Virtual Holonomic Constraints for Euler-Lagrange Systems

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Abstract

This paper investigates *virtual holonomic constraints* for Euler-Lagrange systems with n degrees-of-freedom and $n-1$ controls. In our framework, a virtual holonomic constraint is a relation specifying $n-1$ configuration variables in terms of a single angular configuration variable. The enforcement by feedback of such a constraint induces a desired repetitive behavior in the system. We give conditions under which a virtual holonomic constraint is feasible, i.e, it can be made invariant by feedback, and it is stabilizable. We provide sufficient conditions under which the dynamics on the constraint manifold correspond to an Euler-Lagrange system. These ideas are applied to the problem of swinging up an underactuated pendulum while guaranteeing that the second link does not fall over.

I. Introduction

A virtual holonomic constraint (VHC) for a mechanical system with configuration vector q is a relation of the form $h(q) = 0$ that can be made invariant via feedback. In the past decade, VHC's have emerged as a valuable tool to solve motion control problems, among them the stabilization of walking motion for biped robots pioneered by J. Grizzle and collaborators (see, e.g., [1]–[4]). VHC's can be used to make a haptic interface emulate the presence of an obstacle in a virtual environment or, more generally, to constrain the operator hand to move along preferred directions and to sense the virtual environment [5]. In mobile robotics, VHC's can be enforced to make a group of vehicles move in formation. Camless combustion engines use actuated valves in place of the camshaft to regulate valve phasing, thus allowing valve phasing to be reprogrammed on-the-fly in order to optimize the operation of the engine [6]. Controlling valve phasing corresponds to replacing the mechanical constraint imposed by the camshaft with a VHC enforced by control. The use of VHC's in place of physical constraints allows greater flexibility in the design and operation of mechanical devices.

The concept of virtual holonomic constraint can be traced back to work by Paul Appell in 1911 (see [7]) and was introduced more explicitly in 1922 by Henri Beghin (see [8]) as a constraint that can be enforced through the application of external forces. The reader is referred to the work in [9], [10]. In [11], the authors investigate controlled mechanical systems for which a feedback is to enforce a holonomic constraint. They show that constrained solutions obey a differential-algebraic equation, and propose a technique to solve such equation. Recently,

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Shiriaev and collaborators in [12]–[16] initiated a study of VHC's for underactuated mechanical systems. In [12], the object of study is a system with n degrees-of-freedom and $n-1$ controls. It is shown that the constrained motion is described by an unforced second-order system which possesses an integral of motion. Assuming that this system has a closed orbit γ, a methodology is developed to stabilize γ based on the so-called transverse linearization of the system. The resulting linear time-varying controller yields exponential stability of the closed orbit. In [13], a detailed investigation of the integrals of motion is presented. In [15], the transverse linearization technique is generalized to the case when the degree of underactuation is greater than one. In [14], [17], VHC's are used to select and stabilize desired oscillations of the Furuta pendulum and the pendubot.

A number of key questions pertaining to VHC's remain open. First and foremost, an explicit definition of VHC is missing in the control literature. A proper definition should contain the requirement that the constraint be *feasible* for the dynamics of the control system. For instance, for a point-mass with coordinates (x, y, z) subject to gravity and accelerated by a horizontal control force, the constraint $z = 0$ is not feasible since the control force cannot keep the mass on a horizontal plane. Second, conditions for feasibility of VHC's have not been investigated in the literature. For feasible VHC's, the constrained dynamics have been shown in [12] to have an integral of motion when the degree of underactuation is one. However, as we will show, the presence of this integral of motion does not imply that the constrained dynamics are Euler-Lagrange. This phenomenon, whose existence has not been recognized in the literature, requires investigation.

This paper provides answers to the two questions above. In Section II, we give a definition of VHC which embodies the feasibility requirement, and in Section III we provide conditions under which a VHC is feasible and stabilizable. In Section IV, we first show that the constrained motion of an Euler-Lagrange system with a VHC is generally *not* Euler-Lagrange. We then present sufficient conditions on the system data guaranteeing that the constrained dynamics are Euler-Lagrange. Finally, in Section V we use VHC's to design a controller that swings up the pendubot from the low-high to the high-high equilibrium while simultaneously guaranteeing that during the transient phase the unactuated link does not fall over. In this example, we enforce a VHC to prevent the unactuated link from falling over. We then stabilize a closed orbit on the constraint manifold to meet the swing-up requirement. The technique we present to stabilize the closed orbit has independent interest and it can be generalized, but it is not the main focus of the paper and will be developed elsewhere.

We use the following notation. If $x \in \mathbb{R}$ and $T > 0$, then x modulo T is denoted by $[x]_T$, and the set $\{[x]_T : x \in \mathbb{R}\}$ is denoted by $[\mathbb{R}]_T$. This set can be given a manifold structure which makes it diffeomorphic to the unit circle. If a and b are vectors, then $\mathrm{col}(a, b) := [a^\top \; b^\top]^\top$. If $h : M \to N$ is a smooth map between manifolds, and $q \in M$, we denote by $dh_q : T_q M \to T_{h(q)} N$ the derivative of h at q. Given a function $h : \mathcal{Q} \to \mathbb{R}^k$, then $h^{-1}(0) := \{q \in \mathcal{Q} : h(q) = 0\}$.

II. Problem formulation

In this paper we investigate underactuated Euler-Lagrange systems with an n-dimensional configuration space $\mathcal{Q}$ and $n-1$ controls $\tau \in \mathbb{R}^{n-1}$. The model is given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = B(q)\tau.$$

In the above, $B : \mathcal{Q} \to \mathbb{R}^{n \times n-1}$ is smooth and it has full rank $n-1$. The Lagrangian function $L(q, \dot{q})$ is smooth, and we assume that it has the form $L(q, \dot{q}) = (1/2)\dot{q}^\top D(q)\dot{q} - P(q)$, where $D(q)$, the inertia matrix, is positive definite for all q, and $P(q)$, the potential energy function, is smooth. The system can be rewritten in the standard form

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + \nabla P(q) = B(q)\tau. \tag{1}$$

We assume that $q_1, \ldots, q_p$, $p \leq n-1$, are linear displacements in $\mathbb{R}$ and $q_{p+1}, \ldots, q_n$ are angular variables, with $q_i \in [\mathbb{R}]_{T_i}$ (typically, $T_i = 2\pi$). The configuration space of the system is $\mathcal{Q} = \mathbb{R}^p \times [\mathbb{R}]_{T_{p+1}} \times \cdots \times [\mathbb{R}]_{T_n}$. We assume throughout that $B(q)$ has a left-annihilator $B^\perp(q)$; that is, there is a smooth function $B^\perp : \mathcal{Q} \to \mathbb{R}^{1 \times n} \backslash \{0\}$ such that $B^\perp(q)B(q) = 0$ for all q.

