Virtual Holonomic Constraints for Euler-Lagrange Systems

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Abstract: This paper investigates virtual holonomic constraints for Euler-Lagrange systems with n degrees-of-freedom and $n-1$ controls. The constraints have the form $q_1 = \phi_1(q_n), \ldots, q_{n-1} = \phi_{n-1}(q_n)$, where q_n is a cyclic configuration variable, so their enforcement corresponds to the stabilization of a desired oscillatory motion. We give conditions under which such a set of constraints is feasible, meaning that it can be made invariant by feedback. We show that it is possible to systematically determine feasible virtual constraints as periodic solutions of a scalar differential equation, the virtual constraint generator. Moreover, under a symmetry assumption we show that the motion on the constraint manifold is a Euler-Lagrange system with one degree-of-freedom, and use this fact to complete characterize its dynamical properties. Finally, we show that if the constraint is feasible then the virtual constraint manifold can always be stabilized using input-output feedback linearization.

1. INTRODUCTION

Consider the underactuated Euler-Lagrange system with n degrees-of-freedom and $n-1$ actuators

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = B(q)\tau, \tag{1}$$

with $q \in \mathcal{Q}$, $\tau \in \mathbb{R}^{n-1}$, and such that $B(q)$ has full rank $n-1$ and D is positive definite. Suppose that $q_1, \ldots, q_k$, $k \leq n-1$, are real variables (e.g., linear displacements) and $q_{k+1}, \ldots, q_n$ are cyclic variables in S^1 (e.g., angular displacements), so the state space of the system is $(q, \dot{q}) \in \mathcal{Q} \times \mathbb{R}^n$, where $\mathcal{Q} = \mathbb{R}^k \times S^1 \times \cdots \times S^1$ ($n-k$ times). The energy of the system is

$$E(q,\dot{q}) = \frac{1}{2}\dot{q}^\top D(q)\dot{q} + P(q),$$

and $g(q)$ in the system model (1) is the gradient of P, $g(q) = \nabla P(q)$.

The goal of this paper is to design a feedback stabilizing a virtual holonomic constraint expressed as a function of one of the cyclic configuration variables, without loss of generality q_n. The constraint has the form

$$\begin{aligned} q_1 &= \phi_1(q_n) \\ &\vdots \\ q_{n-1} &= \phi_{n-1}(q_n). \end{aligned}$$

We denote

$$\begin{aligned} \phi(q_n) &:= \mathrm{col}(\phi_1(q_n), \ldots, \phi_{n-1}(q_n)), \\ \hat{\phi}(q_n) &:= \mathrm{col}(\phi(q_n), q_n). \end{aligned}$$

The virtual holonomic constraint $\mathrm{col}(q_1, \ldots, q_{n-1}) = \phi(q_n)$ is said to be **feasible** if the **constraint manifold**

$$\begin{aligned} \Gamma = \{(q,\dot{q}) : \ & \mathrm{col}(q_1, \ldots, q_{n-1}) = \phi(q_n), \\ & \mathrm{col}(\dot{q}_1, \ldots, \dot{q}_{n-1}) = \phi'(q_n)\dot{q}_n\} \\ = \{(q,\dot{q}) : \ & q = \hat{\phi}(q_n),\ \dot{q} = \hat{\phi}'(q_n)\dot{q}_n\} \end{aligned} \tag{2}$$

is controlled invariant, i.e., it can be made invariant by a suitable choice of feedback $\tau(q, \dot{q})$.

The idea of virtual constraint is a useful paradigm for the control of oscillations. It is one of the underlying principles of the design in Plestan et al. [2003] and Westervelt et al. [2003] aimed at stabilizing a limit cycle corresponding to stable walking motion for bipedal robots. In Shiriaev et al. [2007], virtual constraints were used to stabilize desired oscillations of the Furuta pendulum. The recent work in Shiriaev et al. [2005], Freidovich et al. [2008] made considerable progress in the investigation of virtual constraints for Euler-Lagrange systems. There, an integral of motion for the dynamics on the virtual constraint was given explicitly, and a methodology to stabilize desired limit cycles on the virtual constraint manifold was given based on a time-varying linearization of the system on the limit cycle. We refer the reader to Fradkov and Pogromsky [1998] for more information on control of oscillations.

In this paper we present four novel results. Our first result is a condition for feasibility of the constraint manifold. The condition turns out to be generically satisfied for the class of systems considered in this paper. Virtual holonomic constraints meeting this generic condition will be called **regular.**

The second result in this paper is a proof of the fact that regular holonomic constraints are locally exponentially stabilizable (i.e., Γ is locally exponentially stabilizable) with an input-output feedback linearizing feedback, since

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the output functions $e_i = q_i - \phi_i(q_n)$, $i = 1, \ldots, n-1$ yields a well-defined vector relative degree $\{2, \ldots, 2\}$ on Γ.

The third result is a systematic procedure for the determination of regular virtual holonomic constraints which is based on the solutions of a scalar differential equation. The constraint manifold Γ is a two-dimensional manifold diffeomorphic to the cylinder $S^1 \times \mathbb{R}$, the diffeomorphism $S^1 \times \mathbb{R} \to \Gamma$ being given by [2] $(q_n, \dot{q}_n) \mapsto (\hat{\phi}(q_n), \hat{\phi}'(q_n)\dot{q}_n)$. On Γ, therefore, the motion is two-dimensional and is parametrized by $(q_n, \dot{q}_n)$. The fourth result in this paper is a proof of the fact that if the inertia $D(q)$, the potential $P(q)$, and the input matrix $B(q)$ are even functions, then the $(q_n, \dot{q}_n)$ dynamics on Γ are Euler-Lagrange. We also provide an explicit expression for the inertia and potential function of this reduced system. This result complements the theory in Shiriaev et al. [2005], where it was shown that an integral of motion for the reduced dynamics always exists, but the Lagrangian nature of the reduced dynamics was not exposed. Indeed, in the general case when D, P, and B are not even functions, the reduced system on Γ may not be Euler-Lagrange, although it still possesses an integral of motion. Using the potential function of the reduced system we study the dynamical properties of the motion on Γ, specifically the equilibria, their stability type, and the types of oscillations that are possible.

