star} \\ \dot{\eta} &= \nabla J(u) + \mathbb{1}_n\mu \end{aligned} \qquad u = -\eta \qquad (5.8)$$

and, if each J_i is strictly convex

$$\dot{\mu} = \mathbb{1}_n^{\mathsf{T}}u + \mathbb{1}_n^{\mathsf{T}}P^{\star}, \qquad u = (\nabla J)^{-1}(-\mu\mathbb{1}_n)\,. \qquad (5.9)$$

In (5.8) the optimality error ϵ is integrated to zero by the internal model, while in (5.9) we instead instantaneously zero ϵ through selection of u. The former can be considered as a "primal-dual" algorithm (see [68]), while the latter would be called "dual ascent". Both designs are feedforward OSS controllers, in that neither uses feedback from the system dynamics; in this application one is free to add additional negative frequency feedback to the control.

We omit the calculations for the RFS optimality model (4.15) for this formulation of the optimization problem, as the RFS-OM will be illustrated for the second formulation. We note, however, that application of the RFS-OM recovers another control scheme from the literature. Inspired by approaches in multi-agent control, we introduce a connected, weighted and directed communication graph $\mathcal{G}_{\mathrm{c}} = (\{1,\ldots,n\}, \mathcal{E}_{\mathrm{c}})$ between the buses, with associated *Laplacian matrix* $L_{\mathrm{c}} \in \mathbb{R}^{n \times n}$. If the directed graph $\mathcal{G}_{\mathrm{c}}$ contains a globally reachable node[4], then

$$\epsilon = \begin{bmatrix} \mathbb{1}_n^{\mathsf{T}}u + \mathbb{1}_n^{\mathsf{T}}P^{\star} \\ L_{\mathrm{c}}\nabla J(u) \end{bmatrix} \qquad (5.10)$$

---

[3]One may also reach the same conclusion by observing that the full-rank condition of Proposition 4.2.6 holds for this instance of OSS control.

[4]See [12, Chapter 6] for details.

is one option for an optimality model. If, furthermore, $\mathcal{G}_c$ is weight-balanced, then from the optimality model (5.10) we can recover the controller

$$\dot{\eta} = -L_c \nabla J(u) + \frac{1}{n}\left(\mathbb{1}_n^\mathsf{T} u + \mathbb{1}_n^\mathsf{T} P^\star\right)\mathbb{1}_n, \quad u = \eta\,. \tag{5.11}$$

Equation (5.11) is the controller [15, Equation (7)] specialized to an economic dispatch problem with differentiable cost function and without box constraints.

5.2.2 Frequency Constraint Formulation of OFRP

The second formulation of the optimization problem explicitly requires zero steady-state frequency deviations:

$$\begin{aligned} \underset{u,\omega}{\text{minimize}} \quad & J(u) := \sum_{i=1}^{n} J_i(u_i) \\ \text{subject to} \quad & F\omega = \mathbb{0}_r\,. \end{aligned} \tag{5.12}$$

The matrix $F \in \mathbb{R}^{r \times n}$ is assumed to satisfy $\mathbb{1}_n \notin \mathbf{null}\, F$. Options for the matrix F include $F := I_n$, to enforce $\omega = \mathbb{0}_n$, or $F := e_1^\mathsf{T}$, to enforce $\omega_1 = 0$, or $F := \mathsf{c}^\mathsf{T}$, where $\mathsf{c} \in \mathbb{R}^n$ is a vector of convex combination coefficients satisfying $\mathsf{c}_i \geq 0$ and $\sum_{i=1}^{n} \mathsf{c}_i = 1$. We identify the optimization output as $y := \mathrm{col}(u, \omega)$. Therefore

$$C := \begin{bmatrix} \mathbb{0} & \mathbb{0} \\ I_n & \mathbb{0} \end{bmatrix} \quad D := \begin{bmatrix} I_n \\ \mathbb{0} \end{bmatrix}. \tag{5.13}$$

We assume the measured output y_m consists of the inputs u and the term $F\omega$, so that $y_\mathrm{m} = \mathrm{col}(u, F\omega)$. As a consequence, the constraint violation $F\omega$ is measurable.