As the adjective "virtual" suggests, a VHC is a holonomic constraint that is not physically existing, but it can be made by control to appear to do so. In control theoretic terms, the property of the holonomic constraint being "virtual" is embodied by the notion of controlled invariance, as in the next definition.

Definition 2.1: A **virtual holonomic constraint (VHC) of order** k for system (1) is a relation $h(q) = 0$, where $h : \mathcal{Q} \to \mathbb{R}^k$ is smooth, $\operatorname{rank}(dh_q) = k$ for all $q \in h^{-1}(0)$, and the set

$$\Gamma = \{(q, \dot{q}) : h(q) = 0,\ dh_q \dot{q} = 0\} \tag{2}$$

is controlled invariant. That is, there exists a smooth feedback $\tau(q, \dot{q})$ such that Γ is positively invariant for the closed-loop system. The set Γ is called the **constraint manifold** associated with the VHC $h(q) = 0$. A VHC is **stabilizable** if there exists a smooth feedback $\tau(q, \dot{q})$ that asymptotically stabilizes[1] Γ. In this case, the feedback $\tau(q, \dot{q})$ is said to **enforce the VHC** $h(q) = 0$. $\triangle$

In the definition above, the condition that dh_q has full rank on $h^{-1}(0)$ guarantees that the set $h^{-1}(0)$ is an $n-k$-dimensional submanifold of $\mathcal{Q}$. As mentioned earlier, the requirement that Γ is controlled invariant embodies the notion that although the holonomic constraint $h(q) = 0$ does not physically exist, it can be made by control to appear to do so. Specifically, whenever the configuration vector $q(0)$ is initialized on the constraint set $h^{-1}(0)$, and its initial velocity $\dot{q}(0)$ is tangent to $h^{-1}(0)$, then the resulting configuration trajectory $q(t)$ can be made, through appropriate control, to satisfy the constraint for all $t \geq 0$. This matter will be further illustrated in Example 3.4. Another reason for requiring controlled invariance of Γ in Definition 2.1 is that this property is a necessary condition for Γ to be stabilizable[2].

In this paper we investigate VHC's of order $n-1$. In this case each connected component of $h^{-1}(0)$ is a smooth curve without self-intersections. We will consider VHC's with the property that $h^{-1}(0)$ is a closed curve, with the interpretation that the constraint corresponds to a desired repetitive behavior. It is convenient to adopt a parametric description of the VHC, in which $n-1$ configuration variables are expressed as smooth functions of the remaining configuration variable:

$$q_1 = \phi_1(q_n),\ \cdots,\ q_{n-1} = \phi_{n-1}(q_n). \tag{3}$$

[1]We say that Γ is asymptotically stable for the closed-loop system if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $(q(0), \dot{q}(0)) \in \{(q, \dot{q}) : \|(q, \dot{q})\|_\Gamma < \delta\}$ the solution of the closed-loop system remains in $\{(q, \dot{q}) : \|(q, \dot{q})\|_\Gamma < \varepsilon\}$ and asymptotically converges to Γ, where $\|\cdot\|_\Gamma$ denotes the point-to-set distance to Γ.

[2]A necessary condition for a closed set Γ to be asymptotically stable for a dynamical system $\dot{x} = f(x)$ is that Γ is positively invariant (see [18]). This readily implies that given a control system $\dot{x} = f(x, u)$, a necessary condition for a closed set Γ to be stabilizable is that Γ is controlled invariant.

Here, $h(q) = \mathrm{col}(q_1 - \phi_1(q_n), \ldots, q_{n-1} - \phi_{n-1}(q_n))$, and $h^{-1}(0)$ is a closed curve because $\phi_i(q_n + T_n) = \phi_i(q_n)$, since $q_n \in [\mathbb{R}]_{T_n}$. The parametric representation (3) of a VHC is used in the literature to describe a vast array of repetitive behaviors, such as the walking motion in biped robots (see e.g., [1], [12], [16]). We let $\phi(q_n) = \mathrm{col}(\phi_1(q_n), \ldots, \phi_{n-1}(q_n))$ and $\hat{\phi}(q_n) = \mathrm{col}(\phi(q_n), q_n)$, so that we can express the constraint in (3) as $q = \hat{\phi}(q_n)$.

Our investigation of virtual holonomic constraints focuses on two basic problems:

P1 (Section III) Find conditions ensuring that a given relation $h(q) = 0$ or $q = \hat{\phi}(q_n)$ is a VHC which is stabilizable, and design a feedback that enforces the VHC.

P2 (Section IV) Find conditions ensuring that the reduced dynamics are Euler-Lagrange; characterize the qualitative properties of the reduced motion.

In addition to the investigation of the problems above, in Section V we outline through a design example a technique to stabilize closed orbits on the constraint manifold.

III. Regular VHC's and stabilizability of Γ

In this section we address problem **P1** by presenting conditions for a given relation $h(q) = 0$ or $q = \hat{\phi}(q_n)$ to be a VHC. At the same time, we want to characterize the stabilizability of the VHC in question. We begin with a definition.

Definition 3.1: A relation $h(q) = 0$ is a **regular VHC** if the output function $e = h(q)$ yields vector relative degree $\{2, \cdots, 2\}$ everywhere on the set $\Gamma = \{(q, \dot{q}) : h(q) = 0, dh_q \dot{q} = 0\}$. $\triangle$

This definition implies that any output function $e = h(q)$ yielding a vector relative degree $\{2, \cdots, 2\}$ on Γ is a VHC. Indeed, in this case system (1) with output $e = h(q)$ is input-output feedback linearizable, and the associated zero dynamics manifold is precisely Γ (see [19]). Thus, Γ is controlled invariant, and $h(q) = 0$ is a VHC.