Throughout this paper we make the following standing assumption, although only the results in Section 4 strictly rely on it.

Assumption 1. For some $\bar{q} \in \mathcal{Q}$ it holds that, for all $q \in \mathcal{Q}$,

$$\begin{aligned} D(\bar{q} + q) &= D(\bar{q} - q) \\ P(\bar{q} + q) &= P(\bar{q} - q) \\ B(\bar{q} + q) &= B(\bar{q} - q). \end{aligned}$$

By shifting the origin of the coordinate system to $\bar{q}$, without loss of generality we will assume that $\bar{q} = 0$, so that $D(q) = D(-q)$, $P(q) = P(-q)$, and $B(q) = B(-q)$. It is interesting to note that the Furuta pendulum in Shiriaev et al. [2007] and the 5 degrees-of-freedom swing phase model of a biped robot in Plestan et al. [2003] (when the centre of mass of the torso is on-axis) satisfy this assumption.

Example 1.1. Consider the double pendulum in Figure 1, with $q = \mathrm{col}(\theta_1, \theta_2)$. Both configuration variables are cyclic with period $T = 2\pi$. Assuming that the masses and lengths of the two links are equal and unitary, we have for this system

$$D(q) = \begin{bmatrix} 2 & \cos(\theta_1 - \theta_2) \\ \cos(\theta_1 - \theta_2) & 1 \end{bmatrix}$$

$$C(q, \dot{q}) = \begin{bmatrix} 0 & \sin(\theta_1 - \theta_2)\dot{\theta}_2 \\ -\sin(\theta_1 - \theta_2)\dot{\theta}_1 & 0 \end{bmatrix}$$

$$g(q) = \begin{bmatrix} -2g\sin\theta_1 \\ -g\sin\theta_2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The potential energy is $P(q) = 2g\cos(\theta_1) + g\cos(\theta_2)$. We see that Assumption 1 is satisfied for this system with $\bar{q} = (0, 0)$.

We will develop this example further in this paper to illustrate various concepts.

[2] The inverse of the map $(q_n, \dot{q}_n) \mapsto (\hat{\phi}(q_n), \hat{\phi}'(q_n)\dot{q}_n)$ is $(u, v) \in \Gamma \mapsto (q_n, \dot{q}_n) = (u_n, \hat{\phi}'(u_n)^\top v / [\hat{\phi}'(u_n)^\top \hat{\phi}'(u_n)])$.

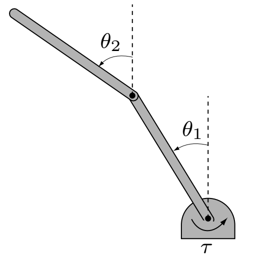

Fig. 1. The double pendulum.

2. FEASIBLE VIRTUAL CONSTRAINTS AND STABILIZATION OF Γ

In this section we derive a condition for a C^1 vector function $\phi(q_n) = \mathrm{col}(\phi_1(q_n), \ldots, \phi_{n-1}(q_n))$ to be a feasible virtual constraint so that the set $\Gamma = \{(q, \dot{q}) : q = \hat{\phi}(q_n),\ \dot{q} = \hat{\phi}'(q_n)\dot{q}_n\}$ is controlled invariant. Define error variables $e_i = q_i - \phi_i(q_n)$, $i = 1, \ldots, n-1$, and let $e = \mathrm{col}(e_1, \ldots, e_{n-1})$. The controlled invariance of Γ amounts to the existence of a feedback $\tau(q, \dot{q})$ such that $\ddot{e}|_{\{e=0, \dot{e}=0\}} = 0$, where derivatives are taken along solutions of the closed-loop system.

Lemma 2.1. If, for all $q_n \in S^1$,

$$\mathrm{Im}[D(\hat{\phi}(q_n))\hat{\phi}'(q_n)] \cap B(\hat{\phi}(q_n)) = \{0\}, \tag{3}$$

then the virtual holonomic constraint $q = \hat{\phi}(q_n)$ is feasible.

Proof. We have

$\ddot{e}|_{\{e=0,\dot{e}=0\}} = [I_{n-1} \quad -\phi'(q_n)]D^{-1}(\hat{\phi}(q_n))B(\hat{\phi}(q_n))\tau + (\star)$

where the term $(\star)$ is a suitable smooth function of $(q_n, \dot{q}_n)$ which is independent of τ. If the matrix-valued function $[I_{n-1} \quad -\phi'(q_n)]D^{-1}(\hat{\phi}(q_n))B(\hat{\phi}(q_n))$, $S^1 \to \mathbb{R}^{n-1 \times n-1}$, is nonsingular, then there exists a smooth feedback τ making Γ invariant. For each $q_n \in S^1$, the matrix $[I_{n-1} \quad -\phi'(q_n)]D^{-1}(\hat{\phi}(q_n))B(\hat{\phi}(q_n))$ is nonsingular if and only if $\ker[I_{n-1} \quad -\phi'(q_n)] \cap \mathrm{Im}[D^{-1}(\hat{\phi}(q_n))B(\hat{\phi}(q_n))] = \{0\}$. Since $\ker[I_{n-1} \quad -\phi'(q_n)] = \mathrm{Im}\,\hat{\phi}'(q_n)$, the condition is equivalent to $\mathrm{Im}[D(\hat{\phi}(q_n))\hat{\phi}'(q_n)] \cap B(\hat{\phi}(q_n)) = \{0\}$, proving the lemma. □