We identify the matrix H of the engineering equality constraints in (4.5) as $H := \begin{bmatrix} \mathbb{0} & F \end{bmatrix}$. Using (5.7) and (5.13), we may calculate $R(\delta) = \begin{bmatrix} C & D \end{bmatrix} N_{AB}(\delta)$ to be

$$R(\delta) = \begin{bmatrix} D(\delta)\mathbb{1}_n & \mathcal{A} \\ \mathbb{1}_n & \mathbb{0} \end{bmatrix}.$$

The subspace $\mathbf{range}\, R(\delta)$ varies with δ, and therefore there cannot exist a fixed matrix R_0 such that $\mathbf{range}\, R(\delta) = \mathbf{range}\, R_0$ for all δ. The robust output subspace property fails by Proposition 4.2.5. However, it is still possible that the robust feasible subspace property holds. To check whether this is the case, we first construct a matrix $R_\perp(\delta) \in \mathbb{R}^{n \times 2n}$ satisfying $\mathbf{null}\, R_\perp(\delta) = \mathbf{range}\, R(\delta)$. We find that selecting

$$R_\perp(\delta) := \begin{bmatrix} \mathbb{1}_n \mathbb{1}_n^\mathsf{T} & -(\mathbb{1}_n^\mathsf{T} D(\delta)\mathbb{1}_n)I_n \end{bmatrix}$$

yields the required property. Following (4.13), we now ask whether there exists a fixed matrix T_0 such that

$$\mathbf{range}\, T_0 = \mathbf{null} \begin{bmatrix} \mathbb{1}_n \mathbb{1}_n^\mathsf{T} & -(\mathbb{1}_n^\mathsf{T} D(\delta) \mathbb{1}_n) I_n \\ \mathbb{0} & F \end{bmatrix} \tag{5.14}$$

for all δ. This null space is spanned by vectors of the form $\mathrm{col}(v, \mathbb{0}_n)$ where $\mathbb{1}_n^\mathsf{T} v = 0$. If L_c is, as before, the Laplacian matrix of a communication digraph $\mathcal{G}_\mathrm{c}$ with a globally reachable node, then

$$T_0 := \begin{bmatrix} L_\mathrm{c}^\mathsf{T} \\ \mathbb{0} \end{bmatrix},$$

is an eligible choice for T_0. Therefore, the optimization problem satisfies the robust feasible subspace property by Proposition 4.2.4. Noting that

$$T_0^\mathsf{T} \nabla g(y, w) = \begin{bmatrix} L_\mathrm{c} & \mathbb{0} \end{bmatrix} \begin{bmatrix} \nabla J(u) \\ \mathbb{0} \end{bmatrix} = L_\mathrm{c} \nabla J(u),$$

we apply Proposition 4.2.9 to obtain the optimality model

$$\epsilon = \begin{bmatrix} F\omega \\ L_\mathrm{c} \nabla J(u) \end{bmatrix}. \tag{5.15}$$

Therefore, one option for the linear-convex OSS controller of Proposition 4.2.11 is

$$\begin{aligned} \dot{\eta}_1 &= F\omega \\ \dot{\eta}_2 &= L_\mathrm{c} \nabla J(u) \\ u &= -K_1 \eta_1 - K_2 \eta_2 - K_3 \omega, \end{aligned}$$

where K_1, K_2, and K_3 are gain matrices that should be selected for closed-loop stability/performance. With $F = I_n$, $K_1 = K_2 = \frac{1}{k} I_n$ for $k > 0$, and $K_3 = \mathbb{0}$, this design reduces to the *distributed-averaging proportional-integral* (DAPI) frequency control scheme; see [1, 21, 66, 73, 80]. Other choices of F with this same stabilizer design lead to various centralized/decentralized controller designs.

The so-called gather-and-broadcast scheme of [22] can be recovered as follows. Assume that each J_i is strictly convex, set $F = \mathsf{c}^\mathsf{T}$ as discussed previously, and retain the integral controller $\dot{\eta} = \mathsf{c}^\mathsf{T} \omega$, which integrates a weighted average of the frequency deviations. Next, select the input u in (5.15) to zero the second component of ϵ:

$$\begin{aligned} L_\mathrm{c} \nabla J(u) = \mathbb{0}_n \quad &\Longleftrightarrow \quad \exists\, \alpha \in \mathbb{R} \text{ s.t. } \nabla J(u) = \alpha \mathbb{1}_n \\ &\Longleftrightarrow \quad \exists\, \alpha \in \mathbb{R} \text{ s.t. } u = (\nabla J)^{-1}(\alpha \mathbb{1}_n). \end{aligned}$$

Selecting $\alpha = \eta$ leads to the hierarchical gather-and-broadcast controller

$$\dot{\eta} = \sum_{i=1}^{n} \mathsf{c}_i \omega_i \,, \quad u_i(t) = (\nabla J_i)^{-1}(\eta(t)) \,. \tag{5.16}$$

In summary, many recent frequency control schemes can be recovered as special cases of our general control framework. The full potential of our methodology for the design of improved power system control will be an area for future study.