Under mild hypotheses, regular VHC's are stabilizable. If there exist strictly increasing functions $\alpha, \beta : [0, r) \to [0, +\infty)$, with $r > 0$, such that the map $H : (q, \dot{q}) \mapsto \mathrm{col}(h(q), dh_q \dot{q})$ is bounded as $\alpha(\|(q, \dot{q})\|_\Gamma) \leq \|H(q, \dot{q})\| \leq \beta(\|(q, \dot{q})\|_\Gamma)$, then an input-output linearizing feedback asymptotically stabilizes Γ, provided that the closed-loop system does not have finite escape times. For parametric VHC's of the form $q = \hat{\phi}(q_n)$, the inequality above is always satisfied, and therefore regular VHC's in parametric form $q = \hat{\phi}(q_n)$ are always stabilizable, and they are enforced by the input-output feedback linearizing feedback

$$
\begin{aligned}
\tau(q, \dot{q}) = \Big\{ [I_{n-1} \quad -\phi'(q_n)] D^{-1}(q) B(q) \Big\}^{-1} \big[& -k_1 e - k_2 \dot{e} \\
& + \phi''(q_n) \dot{q}_n^2 + [I_{n-1} \quad -\phi'(q_n)] D^{-1}(q) (C(q, \dot{q}) \dot{q} + \nabla P(q)) \big],
\end{aligned} \tag{4}
$$

where $k_1, k_2 > 0$ are design parameters and $e = \mathrm{col}(q_1, \ldots, q_{n-1}) - \phi(q_n)$, $\dot{e} = \mathrm{col}(\dot{q}_1, \ldots, \dot{q}_{n-1}) - \phi'(q_n) \dot{q}_n$. Note that, with the definition of e given above, the linearizing feedback (4) yields $\ddot{e} = -k_1 e - k_2 \dot{e}$, so that $e(t) \to 0$ exponentially. The next proposition provides necessary and sufficient conditions for relations $h(q) = 0$ or $q = \hat{\phi}(q_n)$ to be regular VHC's.

Proposition 3.2: Let $h : \mathcal{Q} \to \mathbb{R}^k$ be smooth and such that $\mathrm{rank}\, dh_q = k$ for all $q \in h^{-1}(0)$. Then, $h(q) = 0$ is a regular VHC of order k if and only if

(i) $(\forall q \in h^{-1}(0))$ $\dim \left[\mathrm{Im}(D^{-1}(q) B(q)) \cap \mathrm{Ker}(dh_q) \right] = n - 1 - k$.

A parametric relation $q = \hat{\phi}(q_n)$ is a regular VHC of order $n - 1$ if and only if either one of the following holds

(ii) $(\forall q_n \in [\mathbb{R}]_{T_n})$ $\mathrm{Im}(B(\hat{\phi}(q_n))) \cap \mathrm{Im}(D(\hat{\phi}(q_n)) \hat{\phi}'(q_n)) = \{0\}$.

(iii) $(\forall q_n \in [\mathbb{R}]_{T_n})\ B^{\perp}(\hat{\phi}(q_n))D(\hat{\phi}(q_n))\hat{\phi}'(q_n) \neq 0$,
where $B^{\perp}(q)$ is a left-annihilator of $B(q)$.

Proof: Consider a smooth relation $h(q) = 0$. Letting $e = h(q)$, we have $\ddot{e}|_{\{e=0,\dot{e}=0\}} = (\star) + dh_q D^{-1}(q)B(q)\tau$, where the term $(\star)$ is a suitable smooth function of $(q, \dot{q})$ which is independent of τ. Then, $h(q) = 0$ is a regular VHC if and only if the matrix $dh_q D^{-1}(q)B(q)$ is nonsingular for all $q \in h^{-1}(0)$. This condition is equivalent to (i). Now consider a parametric relation $q = \hat{\phi}(q_n)$. In this case, $k = n-1$, $h(q) = \mathrm{col}(q_1, \ldots, q_{n-1}) - \phi(q_n)$, $dh_q = [I_{n-1} \ \ -\phi'(q_n)]$, and $\mathrm{Ker}(dh_q) = \mathrm{Im}(\mathrm{col}(\phi'(q_n), 1)) = \mathrm{Im}(\hat{\phi}'(q_n))$. Condition (i) becomes $\mathrm{Im}(D^{-1}(q)B(q))|_{q=\hat{\phi}(q_n)} \cap \mathrm{Im}(\hat{\phi}'(q_n)) = \{0\}$. This condition is equivalent to (ii). Finally, if $B^{\perp}(q)$ is a left-annihilator of $B(q)$, then $\mathrm{Im}(B(q)) = \mathrm{Ker}(B^{\perp}(q))$, and therefore condition (ii) can be rewritten as $\mathrm{Ker}(B^{\perp}(\hat{\phi}'(q_n))) \cap \mathrm{Im}(D(\hat{\phi}(q_n)\hat{\phi}'(q_n))) = \{0\}$, which is equivalent to condition (iii). ■

Remark 3.3: The mechanical interpretation of the regularity property is this. In order for $h(q) = 0$ to be a regular VHC of order k, for each $q \in h^{-1}(0)$, $n-1-k$ of the acceleration directions that can be imparted by the control input must be transversal to the tangent space of $h^{-1}(0)$ at q.

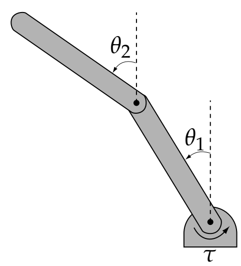

Fig. 1. The pendubot, an underactuated double-pendulum.