Remark 2.2. For each $q_n \in S^1$, the condition (3) involves the intersection of a one-dimensional subspace and an $n-1$ dimensional subspace of $\mathbb{R}^n$. Such intersection is *generically* zero and so, for each q_n, the assumption of the lemma is generic. For this reason, we will say that a constraint $q = \hat{\phi}(q_n)$ satisfying condition (3) is **regular**. Lemma 2.1 can therefore be rephrased as follows: "every regular holonomic constraint is feasible." Letting $B^\perp(q) : \mathcal{Q} \to \mathbb{R}^{1\times n}$ be a left annihilator of $B(q)$, i.e., a nonzero matrix-valued function such that $B^\perp(q)B(q) = 0$ for all $q \in \mathcal{Q}$, condition (3) can be rewritten in the following equivalent form

$$(\forall q_n \in S^1)\ B^\perp(\hat{\phi}(q_n))D(\hat{\phi}(q_n))\hat{\phi}'(q_n) \neq 0. \tag{4}$$

Example 2.3. We return to the double pendulum example. In this case we have $B^\perp = [0 \quad 1]$. Consider the virtual constraint $\theta_1 = \phi(\theta_2) = \sin(2\theta_2)/4$. The function in Lemma 2.1 is $B^\perp D(\hat{\phi}(\theta_2))\hat{\phi}'(\theta_2) = \frac{1}{2}\cos(\theta_1 - \theta_2)\cos(2\theta_2) + 1$ and is clearly nonzero. Therefore, the set $\Gamma = \{(q, \dot{q}) : \theta_1 = \sin(2\theta_2)/4, \dot{\theta}_1 = \cos(2\theta_2)/2\}$ is controlled invariant. The constraint $\theta_1 = \sin(2\theta_2)/4$ is depicted in Figure 2.

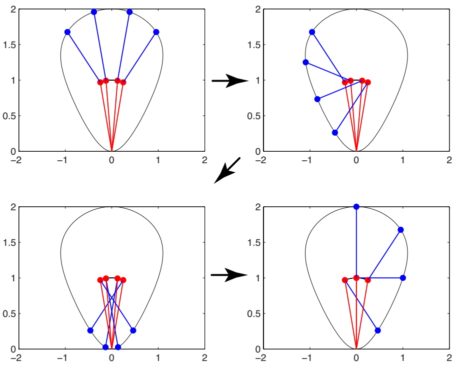

Fig. 2. Configurations of the double pendulum with the virtual constraint $\theta_1 = \sin(2\theta_2)/4$.

It was proved in Lemma 2.1 that the regularity condition (3) implies that system (1) with input τ and output e has a vector relative degree $\{2, \dots, 2\}$ everywhere on Γ. We therefore have the following result.

Lemma 2.4. Regular virtual holonomic constraints (i.e., constraints $q = \hat{\phi}(q_n)$ satisfying (3) or (4)) are locally exponentially stabilizable, i.e., Γ is locally exponentially stabilizable.

We locally exponentially stabilize Γ by setting

$$\tau(q) = \left\{[I_{n-1} \ - \phi'(q_n)]D^{-1}(q)B(q)\right\}^{-1} [-k_1 e - k_2 \dot{e} + \phi''(q_n)\dot{q}_n^2 + [I_{n-1} \ - \phi'(q_n)]D^{-1}(q)(C(q,\dot{q})\dot{q} + g(q))\Big],$$

where $k_1, k_2 > 0$ are design parameters and

$$e = \mathrm{col}(q_1, \dots, q_{n-1}) - \phi(q_n),$$
$$\dot{e} = \mathrm{col}(\dot{q}_1, \dots, \dot{q}_{n-1}) - \phi'(q_n)\dot{q}_n.$$

3. GENERATION OF REGULAR HOLONOMIC CONSTRAINTS

In this section we present a constructive procedure to determining regular virtual constraints $\phi_i(q_n)$, $i = 1, \dots, n-1$, i.e., constraints satisfying (4), i.e., such that

$$B^{\perp}(\hat{\phi}(q_n))D(\hat{\phi}(q_n))\hat{\phi}'(q_n) = \delta(q_n), \tag{5}$$

for some continuous function $\delta(q_n)$ bounded away from zero. Since $D(q)$ is nonsingular, the vector function $B^{\perp}D$ is never zero. Suppose that one of its components, say the first one, is bounded away from zero. This assumption is not essential. We will show in the next example how to overcome it. Pick any C^1 constraints $\phi_2(q_n), \dots, \phi_{n-1}(q_n)$ that are odd functions, i.e., $\phi_j(q_n) = -\phi_j(-q_n)$, $j = 2, \dots, n-1$ and T-periodic, where T is the period of the cyclic variable q_n. We now show that, under mild assumptions, it is possible to choose the remaining virtual constraint $\phi_1(q_n)$ so that (6) holds for some nonzero $\delta(q_n)$ and so, by Lemma 2.1, $\phi_1(q_n), \dots, \phi_n(q_n)$ are feasible. Denote

$$[b_1(\phi_1, q_n) \ \cdots \ b_n(\phi_1, q_n)] := \left[B^{\perp}(q)D(q)\right]_{\substack{q = (\phi_1, \\ \phi_2(q_n), \dots, \phi_{n-1}(q_n), q_n)}} .$$