Chapter 6

Conclusions and Future Work[1]

We have defined and presented a detailed discussion of the optimal steady-state control problem, wherein the goal is to guide a combination of states and inputs of a nonlinear dynamical system to an optimal steady-state, in the presence of exogenous time-varying disturbances and model uncertainty. Necessary and sufficient conditions for solvability of the problem were presented, along with a constructive design framework that revolves around the introduction of an optimality model, the purpose of which is to robustly produce a proxy for the error between the optimized variables and their desired optimal values. This optimality model converts the OSS control problem into an output regulation problem; one designs an output-regulating controller for the plant and optimality model in cascade to solve the OSS control problem.

We then studied in detail the special case of the linear-convex OSS control problem, wherein the plant is an uncertain LTI system, the exogenous disturbances are constant, and the optimization problem is convex. A complete controller design procedure was presented, and two properties — the robust feasible subspace and robust output subspace properties — were identified as important for understanding cases where optimizing robustly with respect to parametric modelling uncertainty is achievable. Applying our linear-convex OSS procedures to a frequency regulation problem from power systems, we recovered a number of existing controller designs from the recent literature.

Immediate future work will present the analogous discrete-time and sampled-data OSS control problems, along with a more detailed study of applications in power system control.

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[1]The contents of this chapter will be incorporated into a publication: L. S. P. Lawrence, J. W. Simpson-Porco, and E. Mallada, "The Linear-Convex Optimal Steady-State Control Problem," to be submitted to IEEE Transactions on Automatic Control.

See Appendix A for an example of current research into discrete-time OSS control.

A large number of open questions remain concerning the presented framework, including but not limited to: construction of OSS controllers for special classes of nonlinear systems, flexibility of the framework for distributed/decentralized control, formulations and solutions of hierarchical and approximate OSS control problems.

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APPENDICES

Appendix A

Future Work

The following section is an example of a future research direction: OSS controller design for discrete-time plants by direct interconnection with optimization algorithms.

A.1 Problem Setup

We consider an LTI system

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k + B_w w \\
y_k &= Cx_k + Du_k + Qw
\end{aligned}
\tag{A.1}
$$

where $A \in \mathbb{R}^{n \times n}$ is Schur, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the optimization output. The vector $w \in \mathbb{R}^{n_w}$ is a constant disturbance. The transfer function from $u \to y$ is

$$
P(z) := C(zI - A)^{-1}B + D
$$

and from $w \to y$ is

$$
P_w(z) := C(zI - A)^{-1}B_w + Q \,.
$$

In steady-state, we have the relations

$$
\bar{y} = P(1)\bar{u} + P_w(1)w \,.
$$

The optimization problem of interest is the convex problem

$$
\underset{\bar{u},\bar{y}}{\text{minimize}} \quad f(\bar{y}) \tag{A.2a}
$$

$$
\text{subject to} \quad \bar{y} = P(1)\bar{u} + P_w(1)w \tag{A.2b}
$$

We assume that $f : \mathbb{R}^m \to \mathbb{R}$ is differentiable and κ-strongly convex with an L-Lipschitz gradient.

Assumption A.1.1 (Optimizer Exists) *An optimizer for the problem* (A.2) *exists.*

Due to strong convexity of f in $\bar{y}$, Assumption A.1.1 is enough to guarantee a *unique* $\bar{y}$ optimizer for (A.2). We denote this unique $\bar{y}$ optimizer by $y^\star$.

We can rewrite the optimization problem (A.2) by directly substituting for $\bar{y}$ using the constraint equation. We obtain the equivalent unconstrained problem

$$\underset{\bar{u}}{\text{minimize}} \quad f(P(1)\bar{u} + P_w(1)w) \, . \tag{A.3}$$

We are interested in designing a dynamic feedback controller of the form

$$\begin{aligned} \xi_{k+1} &= f_\text{c}(\xi_k, y_k) \, , \qquad \xi_0 \in \mathbb{R}^{n_\xi} \\ u_k &= h_\text{c}(\xi_k, y_k) \, , \end{aligned} \tag{A.4}$$

for the system.