Example 3.4: Consider the pendubot system in Figure 1. This is a double-pendulum in which the shoulder is actuated while the elbow is not. The configuration variables (θ_1, θ_2) are in $[\mathbb{R}]_{2\pi} \times [\mathbb{R}]_{2\pi}$. Assuming that the masses and lengths of the two links are equal and unitary, and neglecting mechanical friction, we have for this system

$$\begin{aligned} D(q) &= \begin{bmatrix} 2 & \cos(\theta_1 - \theta_2) \\ \cos(\theta_1 - \theta_2) & 1 \end{bmatrix}, \\ C(q, \dot{q}) &= \begin{bmatrix} 0 & \sin(\theta_1 - \theta_2)\dot{\theta}_2 \\ -\sin(\theta_1 - \theta_2)\dot{\theta}_1 & 0 \end{bmatrix} \\ P(q) &= 2g\cos\theta_1 + g\cos\theta_2, \ B = \mathrm{col}(1, 0). \end{aligned} \tag{5}$$

Consider the relation $\theta_2 = \phi(\theta_1) = 2\theta_1$. To check whether such a relation is a regular VHC, we set $\hat{\phi}(\theta_1) = \mathrm{col}(\theta_1, \phi(\theta_1)) = \mathrm{col}(\theta_1, 2\theta_1)$. Letting $B^{\perp} = [0 \ \ 1]$, we have $B^{\perp}D(\hat{\phi}(\theta_1))\hat{\phi}'(\theta_1) = 2 + \cos(\theta_1) \neq 0$ for all $\theta_1 \in [\mathbb{R}]_{2\pi}$, proving that the relation $\theta_2 = 2\theta_1$ is a regular VHC. Being regular, this VHC is also stabilizable. On other hand, the relation $\theta_2 = \theta_1/2$ is not a regular VHC, since $B^{\perp}D(\hat{\phi}(\theta_1))\hat{\phi}'(\theta_1) = 1/2 + \cos(\theta_1/2)$ is not bounded away from zero. Actually, the relation $\theta_2 = \theta_1/2$ is not even a VHC. To illustrate, let $e = \theta_2 - \theta_1/2$. For any value of the variable $\dot{\theta}_1$, the state $(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = (8\pi/3, 4\pi/3, \dot{\theta}_1, \dot{\theta}_1/2)$ belongs to the set $\Gamma = \{\theta_2 = \theta_1/2, \dot{\theta}_2 = \dot{\theta}_1/2\}$, but one can check that $\ddot{e}|_{(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2)=(8\pi/3,4\pi/3,\dot{\theta}_1,\dot{\theta}_1/2)} = -(\sqrt{3}/2)(g + \dot{\theta}_1^2) \neq 0$. Therefore, there is no feedback making $\ddot{e} = 0$ everywhere on Γ, and Γ is not controlled invariant. In other words, no matter how one initializes the joint velocities of the pendulum, when the joint angles are initialized as $(\theta_1(0), \theta_2(0)) = (8\pi/3, 4\pi/3)$, no choice of τ can make the pendulum satisfy the constraint $\theta_2 = \theta_1/2$. This example illustrates the importance of requiring controlled invariance of Γ in the definition of VHC. $\triangle$

IV. Motion on the constraint manifold

In this section we investigate problem **P2**. Given a regular VHC in parametric form $q = \hat{\phi}(q_n)$, we determine the reduced dynamics of system (1) on the virtual constraint manifold $\Gamma = \{(q, \dot{q}) : q = \hat{\phi}(q_n),\ \dot{q} = \hat{\phi}'(q_n)\dot{q}_n\}$. In Section IV-A, we present conditions under which the reduced motion on Γ is Euler-Lagrange. In Section IV-B, we characterize the qualitative properties of the reduced motion on Γ: equilibria, their stability type, and closed orbits.

By left-multiplying both sides of (1) by $B^{\perp}(q)$, evaluating the result on Γ, and using the identity $B^{\perp}(\hat{\phi})D(\hat{\phi})\hat{\phi}' = \delta \neq 0$ (which follows from part (iii) of Proposition 3.2), we get

$$\ddot{q}_n = -\frac{B^{\perp}(\hat{\phi}(q_n))}{\delta(q_n)} \Big[D\,\hat{\phi}''(q_n)\dot{q}_n^2 + C\,\dot{q} + \nabla P(q)\Big]_{\substack{q = \hat{\phi}(q_n), \\ \dot{q} = \hat{\phi}'(q_n)\dot{q}_n}}.$$

The product $B^{\perp}(q)C(q,\dot{q})\dot{q}$ is given by $B^{\perp}(q)C(q,\dot{q})\dot{q} = \sum_{i=1}^{n} B_i^{\perp}(q)\dot{q}^{\top}Q_i(q)\dot{q}$, where $B_i^{\perp}$ is the i-th element of $B^{\perp}$ and $Q_i(q)$ is a symmetric matrix whose (j,k) entry is the Christoffel coefficient $(Q_i)_{jk} = (1/2)\left\{\partial D_{ij}/\partial q_k + \partial D_{ik}/\partial q_j - \partial D_{kj}/\partial q_i\right\}$. If $q = \hat{\phi}(q_n)$ and $\dot{q} = \hat{\phi}'(q_n)\dot{q}_n$, we have $\dot{q}^{\top}Q_i(q)\dot{q} = \hat{\phi}'(q_n)^{\top}Q_i(\hat{\phi}(q_n))\hat{\phi}'(q_n)\dot{q}_n^2$, and so letting

$$\begin{aligned}
\Psi_1(q_n) &= -\frac{B^{\perp}(\hat{\phi}(q_n))}{\delta(q_n)}\nabla P(\hat{\phi}(q_n)) \\
\Psi_2(q_n) &= -\frac{1}{\delta(q_n)}\Bigg[B^{\perp}(\hat{\phi}(q_n))D(\hat{\phi}(q_n))\hat{\phi}''(q_n) \\
&\quad + \sum_{i=1}^{n} B_i^{\perp}(\hat{\phi}(q_n))\hat{\phi}'(q_n)^{\top}Q_i(\hat{\phi}(q_n))\hat{\phi}'(q_n)\Bigg]
\end{aligned} \tag{6}$$

we have

$$\ddot{q}_n = \Psi_1(q_n) + \Psi_2(q_n)\dot{q}_n^2. \tag{7}$$

System (7) represents the dynamics of system (1) on the constraint manifold subject to a feedback $\tau(q,\dot{q})$ making Γ invariant[3]. We call system (7) the **reduced dynamics** induced by the VHC $q = \hat{\phi}(q_n)$. Note that the reduced dynamics have no control input. This is not surprising, since the set $\{q : q = \hat{\phi}(q_n)\}$ is a one-dimensional submanifold of $\mathcal{Q}$, and all $n-1$ control directions are used to enforce the VHC. Letting

$$\begin{aligned}
M(q_n) &= \exp\left\{-2\int_0^{q_n}\Psi_2(\tau)d\tau\right\}, \\
V(q_n) &= -\int_0^{q_n}\Psi_1(\mu)M(\mu)d\mu,
\end{aligned} \tag{8}$$

[3]On Γ, such a feedback is unique, and it is given by (4) with $e = \dot{e} = 0$.