The arguments of b_i highlight the independent variables, namely the function ϕ_1 and the configuration variable q_n. Since $b_1 \neq 0$ we may rewrite (5) as

$$\frac{d\phi_1}{dq_n} = \frac{1}{b_1(\phi_1, q_n)} \left[-\sum_{i=2}^{n-1} b_i(\phi_1, q_n)\phi_i'(q_n) + \delta(q_n) \right]. \tag{6}$$

The above is a scalar differential equation with state ϕ_1 whose right-hand side is T-periodic because q_n is a cyclic variable of period T. In order for a solution ϕ_1 to be a well-defined function of the cyclic variable q_n, it must be T-periodic. The problem then is to find $\delta(q_n)$, bounded away from zero and T-periodic, so that (6) has a T-periodic solution $\phi_1(q_n)$. Once that is done, $\phi_1(q_n), \dots, \phi_{n-1}(q_n)$ will be a set of feasible virtual constraints. For this reason, we call (6) the **virtual constraint generator**. The next lemma gives conditions for the existence of the desired $\delta(q_n)$ and $\phi_1(q_n)$.

Lemma 3.1. Suppose that $b_1 \neq 0$ and the solution of the initial value problem

$$\frac{d\phi_1}{dq_n} = \frac{1}{b_1(\phi_1, q_n)} \left[-\sum_{i=2}^{n-1} b_i(\phi_1, q_n)\phi_i'(q_n) \right]$$
$$\phi_1(q_n^0) = \phi_1^0$$

is not T-periodic and that for all $q_2, \dots, q_n$ the function $q_1 \mapsto D(q_1, q_2, \dots, q_n)$ is bounded. Let $\mu(q_n)$ be an arbitrary locally Lipschitz and T-periodic function which is positive and even, and set $\delta(q_n) = \epsilon\mu(q_n)$. Then, there exists a unique real number $\epsilon \neq 0$ such that the solution of (6) with initial condition, $\phi_1(q_n^0) = \phi_1^0$, is T-periodic.

Remark 3.2. The lemma above states that under a mild assumption on the inertia matrix[3] and on the solution of (6) with $\delta = 0$, there exists a T-periodic function $\phi_1(q_n)$ guaranteeing controlled invariance of Γ, and thus feasibility of the virtual constraint $\mathrm{col}(q_1, \dots, q_{n-1}) = \phi(q_n)$. The function ϕ_1 can be found systematically by choosing an even function $\mu(q_n) > 0$ and looking for the unique value of ϵ giving T-periodicity. This latter operation can be performed quite easily using numerical integration and a one-dimensional search. Of course, different selections of $\mu(q_n)$ will yield different solutions $\phi_1(q_n)$, so there is some freedom in the choice of ϕ_1.

Proof. Consider (6) with $\delta = \epsilon\mu(q_n)$. Since for each $(q_2, \dots, q_n)$, $q_1 \mapsto D(q_1, q_2, \dots, q_n)$ is bounded, we have that for each $q_n \in S^1$ $\phi_1 \mapsto D(\phi_1, \phi_2(q_n), \dots, \phi_{n-1}(q_n), q_n)$ is bounded as well. This property, the fact that $\|B^{\perp}(q)\| = 1$, and the compactness of S^1 together imply that the functions $b_i(\phi_1, q_n)$ are bounded. The functions $\phi'(q_n)$ and $\mu(q_n)$, being continuous and defined on the compact set S^1, are also bounded. Since the right-hand side of (6) is bounded, all solutions are globally defined. For a given value of the parameter ϵ, let $\Phi(q_n, q_n^0, \phi_1^0, \epsilon)$ denote the solution of (6) at "time" q_n with initial condition $\phi(q_n^0) = \phi_1^0$. Since the differential equation is T-periodic, in order to show that the solution with initial condition $\phi_1(q_n^0) = \phi_1^0$ is T-periodic it suffices to show that $\Phi(q_n^0 + T, q_n^0, \phi_1^0, \epsilon) = \phi_1^0$. Consider the continuous map $f(\epsilon) = \Phi(q_n^0 + T, q_n^0, \phi_1^0, \epsilon)$. The function $(\phi_1, q_n) \mapsto \left|\frac{\mu(q_n)}{b_1(\phi_1, q_n)}\right|$ is bounded from below because both b_1 and μ are bounded and $b_1 \neq 0$. This property implies that the right-hand side of (6) tends to infinity as $\epsilon \to \infty$ and so $f(\epsilon) \to \infty$ as $\epsilon \to \infty$ as well. This fact and the continuity of f imply that $f : \mathbb{R} \to \mathbb{R}$ is surjective, and there exists $\epsilon \in \mathbb{R}$ such that $f(\epsilon) = \phi_1^0$. By assumption, when $\epsilon = 0$ the solution with initial condition

[3] The class of robot manipulators with bounded inertia matrix is very large as it includes all manipulators with revolute joints. Necessary and sufficient conditions for the uniform boundedness of D are found in Ghorbel et al. [1998].

$\phi_1(q_1^0) = \phi_1^0$ is not T-periodic, or $f(0) \neq \phi_1^0$, so it must be that $\epsilon \neq 0$, implying that $\delta(q_n) = \epsilon\mu(q_n)$ is bounded away from zero. Finally, the right-hand side of (6) is strictly monotonic with respect to δ and therefore, by the Comparison lemma, the function $f(\epsilon)$ is strictly monotonic as well, and hence injective. In conclusion, there is a unique $\epsilon \neq 0$ such that $f(\epsilon) = \phi_1^0$. $\square$

The next lemma shows that if in Lemma 3.1 the initial condition is $\phi_1(0) = 0$, then the resulting solution $\phi_1(q_n)$ is odd. This fact will be useful in the next section.