Problem A.1.2 (Optimal Steady-State Control) *Design, if possible, a dynamic feedback controller* (A.4) *for the dynamic system* (A.1) *such that the closed-loop system meets the following criteria:*

(i) well-posedness: *the closed-loop system is well-posed;*

(ii) bounded trajectories: *for each initial condition* (x_0, ξ_0) *and exogenous input* $w \in \mathbb{R}^{n_w}$, *the trajectories* $(x(t), \xi(t))$ *of the closed-loop system* (A.1) *and* (A.4) *remain bounded for all* $t \geq 0$;

(iii) asymptotic optimality: *For every initial condition* (x_0, ξ_0) *of the closed-loop system and every constant disturbance* $w \in \mathbb{R}^{n_w}$, *the optimization output* y_k *is asymptotically brought into agreement with the optimizer* $y^\star$

$$\lim_{k \to \infty} y_k = y^\star \, .$$

A.2 Stability of OSS Control with First-Order Optimization Algorithms

As shown in our recent paper, the class of controllers which solves Problem A.1.2 is very broad. In this document we want to focus in on the application of standard first-order optimization methods for OSS control. For example, applying the gradient method to (A.3) yields the algorithm

$$\mathsf{u}_{k+1} = \mathsf{u}_k - \alpha \left[P(1)^\mathsf{T} \nabla f(P(1)\mathsf{u}_k + P_w(1)w) \right] .$$

This algorithm produces a sequence of feedforward inputs $\{\mathsf{u}_1, \mathsf{u}_2, \ldots\}$, which we apply to the system, i.e., $u_k = \mathsf{u}_k$. Under mild assumptions we expect that $\mathsf{u}_k \to u^\star$ and hence that $y_k \to y^\star$ by internal stability of (A.1). However, this implementation requires knowledge of the (possibly unknown) disturbance w as well as the steady-state gain matrix $P_w(1)$ from disturbance to output. If we replace the argument of the gradient with the real-time measurement y_k, then we have the feedback algorithm

$$\mathsf{u}_{k+1} = \mathsf{u}_{k+1} - \alpha \left[P(1)^\mathsf{T} \nabla f(y_k) \right]$$
$$u_k = \mathsf{u}_k .$$

We can now generalize this easily. Following the methodology of Lessard et al, we represent the first order optimization algorithm by a linear time-invariant system of the form

$$
\begin{aligned}
v_k &= \nabla f(y_k) \\
\xi_{k+1} &= F\xi_k + GP(1)^\mathsf{T} v_k \\
u_k &= H\xi_k
\end{aligned}
\tag{A.5}
$$

which has input y_k and output u_k. The formulas for some standard methods are

$$
\begin{aligned}
\text{Gradient descent:} \quad & \left[\begin{array}{c|c} F & G \\ \hline H & \mathbb{0} \end{array} \right] = \left[\begin{array}{c|c} I & -\alpha I_m \\ \hline I_m & \mathbb{0} \end{array} \right] \\[2em]
\text{Heavy ball:} \quad & \left[\begin{array}{c|c} F & G \\ \hline H & \mathbb{0} \end{array} \right] = \left[\begin{array}{cc|c} (1+\beta)I_m & -\beta I_m & -\alpha I_m \\ I_m & \mathbb{0} & \mathbb{0} \\ \hline I_m & \mathbb{0} & \mathbb{0} \end{array} \right] \\[2em]
\text{Nesterov:} \quad & \left[\begin{array}{c|c} F & G \\ \hline H & \mathbb{0} \end{array} \right] = \left[\begin{array}{cc|c} (1+\beta)I_m & -\beta I_m & -\alpha I_m \\ I_m & \mathbb{0} & \mathbb{0} \\ \hline (1+\beta)I_m & -\beta I_m & \mathbb{0} \end{array} \right]
\end{aligned}
\tag{A.6}
$$