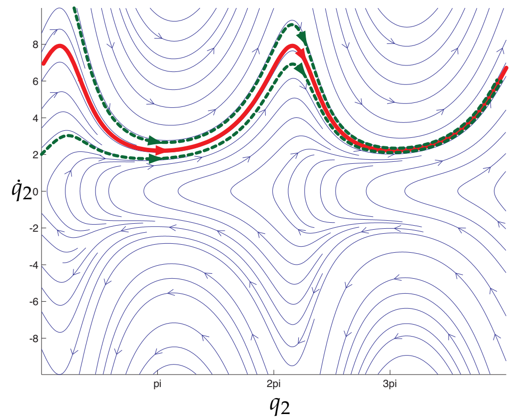

Fig. 2. Phase portrait of the reduced system (10). Since $q_2 \in [\mathbb{R}]_{2\pi}$, the axes $q_2 = 0$, $q_2 = 2\pi$ are identified. The solid highlighted trajectory is an isolated closed orbit. Two sample solutions approaching the closed orbit are depicted by dashed lines.

it can be easily verified that the function $E(q_n, \dot{q}_n) = \frac{1}{2} M(q_n) \dot{q}_n^2 + V(q_n)$, is an integral of motion[4] for system (7) in that, along solutions $(q_n(t), \dot{q}_n(t))$ of (7), we have that $(d/dt)E(q_n(t), \dot{q}_n(t)) = 0$. This fact may seem to imply that the reduced dynamics are Euler-Lagrange. Note, however, that since q_n is an angular variable in $[\mathbb{R}]_{T_n}$, and since $M(q_n)$ and $V(q_n)$ in (8) are not necessarily T_n-periodic functions, the function $E(q_n, \dot{q}_n)$ is generally multivalued. Therefore, the integral of motion $E(q_n, \dot{q}_n)$ cannot be used to deduce that the reduced dynamics are Euler-Lagrange. As a matter of fact, generally they are not, as we show in the next example.

Example 4.1: Consider the Euler-Lagrange control system of the form (1) with $q = (q_1, q_2) \in \mathbb{R} \times [\mathbb{R}]_{2\pi}$, and

$$\begin{aligned} &D(q) = \begin{bmatrix} 1 & 0 \\ 0 & 1 - (\sin q_2)/2 \end{bmatrix}, \ C(q, \dot{q}) = \begin{bmatrix} 0 & 0 \\ 0 & -(1/4)\cos q_2 \dot{q}_2 \end{bmatrix}, \\ &P(q) = \frac{1}{2} q_1^2, \ B = \mathrm{col}(1,1). \end{aligned} \tag{9}$$

Let $B^\perp = [1 \quad -1]$. The relation $q_1 = \phi(q_2) = 1 + (\sin q_2)/2$ is a regular VHC because $B^\perp D \hat{\phi}' = [\cos(q_2) + \sin(q_2)]/2 - 1$, which is bounded away from zero. The reduced dynamics are given by

$$\ddot{q}_2 = \frac{4 + 2\sin q_2}{4 - 2\sin q_2 - 2\cos q_2} + \frac{\cos q_2 - 2\sin q_2}{4 - 2\sin q_2 - 2\cos q_2} \dot{q}_2^2. \tag{10}$$

The phase portrait of this system, shown in Figure 2, reveals the existence of an isolated closed orbit which is attractive. By the Poincarè-Lyapunov theorem (see [20]), nontrivial periodic orbits of Euler-Lagrange systems cannot be isolated. Therefore, the reduced system (10) is not Euler-Lagrange. The existence of the isolated closed orbit is rigorously proved in [21, Proposition 3.1]. △

[4]This result is not new. In [12], [13], [15], the authors use an integral of motion which depends on initial conditions, but appears to be equivalent to the one presented here.

The source of the problem in the example above is the fact that the functions $M(q_n)$, $V(q_n)$ are multivalued. If q_n were a displacement in $\mathbb{R}$ rather than an angle in $[\mathbb{R}]_{T_n}$, then $E(q_n, \dot{q}_n)$ would be a single-valued function, and the reduced system (7) would always be Euler-Lagrange. Although the reduced dynamics are generally not Euler-Lagrange when $q_n \in [\mathbb{R}]_{T_n}$, it is possible that in a neighborhood of an equilibrium $(q_n^\star, 0)$ of (7), the reduced dynamics are *locally* Euler-Lagrange. This occurs when one of the branches of the multi-valued function $V(q_n)$ has a local minimum at $q_n^\star$. In this case, in a neighborhood $(q_n^\star, 0)$ there are closed-orbits $(q_n(t), \dot{q}_n(t))$ along which $q_n(t)$ does not perform complete revolutions and therefore $(q_n(t), \dot{q}_n(t))$ remains on one branch of $E(q_n, \dot{q}_n)$.

A. Conditions for the reduced dynamics to be Euler-Lagrange

Motivated by the discussion above, we now present conditions under which the functions $M(q_n)$ and $V(q_n)$ in (8) are T_n-periodic, so that the reduced system (7) is Euler-Lagrange. To this end, we make the following symmetry assumption on the inertia matrix, potential function, and input matrix of system (1).

Assumption 4.2: For some $\bar{q} \in \mathcal{Q}$, it holds that $D(q)$, $P(q)$, and $B(q)$ in (1) are even with respect to $\bar{q}$, i.e., for all $q \in \mathcal{Q}$, $D(\bar{q}+q) = D(\bar{q}-q),\ P(\bar{q}+q) = P(\bar{q}-q),\ B(\bar{q}+q) = B(\bar{q}-q)$.