Lemma 3.3. If in Lemma 3.1 we set $q_n^0 = 0$ and $\phi_1^0 = 0$, then the corresponding T-periodic solution $\phi_1(q_n)$ of (6) is odd, i.e., such that $\phi_1(q_n) = -\phi_1(-q_n)$.

Proof. Let $\phi_1(q_n)$ be the solution of (6) with initial condition $\phi_1(0) = 0$, and define $e(q_n) := \phi_1(q_n) + \phi_1(-q_n)$. We want to show that $e(q_n) \equiv 0$.

Write (6) as $d\phi_1/dq_n = h(\phi_1, q_n)$. We claim that $h(-\phi_1, -q_n) = h(\phi_1, q_n)$. In Lemma 3.1 $\delta(q_n)$ is an even function, $\delta(-q_n) = \delta(q_n)$. Moreover, $\phi_2, \ldots, \phi_{n-1}$ were chosen to be odd functions, so their derivatives $\phi_i'(q_n)$, $i = 2, \ldots, n-1$ are even. If we show that $b_i(\phi_1, q_n) = b_i(-\phi_1, -q_n)$ then the claim follows. To this end, recall that $B^\perp(q)$ and $D(q)$ are even functions, so $B^\perp(q)D(q) = B^\perp(-q)D(-q)$ and

$$\begin{aligned}[b_1(\phi_1, q_n) \ \cdots \ b_n(\phi_1, q_n)] &= \left[B^\perp(q)D(q)\right]_{\substack{q = (\phi_1, \phi_2(q_n), \\ \ldots, \phi_{n-1}(q_n), q_n)}} \\ &= \left[B^\perp(q)D(q)\right]_{\substack{q = (-\phi_1, -\phi_2(q_n), \\ \ldots, -\phi_{n-1}(q_n), -q_n)}}.\end{aligned}$$

Since $\phi_2, \ldots, \phi_{n-1}$ are odd functions, the latter expression is equal to

$$\left[B^\perp(q)D(q)\right]_{\substack{q = (-\phi_1, \phi_2(-q_n), \\ \ldots, \phi_{n-1}(-q_n), -q_n)}} = \begin{matrix}[b_1(-\phi_1, -q_n) \ \cdots \\ b_n(-\phi_1, -q_n)],\end{matrix}$$

which proves the claim that $h(-\phi_1, -q_n) = h(\phi_1, q_n)$.

Now consider the derivative of $e(q_n)$

$$\begin{aligned}\frac{de}{dq_n} &= \frac{d\phi_1}{dq_n}(q_n) - \frac{d\phi_1}{dq_n}(-q_n) \\ &= h(e(q_n) - \phi_1(-q_n), q_n) - h(-\phi_1(-q_n), q_n).\end{aligned}$$

The function $(\phi_1, e, q_n) \mapsto h(e - \phi_1, q_n) - h(-\phi_1, q_n)$ is continuous and identically zero when $e = 0$, so there exists a continuous function $\tilde{h}(\phi_1, e, q_n)$ such that $h(e - \phi_1, q_n) - h(-\phi_1, q_n) = \tilde{h}(\phi_1, e, q_n)e$. Using this identity in the derivative of e we get $\frac{de}{dq_n} = \tilde{h}(\phi_1(q_n), e(q_n), q_n)e(q_n)$. Since q_n belongs to the compact set S^1 and $q_n \mapsto \tilde{h}(\phi_1(q_n), e(q_n), q_n)$ is continuous, there exists $M > 0$ such that $|\tilde{h}(\phi_1(q_n), e(q_n), q_n)| \leq M$, and thus $|de/dq_n| \leq M|e(q_n)|$. By Gronwall's lemma, $|e(q_n)| \leq |e(0)| \exp(Mq_n)$. However, $e(0) = 0$ because by assumption $\phi_1(0) = 0$, and therefore $e(q_n) = 0$. $\square$

Procedure: determination of regular holonomic constraints

We want to find virtual constraints ϕ_i, $i = 1, \ldots, n-1$, in terms of one of the cyclic configuration variables, say q_n.

(i) Assume that one of the first $n-1$ coefficients of the row vector $B^\perp(q)D(q)$ is never zero. Suppose, without loss of generality, that it is the first one. If this assumption does not hold, then it may be possible to modify the procedure. This issue is explored in the second part of the next example.

(ii) Choose arbitrary functions $\phi_2(q_n), \ldots, \phi_{n-1}(q_n)$ that are T-periodic and odd, i.e., $\phi_i(q_n) = -\phi_i(-q_n)$.

(iii) Write the scalar T-periodic differential equation (6). Check whether the solution of the differential equation with initial condition $\phi_1(0) = 0$ and with $\delta = 0$ is not T-periodic.

(iv) Set $\delta(q_n) = \epsilon\mu(q_n)$, where $\mu(q_n)$ is any locally Lipschitz and T-periodic function which is positive and even. Find $\epsilon \in \mathbb{R}$ such that the solution $\phi_1(q_n)$ of (6) with initial condition $\phi_1(0) = 0$ is such that $\phi_1(T) = 0$. Such value of ϵ is unique, and guaranteed to exist by Lemma 3.1. The corresponding solution $\phi_1(q_n)$ will be odd and T-periodic.

(v) The virtual constraints $q_1 = \phi_1(q_n), \ldots, q_{n-1} = \phi_{n-1}(q_n)$ so obtained are feasible.