Note that in all cases, F has p simple eigenvalues at $z = 1$ and that (F, G, H) is controllable and observable. We denote the optimization algorithm's transfer function by $O(z)$, with

$$O(z) := H(zI - F)^{-1}G.$$

Closing the loop between (A.1) and (A.5), the system is described by

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k + B_w w_k \\
y_k &= Cx_k + Du_k + Qw_k \\
\xi_{k+1} &= F\xi_k + GP(1)^{\mathsf{T}} v_k \\
v_k &= \nabla f(y_k) \\
u_k &= H\xi_k
\end{aligned}
\tag{A.7}
$$

Let $(x^\star, \xi^\star, y^\star, u^\star, v^\star)$ denote a fixed point of this closed-loop system, i.e., a point satisfying

$$
\begin{aligned}
x^\star &= Ax^\star + Bx^\star + B_w w \\
y^\star &= P(1)u^\star + P_w(1)w \\
\xi^\star &= F\xi^\star + GP(1)^{\mathsf{T}} v^\star \\
u^\star &= H\xi^\star \\
v^\star &= \nabla f(y^\star)
\end{aligned}
$$

We assume the point is unique and optimal.

In terms of deviation variables

$$\tilde{x} = x - x^\star, \quad \tilde{y} = y - y^\star, \ldots$$

with respect to the equilibrium point, we have

$$
\text{Plant:} \quad
\begin{cases}
\tilde{x}_{k+1} &= A\tilde{x}_k + B\tilde{u}_k \\
\tilde{y}_k &= C\tilde{x}_k + D\tilde{u}_k
\end{cases}
$$

$$
\text{Controller:} \quad
\begin{cases}
\tilde{v}_k &= \nabla f(\tilde{y}_k + y^\star) := \psi(\tilde{y}_k) \\
\tilde{\xi}_{k+1} &= F\tilde{\xi}_k + GP(1)^{\mathsf{T}} \tilde{v}_k \\
\tilde{u}_k &= H\tilde{\xi}_k
\end{cases}
\tag{A.8}
$$

where now $\psi(\mathbb{0}_p) = \mathbb{0}_p$.

A.3 Combined Plant and Algorithm Analysis

We collect the LTI components of the closed-loop system (plant and optimization algorithm) into the augmented linear system Ψ

$$\Psi = \left[\begin{array}{cc|c} A & BH & \mathbb{0} \\ \mathbb{0} & F & GP(1)^\mathsf{T} \\ \hline C & DH & \mathbb{0} \end{array} \right]$$

with transfer function $\Psi(z) = P(z)O(z)P(1)^\mathsf{T}$. We shall apply standard robust stability analysis to the interconnection of Ψ and a nonlinearity $\Delta \in \sec[\kappa, L]$ (the gradient).

The transfer functions for common optimization algorithms are:

$$\begin{aligned} \text{Gradient descent:} \quad & O_{\text{grad}}(z) = \frac{-\alpha}{z-1} I_m \\ \text{Heavy ball:} \quad & O_{\text{heavy}}(z) = \frac{-\alpha}{z-1} \frac{z}{z-\beta} I_m \\ \text{Nesterov:} \quad & O_{\text{nesterov}}(z) = \frac{-\alpha}{z-1} \frac{(1+\beta)z - \beta}{z-\beta} I_m \, , \end{aligned} \tag{A.9}$$

where $\alpha > 0$ and $\beta \in (0,1)$ are tuning parameters. The parameter α is the "step size" in the language of optimization algorithms, or the "integral gain" in the language of control theory; we will emphasize the dependence of the LTI system Ψ on α by writing Ψ_α hereafter. Notice that each optimization algorithm consists of an integrator and a stable filter in series. Furthermore, the DC gain matrix of the filter for each algorithm is positive definite (the importance of this point will become clear shortly).

We state and prove the main theorem.

Theorem A.3.1 (Low-Gain First-Order Optimization) *Consider the feedback interconnection of the LTI plant* (A.1) *and a first-order optimization algorithm of the form* (A.5) *whose transfer matrix $O(z) = H(zI - F)^{-1}G$ can be written in the form*

$$O(z) = \frac{-\alpha}{z-1} M(z) \, ,$$

with $M(z)$ a stable transfer matrix satisfying $M(1) \succ \mathbb{0}$. There exists an $\alpha^\star > 0$ such that for any $\alpha \in (0, \alpha^\star)$, the controller (A.5) *solves the optimal steady-state control problem.*

Appendix B

Continuity of Optimal Solution Function

Proposition B.0.1 (Sufficient Conditions for Properties of $\boldsymbol{y^\star}$) *Suppose the following hold:*