Remark 4.3: (i) Henceforth, for notational simplicity we will assume that $\bar{q} = 0$ so that $D(q) = D(-q)$, $P(q) = P(-q)$, and $B(q) = B(-q)$. There is no loss of generality in this assumption, for by defining $\tilde{D}(q) := D(q+\bar{q})$, one gets $\tilde{D}(q) = \tilde{D}(-q)$. The same observation holds for P and B.

(ii) The double pendulum in Example 3.4, the Furuta pendulum in [17], and the 5-DOF swing phase model of a biped robot in [1] (when the centre of mass of the torso is on-axis) satisfy Assumption 4.2.

Proposition 4.4: If Assumption 4.2 holds, and if $q = \hat{\phi}(q_n)$ is a regular and odd VHC, i.e., such that

$$(\forall q_n \in [\mathbb{R}]_{T_n})(\forall i \in \{1, \dots, n-1\})\ \phi_i(q_n) = -\phi_i(-q_n),$$

then the reduced system (7) is Euler-Lagrange with Lagrangian $\mathcal{L}(q_n, \dot{q}_n) = (1/2)M(q_n)\dot{q}_n^2 - V(q_n)$, where M and V are defined in (8).

Proof: If $M(q_n)$ and $V(q_n)$ are T_n-periodic functions, then (7) is Euler-Lagrange with Lagrangian $\mathcal{L}(q_n, \dot{q}_n) = (1/2)M(q_n)\dot{q}_n^2 - V(q_n)$. To prove that $M(q_n)$ and $V(q_n)$ are T_n-periodic, we will show that, besides being T_n-periodic, the functions $\Psi_1(q_n)$ and $\Psi_2(q_n)$ in (6) are also odd. Suppose for a moment that this is the case (i.e., Ψ_1, Ψ_2 are odd). Since the antiderivative of an odd T_n-periodic function is an even T_n-periodic function, $M(q_n)$ is even and T_n-periodic. This fact implies that the product $\Psi_1(q_n)M(q_n)$ is odd and T_n-periodic, and therefore $V(q_n)$ is even and T_n-periodic. These considerations show that in order to prove that the reduced system (7) is Euler-Lagrange it suffices to show that, besides being T_n-periodic, the functions $\Psi_1(q_n)$, $\Psi_2(q_n)$ in (6) are odd. To this end, note that $\hat{\phi}(q_n)$ is odd and T_n-periodic and therefore in light of Assumption 4.2 the functions $D(\hat{\phi}(q_n))$, $P(\hat{\phi}(q_n))$, and $B(\hat{\phi}(q_n))$ are even and T_n-periodic (for instance $D(\hat{\phi}(-q_n)) = D(-\hat{\phi}(q_n)) = D(\hat{\phi}(q_n))$). Moreover, since $B(q)$ is even, $B^\perp(\hat{\phi}(q_n))$ is even as well. The derivative of an even function is odd, while the derivative of an odd function is even. These facts imply that $\nabla P(\hat{\phi}(q_n))$, $\hat{\phi}''(q_n)$, and $Q_i(\hat{\phi}(q_n))$ are odd functions and that $\delta(q_n) = B^\perp(\hat{\phi}(q_n))D(\hat{\phi}(q_n))\hat{\phi}'(q_n)$ is even. Therefore, the functions $\Psi_1(q_n)$, $\Psi_2(q_n)$ in (6) are odd. ■

Example 4.5: Consider the pendubot system of Example 3.4. We have shown that the relation $\theta_2 = 2\theta_1$ is a regular VHC. Since the function $\phi(\theta_1) = 2\theta_1$ is odd, it follows from Proposition 4.4 that the reduced dynamics are Euler-Lagrange. As a matter of fact, one can check that the reduced dynamics are given by

$$\ddot{\theta}_1 = \frac{1}{2+\cos\theta_1}\left[g\sin(2\theta_1) - \sin\theta_1\dot{\theta}_1^2\right], \tag{11}$$

and the functions $M(\theta_1), V(\theta_1)$ are given by $M(\theta_1) = \frac{9}{(\cos\theta_1+2)^2}$, $V(\theta_1) = 4g - \frac{18g(\cos\theta_1+1)}{(\cos\theta_1+2)^2}$. Since M and V are 2π-periodic, the Lagrangian $\mathcal{L}(q_1,\dot{q}_1) = \frac{1}{2}M(\theta_1)\dot{\theta}_1^2 - V(\theta_1)$ is a well-defined single-valued function, and therefore system (11) is Euler-Lagrange.

Next, consider system (9) in Example 4.1, which was shown not to be Euler-Lagrange. The VHC in that example, $q_1 = \phi(q_2) = 1 + (\sin q_2)/2$ is not odd. $\triangle$

B. Qualitative properties of Euler-Lagrange reduced motion

Suppose that we have found a regular VHC which is odd, so that (7) is a one degree-of-freedom Euler-Lagrange system with energy function

$$E(q_n,\dot{q}_n) = \frac{1}{2}M(q_n)\dot{q}_n^2 + V(q_n), \tag{12}$$

where $M(q_n)$ and $V(q_n)$ are defined in (8). As is well-known, the properties of solutions of such a system can be completely characterized in terms of the potential function $V(q_n)$.

Definition 4.6: A closed orbit γ of the reduced dynamics (7) is said to be a **rotation of** q_n if γ is homeomorphic to a circle $\{(q_n,\dot{q}_n) \in [\mathbb{R}]_{T_n} \times \mathbb{R} : \dot{q}_n = \text{constant}\}$ via a homeomorphism of the form $(q_n,\dot{q}_n) \mapsto (q_n, T(q_n)\dot{q}_n)$; γ is an **oscillation of** q_n if it is homeomorphic to a circle $\{(q_n,\dot{q}_n) \in [\mathbb{R}]_{T_n} \times \mathbb{R} : q_n^2 + \dot{q}_n^2 = \text{constant}\}$ via a homeomorphism of the form $(q_n,\dot{q}_n) \mapsto (q_n, T(q_n)\dot{q}_n)$. $\triangle$

In other words, rotations of q_n are closed orbits of (7) along which q_n performs complete revolutions. On the other hand, oscillations of q_n are closed orbits along which q_n exhibits a rocking motion without performing complete revolutions.