The check in step (iii) can be done by a simple numerical integration, and the value of ϵ in step (iv) can be efficiently computed with a one dimensional search and numerical integration of (6).

Example 3.4. We will present two different classes of virtual constraints for the double pendulum. Since $n = 2$, we have to find one virtual constraint. We begin by looking for a constraint of the form $\theta_2 = \phi(\theta_1)$. We have $B^\perp(q)D(q) = [\cos(\theta_1 - \theta_2) \quad 1]$. The second coefficient is never zero. We have $\hat{\phi}(\theta_1) = \mathrm{col}(\theta_1, \phi(\theta_1))$ and the virtual constraint generator (6) reads as $\frac{d\phi}{d\theta_1} = -\cos(\theta_1 - \phi(\theta_1)) + \delta(\theta_1)$. When $\delta = 0$, the solution with initial condition $\phi(0) = 0$ is $\phi(\theta_1) = \theta_1 - 2\arctan\theta_2$, which is not 2π-periodic. Pick, for instance, $\mu(\theta_1) = 1$ (obviously positive, even, and T-periodic) so set $\delta(\theta_1) = \epsilon$. The virtual constraint generator becomes $\frac{d\phi}{d\theta_1} = -\cos(\theta_1 - \phi(\theta_1)) + \epsilon$. The solution with zero initial condition is periodic when $\epsilon = 1 - \sqrt{2}$, and is given[4] by $\phi(\theta_1) = \theta_1 + 2\arctan[\tan(-\theta_1/2)(1 + \sqrt{2})]$. This constraint is, by construction, feasible and odd. It is depicted in Figure 3. The interesting feature of this

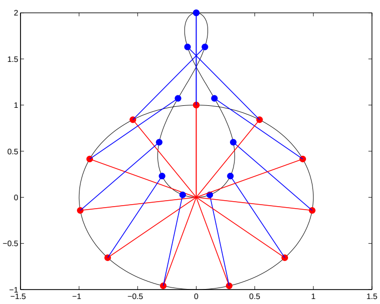

Fig. 3. Configurations of the double pendulum with the virtual constraint $\theta_2 = \theta_1 + 2\arctan[\tan(-\theta_1/2)(1 + \sqrt{2})]$.

constraint is that, on it, the second pendulum does not perform complete revolutions and its angle has average value 0. Here we have taken the simplest choice for $\mu(\theta_1)$. Different choices of μ lead to different constraints.

[4] This function has 2π jumps $\theta_1 = k\pi$, but its value modulo 2π is in fact smooth and 2π-periodic.

Next, we explore a second type of constraint, $\theta_1 = \phi(\theta_2)$. Since the first component of $\cos(\theta_1 - \theta_2)$ is zero when $\theta_2 - \theta_2 = \pm\pi/2$ modulo 2π, the assumption in step (i) of the procedure fails. Nonetheless, a simple modification overcomes this obstacle. Relation (6) is given by $\cos(\phi(\theta_2) - \theta_2)\phi'(\theta_2) + 1 = \delta(\theta_2)$. Set $\delta(\theta_2) = 1 + \cos(\phi(\theta_2) - \theta_2)\mu(\theta_2)$, where $\mu(\theta_2)$ is any 2π-periodic C^1 function that is even, has zero average, and satisfies $|\mu(\theta_2)| < 1$. This choice guarantees that $\delta(\theta_2)$ is bounded away from zero and the above equation holds if $\frac{d\phi}{d\theta_2} = \mu(\theta_2)$. The solution with $\phi(0) = 0$ is $\phi(\theta_2) = \int_0^{\theta_2} \mu(\tau)d\tau$. Since μ is 2π-periodic and even with zero average, its antiderivative $\phi(\theta_2)$ is 2π-periodic and odd, as required. We have thus found a second class of feasible virtual constraints $\theta_1 = \phi(\theta_2)$. Choosing, for instance, $\mu(\theta_2) = \cos(2\theta_2)/2$, we get $\phi(\theta_2) = \sin(2\theta_2)/4$, which is precisely the constraint examined in Example 2.3 and depicted in Figure 2.

4. MOTION ON THE CONSTRAINT MANIFOLD

Suppose that through the procedure in the previous section we have determined a set of regular virtual holonomic constraints. In order to determine the dynamics on the constraint manifold $\Gamma = \{(q, \dot{q}) : q_1 = \phi_1(q_n), \ldots, q_{n-1} = \phi_{n-1}(q_n)\}$, we multiply both sides of (1) by $B^\perp(q)$ to the left, obtaining

$$\left\{ B^\perp D\,\hat{\phi}'\ddot{q}_n + B^\perp \left[D\,\hat{\phi}''\dot{q}_n^2 + C\,\hat{\phi}'\dot{q}_n + g \right] \right\}_{\substack{q\,=\,\hat{\phi}(q_n),\\ \dot{q}\,=\,\hat{\phi}'(q_n)\dot{q}_n}} = 0.$$

Using (5), rewrite the above as

$$\ddot{q}_n = -\frac{B^\perp(\hat{\phi}(q_n))}{\delta(q_n)} \left[D\,\hat{\phi}''(q_n)\dot{q}_n^2 + C\,\hat{\phi}'(q_n)\dot{q}_n + g \right]_{\substack{q\,=\,\hat{\phi}(q_n),\\ \dot{q}\,=\,\hat{\phi}'(q_n)\dot{q}_n}}.$$