(i) *the inequality constraint functions of the optimization problem* (4.5) *are all affine, so that there exist matrices $J \in \mathbb{R}^{n_{ec} \times p}$ and $M \in \mathbb{R}^{n_{ec} \times n_w}$ such that the inequality constraints* (4.5d) *may be written as $Jy \leq Mw$;*

(ii) *the optimization problem* (4.5) *has a solution for each (w, δ);*

(iii) *the objective function $g(y; w)$ of the optimization problem* (4.5) *is strictly convex in y for each w;*

(iv) *the LTI plant* (4.1) *has a robust output subspace;*

(v) $\mathbf{rank}\begin{bmatrix} A(\delta) & B(\delta) \end{bmatrix} = n$ *for all $\delta \in \boldsymbol{\delta}$.*

Then $y^\star$ is continuous and $y^\star(w, \delta)$ is single-valued for every (w, δ). △

Proof: Uniqueness of the solution to (4.5) follows from strict convexity of g [10, Section 4.2.1]. By the assumption that the optimization problem has a solution for each (w, δ), it follows that $y^\star(w, \delta)$ is single-valued for every (w, δ).

We now show continuity of the mapping $y^\star : W \to Y$. The optimization problem (4.5) can be written as

$$\underset{y \in Y}{\text{minimize}} \quad g(y; w)$$
$$\text{subject to} \quad y \in \mathcal{C}(w, \delta),$$

where $\mathcal{C} : W \times \delta \rightrightarrows Y$ is a set-valued mapping determining the feasible region. Specifically,

$$\mathcal{C}(w, \delta) := \{y \in Y : y \in \overline{Y}(w, \delta), \ Hy = Lw, \ Jy \le Mw\}.$$

By definition, $\mathcal{C}(w, \delta)$ is closed and convex for all (w, δ). If, futhermore, $\mathcal{C}$ is a continuous set-valued mapping then by [71, Theorem 3.1] $y^\star$ is continuous.

Suppose the LTI plant has a robust output subspace, so that

$$\overline{Y}(w, \delta) = \mathsf{y}(w, \delta) + V_0$$

for all (w, δ). Recall that the subspace V_0 is uniquely determined by the plant matrices, while the offset vector $\mathsf{y}(w, \delta)$ is non-unique. The remainder of the proof depends on the fact that it is always possible to select the offset vector $\mathsf{y}(w, \delta)$ as a continuous function of (w, δ).

Lemma B.0.2 (Continuous Offset Vector) *If the LTI plant has a robust output subspace, then there exists a continuous mapping* $\mathsf{y} : W \times \boldsymbol{\delta} \to Y$ *such that* $\overline{Y}(w, \delta) = \mathsf{y}(w, \delta) + V_0$

Proof: Recall from the proof of Lemma 4.2.3 that we can think of the affine space $\overline{Y}(w, \delta)$ as being constructed by a two-step process. First, determine functions $\tilde{x}(w, \delta)$ and $\tilde{u}(w, \delta)$ satisfying

$$A(\delta)\tilde{x}(w, \delta) + B(\delta)\tilde{u}(w, \delta) + B_w(\delta)w = \mathbb{0}_n. \tag{B.1}$$

for all (w, δ). Then $\overline{Y}(w, \delta) = \mathsf{y}(w, \delta) + V(\delta)$ where

$$\mathsf{y}(w, \delta) = C(\delta)\tilde{x}(w, \delta) + D(\delta)\tilde{u}(w, \delta) + Q(\delta)w$$
$$V(\delta) = \begin{bmatrix} C(\delta) & D(\delta) \end{bmatrix} \left(\mathbf{null} \begin{bmatrix} A(\delta) & B(\delta) \end{bmatrix} \right).$$

It follows that if $\tilde{x}(\cdot, \cdot)$ and $\tilde{u}(\cdot, \cdot)$ can be chosen as continuous mappings then the $\mathsf{y}(w, \delta)$ generated by the above equation is continuous.