Proposition 4.7: Suppose that $q = \hat{\phi}(q_n)$ is a regular and odd VHC, and that Assumption 4.2 holds. Consider the dynamics (7) on the constraint manifold Γ. The equilibrium configurations are the points $(q_n^\star, 0)$ such that $\Psi_1(q_n^\star) = 0$, or equivalently $\nabla P(\hat{\phi}(q_n^\star)) \in \operatorname{Im}(B(\hat{\phi}(q_n^\star)))$, where $P(q)$ is the potential of the original system (1). There are at least two equilibria at $q_n^\star = [0]_{T_n}$ and $q_n^\star = [T_n/2]_{T_n}$. The stability type of an equilibrium $(q_n^\star, 0)$ is determined by the sign of the expression $\delta(q_n^\star)(d/dq_n)\left[B^\perp(\hat{\phi}(q_n))\nabla P(\hat{\phi}(q_n))\right]\big|_{q_n=q_n^\star}$ (positive $\Longrightarrow$ stable, negative $\Longrightarrow$ unstable, $0 \Longrightarrow$ no conclusion). Let $\underline{V} = \min_{q_n\in[\mathbb{R}]_{T_n}} V(q_n)$ and $\bar{V} = \max_{q_n\in[\mathbb{R}]_{T_n}} V(q_n)$. Then, all phase curves of (7) in the set $\{(q_n,\dot{q}_n) \in [\mathbb{R}]_{T_n} \times \mathbb{R} : 1/2M(q_n)\dot{q}_n^2 + V(q_n) > \bar{V}\}$ are rotations of q_n. Almost all (in the Lebesgue sense) phase curves in the set $\{(q_n,\dot{q}_n) \in [\mathbb{R}]_{T_n} \times \mathbb{R} : \underline{V} < 1/2M(q_n)\dot{q}_n^2 + V(q_n) < \bar{V}\}$ are oscillations of q_n.

Proof Sketch: We only prove the first part of the proposition concerning equilibria of (7). The second part of the proof, concerning the types of closed orbits, is identical to the proof of Lemma 3.12 in [22]. The equilibria of (7) are the critical points of the potential $V(q_n)$. Since $M(q_n) > 0$, we have that $V'(q_n) = -\Psi_1(q_n)M(q_n)$ is equal to zero when $\Psi_1(q_n) = 0$, or $B^\perp(\hat{\phi}(q_n))\nabla P(\hat{\phi}(q_n)) = 0$, which can only happen if $\nabla P(\hat{\phi}(q_n)) \in \operatorname{Im}(B(\hat{\phi}(q_n)))$. Using the fact that $\Psi_1(q_n)$ is odd, we have $\Psi_1(0) = -\Psi_1(0)$, or $\Psi_1(0) = 0$. Moreover, $[T_n/2]_{T_n} = [-T_n/2]_{T_n}$, and so $\Psi_1([T_n/2]_{T_n}) = \Psi_1([-T_n/2]_{T_n}) = -\Psi_1([T_n/2]_{T_n})$, where the last identity follows from the fact that Ψ_1 is an odd function. Thus, $\Psi_1([T_n/2]_{T_n}) = 0$, and so there are at least two equilibria of (7) at $(q_n^\star, 0)$, $q_n^\star = [0]_{T_n}$, $q_n^\star = [T_n/2]_{T_n}$. The stability type of equilibria is determined by the sign of $V''(q_n^\star)$. We have $V''(q_n^\star) = -\Psi_1'(q_n^\star)M(q_n^\star) - \Psi_1(q_n^\star)M'(q_n^\star)$. Since $\Psi_1(q_n^\star) = 0$ and $M(q_n^\star) > 0$, we have $\operatorname{sgn}(V''(q_n^\star)) = \operatorname{sgn}(-\Psi_1'(q_n^\star)) = \operatorname{sgn}\left\{\delta(q_n^\star)(d/dq_n)\left[B^\perp(\hat{\phi}(q_n))\nabla P(\hat{\phi}(q_n))\right]\right\}\big|_{q_n=q_n^\star}$. The latter equality follows from the fact that the term $B^\perp(\hat{\phi}(q_n))\nabla P(\hat{\phi}(q_n))$ in the expression of $\Psi_1'(q_n)$ is zero at all equilibria of (7). ■

Example 4.8: Consider again the pendubot of Example 3.4 with regular and odd VHC $\theta_2 = 2\theta_1$. We now apply Proposition 4.7 to characterize the qualitative properties of the reduced motion. The condition $\nabla P(\hat{\phi}(\theta_1)) \in \operatorname{Im}(B)$ gives $\operatorname{col}(-2g\sin\theta_1, -g\sin(2\theta_1)) \in \operatorname{span}\{\operatorname{col}(1,0)\}$, or $\sin(2\theta_1) = 0$, from which we get that the reduced motion has four equilibria, $(\theta_1, \dot{\theta}_1) = ([k\pi/2]_{2\pi}, 0), k = 0, \ldots, 3$. To determine their stability type, recall that $\delta(\theta_1) = B^\perp D(\hat{\phi}(\theta_1))\hat{\phi}'(\theta_1) = 2 + \cos\theta_1$. We have $\operatorname{sgn}\left(\delta(\theta_1)B^\perp \frac{d}{d\theta_1}[B^\perp \nabla P(\hat{\phi}(\theta_1))]\right) = \operatorname{sgn}(-\cos(2\theta_1))$, from which it follows that the equilibria $(\theta_1, \dot{\theta}_1) = ([k\pi/2]_{2\pi}, 0)$, $k = 1, 3$ are stable, while the equilibria $(\theta_1, \dot{\theta}_1) = ([k\pi/2]_{2\pi}, 0)$, $k = 0, 2$ are unstable. Recall from Example 4.5 that the reduced dynamics have a potential function $V(\theta_1) = 4g - 18g(\cos\theta_1 + 1)/(\cos\theta_1 + 2)^2$. We have $\underline{V} = \min V(\theta_1) = -g/2$ and $\bar{V} = \max V(\theta_1) = 4g$. The energy function of the reduced dynamics is $E(\theta_1, \dot{\theta}_1) = \left(9/2(\cos\theta_1 + 2)^2\right)\dot{\theta}_1^2 + 4g - 18g(\cos\theta_1 + 1)/(\cos\theta_1 + 2)^2$. All level sets of $E = c$ with $c \in (\underline{V}, \bar{V})$ are oscillations of θ_1, while for $c > \bar{V}$ they are rotations of θ_1. △

V. Application: Swing-up of the pendubot

In this section we apply the technique developed in the previous section to a swing-up problem for the pendubot. We wish to design a feedback law yielding the following two properties:

1) **Low-high to high-high swing-up:** For any neighborhood U of the high-high equilibrium $(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = (0, 0, 0, 0)$, there exists a punctured neighborhood V of the low-high equilibrium $(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = (\pi, 0, 0, 0)$ such that for each initial condition in V, the solution enters U in finite time.
2) **Boundedness:** For any initial condition in V, the solution has the property that $\theta_2(t) \in (-\pi/2, \pi/2)$ for all $t \geq 0$. In other words, the unactuated link does not fall over during transient.