The product $B^\perp(q)C(q,\dot{q})$ is given by $B^\perp(q)C(q,\dot{q}) = \sum_{i=1}^n B_i^\perp(q)\dot{q}^\top Q_i(q)\dot{q}$, where $Q_i(q)$ is a symmetric matrix whose (j,k) entry is the Christoffel coefficient

$$(Q_i)_{jk} = \frac{1}{2}\left\{ \frac{\partial D_{ij}}{\partial q_k} + \frac{\partial D_{ik}}{\partial q_j} - \frac{\partial D_{kj}}{\partial q_i} \right\}.$$

By Assumption 1, all entries of $D(q)$ are even, and thus the partial derivatives in the above definition of $(Q_i)_{jk}$ are odd functions of q. We have

$$\dot{q}^\top Q_i(q)\dot{q}\Big|_{q=\hat{\phi}(q_n),\ \dot{q}=\hat{\phi}'(q_n)\dot{q}_n} = \hat{\phi}'(q_n)^\top Q_i(\hat{\phi}(q_n))\hat{\phi}'(q_n)\dot{q}_n^2$$

and so letting

$$\begin{aligned} \Psi_1(q_n) &= -\frac{B^\perp(\hat{\phi}(q_n))}{\delta(q_n)} g(\hat{\phi}(q_n)) \\ \Psi_2(q_n) &= -\frac{1}{\delta(q_n)} \Bigg[B^\perp(\hat{\phi}(q_n))D(\hat{\phi}(q_n))\hat{\phi}''(q_n) \\ &\quad + \sum_{i=1}^n B_i^\perp(\hat{\phi}(q_n))\hat{\phi}'(q_n)^\top Q_i(\hat{\phi}(q_n))\hat{\phi}'(q_n) \Bigg] \end{aligned} \tag{7}$$

we have

$$\ddot{q}_n = \Psi_1(q_n) + \Psi_2(q_n)\dot{q}_n^2. \tag{8}$$

System (8) represents the dynamics on the constraint manifold. Remarkably, this is a one degree-of-freedom Euler-Lagrange system with Lagrangian $L(q_n,\dot{q}_n) = \frac{1}{2}M(q_n)\dot{q}_n^2 - V(q_n)$, where

$$\begin{aligned} M(q_n) &= \exp\left\{ -2\int_0^{q_n} \Psi_2(\tau)d\tau \right\} \\ V(q_n) &= -\int_0^{q_n} \Psi_1(\mu)M(\mu)d\mu. \end{aligned} \tag{9}$$

In order for the mass $M(q_n)$ and potential $V(q_n)$ in (9) to be well-defined functions of the cyclic variable $q_n \in S^1$, they must be T-periodic. The fact that they are, shown below, is a consequence of Assumption 1.

Lemma 4.1. Consider the second-order differential equation (8). The functions $\Psi_1(q_n)$ and $\Psi_2(q_n)$ in (7) are C^1, T-periodic, and odd. Therefore, $M(q_n)$ and $V(q_n)$ in (9) are T-periodic and even, i.e., $M(q_n) = M(-q_n)$ and $V(q_n) = V(-q_n)$.

Proof. The functions $\Psi_1(q_n)$ and $\Psi_2(q_n)$ in (7) are obviously C^1 and T-periodic. It is easy to see that they are also odd. The lemma follows from the elementary fact that the antiderivative of a T-periodic odd function is a T-periodic even function. Indeed, from this fact we get that $M(q_n)$ is even and T-periodic, and thus $\Psi_1(q_n)M(q_n)$ is odd and T-periodic. Using the fact again we get that V is even and T-periodic. □

The fact that (9) is a one degree-of-freedom Euler-Lagrange system with known Lagrangian makes it possible to completely characterize the properties of the motion on Γ in terms of the potential function $V(q_n)$. The next result presents the main properties of the motion. Its proof is omitted.

Proposition 4.2. Consider the dynamics (8) on the constraint manifold Γ. The equilibrium configurations are the points q_n such that $\nabla P(\hat{\phi}(q_n)) \in \mathrm{Im}(B(\hat{\phi}(q_n)))$, where $P(q)$ is the potential of the original system (1). There are at least two equilibria at $q_n = 0$ and $q_n = T/2$. Their stability type is determined by the sign of the expression

$$\delta(q_n)\frac{d}{dq_n}\left[B^\perp(\hat{\phi}(q_n))g(\hat{\phi}(q_n)) \right]\Big|_{q_n=0,T/2}$$

(positive $\implies$ stable, negative $\implies$ unstable, 0 $\implies$ no conclusion). Let $\underline{V} = \min_{q_n\in S^1} V(q_n)$ and $\bar{V} = \max_{q_n\in S^1} V(q_n)$. Then, all phase curves of (8) in the set $\{(q_n,\dot{q}_n) \in S^1\times\mathbb{R} : 1/2M(q_n)\dot{q}_n^2 + V(q_n) > \bar{V}\}$ are homeomorphic to circles $\{(q_n,\dot{q}_n) \in S^1\times\mathbb{R} : \dot{q}_n = \text{ constant }\}$, while almost all (in the Lebesgue sense) phase curves in the set $\{(q_n,\dot{q}_n) \in S^1\times\mathbb{R} : \underline{V} < 1/2M(q_n)\dot{q}_n^2 + V(q_n) < \bar{V}\}$ are homeomorphic to circles $\{(q_n,\dot{q}_n) : q_n^2 + \dot{q}_n^2 = \text{ constant }\}$.

The second part of the above proposition can be rephrased as follows. For initial conditions in Γ with energy greater than $\bar{V}$, the resulting phase curve of the system traverses the entire curve $q = \hat{\phi}(q_n)$ in either direction, depending on the sign of $\dot{q}_n(0)$. On the other hand, almost all solutions on Γ with energy in the interval $(\underline{V},\bar{V})$ correspond to rocking motions whereby the curve $q = \hat{\phi}(q_n)$ is not traversed in its entirety.