By the assumption that **rank** $\begin{bmatrix} A(\delta) & B(\delta) \end{bmatrix}$ is the same for all δ, let $M_{AB}(\delta) \in \mathbb{R}^{(n+m) \times n}$ denote the Moore-Penrose pseudoinverse of $\begin{bmatrix} A(\delta) & B(\delta) \end{bmatrix}$ for each δ. The mapping $M_{AB}(\cdot)$ is continuous since $A(\cdot)$ and $B(\cdot)$ are continuous, and we can define $\tilde{x}(w, \delta)$ and $\tilde{u}(w, \delta)$ as

$$\begin{bmatrix} \tilde{x}(w, \delta) \\ \tilde{u}(w, \delta) \end{bmatrix} := -M_{AB}(\delta) B_w(\delta) w \,.$$

Therefore, it is possible to choose the mapping $\mathsf{y}(w, \delta)$ as a continuous function of (w, δ). $\square$

We now rewrite the steady-state constraint (4.5b), $\bar{y} \in \overline{Y}(w, \delta) = \mathsf{y}(w, \delta) + V_0$, in a more standard form as a set of equality constraints. We suppose we have chosen $\mathsf{y}(w, \delta)$ as a continuous function of (w, δ), possible because of Lemma B.0.2. Let Γ be any matrix such that **null** $\Gamma = V_0$, and define $b(w, \delta) := \Gamma \mathsf{y}(w, \delta)$. Then the constraint $\bar{y} \in \overline{Y}(w, \delta)$ is equivalent to

$$\Gamma \bar{y} = b(w, \delta) \,.$$

The constraint set of the optimization problem (4.5) therefore consists of linear equality and inequality constraints with the parameters (w, δ) *only appearing in continuous functions on the right-hand side*. It is straightforward to show from [55, Theorem 2.2] that $\mathcal{C}(\cdot, \cdot)$ is a continuous set-valued mapping, which completes the proof. $\square$

Proposition B.0.3 (Perturbed Problem) *Consider the convex optimization problem* (4.5) *with a perturbed cost function*

$$\begin{aligned} \underset{y \in Y}{\text{minimize}} \quad & g_\varepsilon(y; w) \\ \text{subject to} \quad & y \in \overline{Y}(w, \delta) \\ & Hy = Lw \\ & Jy \le Mw \,, \end{aligned} \tag{B.2}$$

where $\varepsilon \in \mathbb{R}$ is an additional parameter and $g_\varepsilon(y; w) := g(y; w) + \varepsilon^2 \|y\|^2$. Let $g_\varepsilon^\star(w, \delta)$ denote the optimal value of the perturbed problem (B.2) *for each $(w, \delta, \varepsilon) \in W \times \boldsymbol{\delta} \times \mathbb{R}$ and let $g^\star(w, \delta)$ denote the optimal value of the original problem* (4.5) *for each (w, δ). Note that $g_0^\star(w, \delta) = g^\star(w, \delta)$. If the conditions of Proposition B.0.1 hold, then*

  (i) *the optimization problem* (B.2) *has a unique optimizer for each (w, δ, ε) such that $\varepsilon \ne 0$, and*

  (ii) $\lim_{\varepsilon \to 0} g_\varepsilon^\star(w, \delta) = g^\star(w, \delta)$ *for all (w, δ).*

Proof: Since $g(y; w)$ is convex in y for each $w \in W$, $g_\varepsilon(y; w)$ is *strictly convex* in y for each $(w, \epsilon) \in W \times \mathbb{R}$ such that $\varepsilon \neq 0$. Uniqueness of the optimizer follows from strict convexity of the objective function for a convex problem [10, Section 4.2.1]. Furthermore, if the assumptions of Proposition B.0.1 hold, then it follows from [71, Theorem 3.1] that the optimal value function $g_\varepsilon^\star(w, \delta)$ is continuous in ε. Also noting that $g_0^\star(w, \delta) = g^\star(w, \delta)$ yields the identity $\lim_{\varepsilon \to 0} g_\varepsilon^\star(w, \delta) = g^\star(w, \delta)$.

$\square$

Remark B.0.4 (Perturbing for Uniqueness) *Proposition B.0.3 justifies perturbing the cost function of the original optimization problem* (4.5) *by $\varepsilon^2 \|y\|^2$ for small ε to enforce a unique optimizer if g is not already strictly convex. We can be assured that this procedure does not affect the optimal cost significantly, since the optimal value of the perturbed problem can be made arbitrarily close to the optimal value of the original problem by choosing small enough ε because of the property $\lim_{\varepsilon \to 0} g_\varepsilon^\star(w, \delta) = g^\star(w, \delta)$.* $\triangle$