Example 4.3. We will now use the result of Proposition 4.2 to investigate the motion of the double pendulum on the constraint manifolds for the two virtual constraints presented in Example 3.4. For the constraint $\theta_2 = \phi(\theta_1) = \theta_1 + 2\arctan[\tan(-\theta_1/2)(1+\sqrt{2})]$, depicted in Figure 3, the equilibrium configurations θ_1 are given by the condition $\nabla P(\hat{\phi}(\theta_1))) \in \mathrm{Im}\,B \iff \phi(\theta_1) = 0,\pi \iff \theta_1 = 0,\pi.$

These are precisely the equilibria predicted by Proposition 4.2. Their stability type is determined by the sign of $d/dq_n(B^{\perp}g(\hat{\phi}(\theta_1)) = \cos\phi\phi'|_{\theta_1=0,\pi} = \phi'|_{\theta_1=0,\pi}$. We have $\phi'(0) < 0$ and $\phi'(\pi) > 0$, so the configuration $\theta_1 = \pi$ is stable, while $\theta_1 = 0$ is unstable. The level sets of the energy (and hence the phase portrait) of the reduced Euler-Lagrange system on the constraint manifold Γ are depicted in Figure 4. The shaded area in the figure is the region where θ_1 oscillates without performing complete revolutions. The remaining region corresponds to full revolutions of θ_1 in either direction.

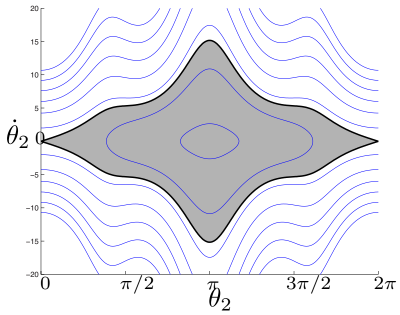

Fig. 4. Energy level sets for double pendulum on the virtual constraint $\theta_1 = \sin(2\theta_2)/4$.

Now consider the constraint $\theta_1 = \phi(\theta_2) = \sin(2\theta_2)/4$ depicted in Figure 2. The condition $\nabla P(\hat{\phi}(\theta_2))) \in \operatorname{Im} B$ gives equilibrium configurations $\theta_2 = 0, \pi$ and it is easily seen that $\theta_2 = 0$ is unstable, while $\theta_2 = \pi$ is stable. The level sets of the energy for the motion on Γ are depicted in Figure 5 where, once again, the shaded region indicates a rocking motion of θ_1 around its stable equilibrium while the unshaded area corresponds to full revolutions.

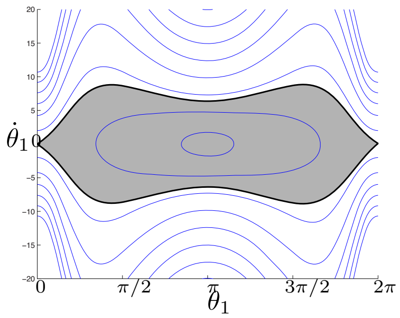

Fig. 5. Energy level sets for double pendulum on the virtual constraint $\theta_2 = \theta_1 + 2\arctan[\tan(-\theta_1/2)(1+\sqrt{2})]$.

Simulation results for the stabilization of the two virtual constraints $\theta_1 = \sin(2\theta_2)/4$ and $\theta_2 = \theta_1 + 2\arctan[\tan(-\theta_1/2)(1+\sqrt{2})]$ are depicted in Figures 6 and 7. It is particularly interesting to observe, in parts 1 and 4 of Figure 7, the "throwing" motion that the first link performs in order to make the second link approach the set of configurations corresponding to the virtual constraint.

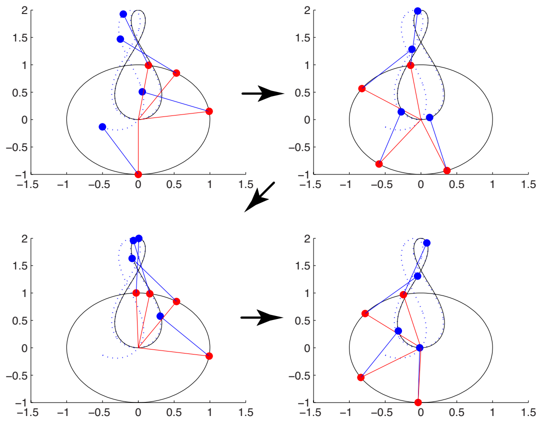

Fig. 6. Stabilization of the constraint $\theta_2 = \theta_1 + 2\arctan[\tan(-\theta_1/2)(1+\sqrt{2})]$.

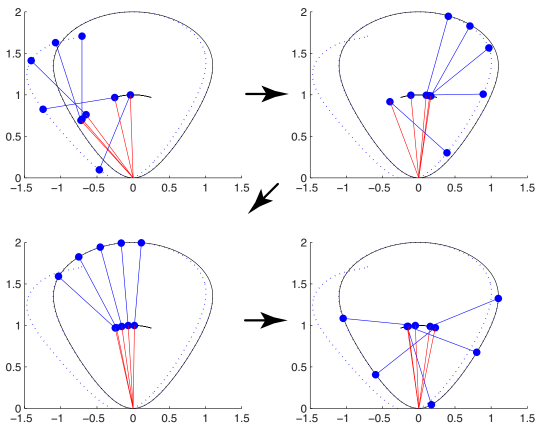

Fig. 7. Stabilization of the constraint $\theta_1 = \sin(2\theta_2)/4$.

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