Owing to space limitations, we will only sketch the methodology of solution of the swing-up problem. The ideas outlined in this section will be presented in a general form elsewhere.

To meet the boundedness requirement, we will enforce a VHC that constrains the second link of the pendulum to remain in a neighborhood of the vertical configuration. The swing-up requirement can then be met by stabilizing a closed orbit on the constraint manifold. The challenge with this approach is that the reduced dynamics in (7) have no control input, so it may seem impossible to stabilize a closed orbit while simultaneously enforcing the VHC. To obviate this problem, we dynamically change the geometry of the VHC while preserving its invariance. The idea is to make the VHC depend on a parameter s. Variations of s affect the dynamics on the constraint manifold, so by controlling s it is possible to stabilize a desired closed orbit. Define the dynamic compensator $\ddot{s} = v$, where v is a scalar control input, and denote by $\bar{q} := \operatorname{col}(q, s)$ the configuration vector of the augmented system. Consider the relation

$$\theta_2 = \phi^s(\theta_1) = \theta_1 + 2\arctan[\tan(s - \theta_1/2)(1 + \sqrt{2})]. \tag{13}$$

Let $\hat{\phi}^s(\theta_1) = \mathrm{col}(\theta_1, \phi^s(\theta_1))$. One can readily check that $B^\perp D(\hat{\phi}^s(\theta_1))\partial_{q_n}\hat{\phi}^s(\theta_1) = 1 - \sqrt{2}$, from which it follows that for any piecewise continuous signal $v(t)$, the relation $\theta_2 = \phi^s(\theta_1)$ is a regular VHC for the pendubot augmented with $\ddot{s} = v(t)$. Moreover, when $s = 0$ the VHC is odd, i.e., the function $\phi^0(\theta_1)$ is odd, so by Proposition 4.4 the reduced dynamics when $s = \dot{s} = 0$ and $v = 0$ are Euler-Lagrange with energy function $E(\theta_1, \dot{\theta}_1)$ as in (12). The configurations of the pendubot satisfying the constraint $\theta_2 = \phi^0(\theta_1)$ are depicted in Figure 3, together with the phase portrait of the reduced dynamics on the plane $s = \dot{s} = 0$. Note that the VHC prevents the second link from falling over.

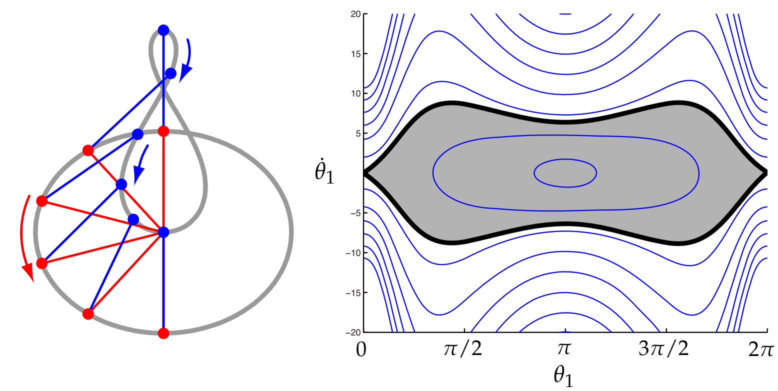

Fig. 3. The left-hand side figure shows configurations of the pendubot satisfying the VHC in (13) with $s = 0$. Constrained configurations are shown for values of $\theta_1 \in [0, \pi]$. The configurations for $\theta_1 \in [\pi, 2\pi]$ are symmetric. The right-hand figure shows the phase portrait of the reduced motion on the constraint manifold when $s = \dot{s} = 0$. The shaded area is filled with oscillations of θ_1, while the unshaded area is filled with rotations of θ_1.

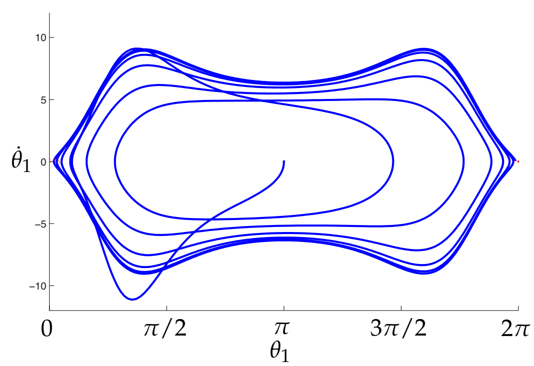

Fig. 4. Phase portrait of $(\theta_1, \dot{\theta}_1)$ for the pendubot.

As a matter of fact, the VHC (13) has the property that $\phi^0(0) = \phi^0(\pi) = 0$, and $|\phi^0(\cdot)| < \pi/2$. Since the map $(s, \theta_1) \mapsto \phi^s(\theta_1)$ is continuous, and θ_1 is in a compact set, there exists $\epsilon > 0$ such that $\phi^s(\cdot) < \pi/2$ for all $|s| < \epsilon$. Therefore, VHC (13) is a suitable candidate to meet the boundedness requirement, provided that s is kept in a neighborhood of zero. Accordingly, referring to the pendubot system with compensator $\ddot{s} = v$, control input τ, and output $e = \theta_2 - \phi^s(\theta_1)$, we will let $\tau(q, \dot{q}, s, \dot{s}, v)$ be an input-output feedback linearizing controller parametrized by v. This

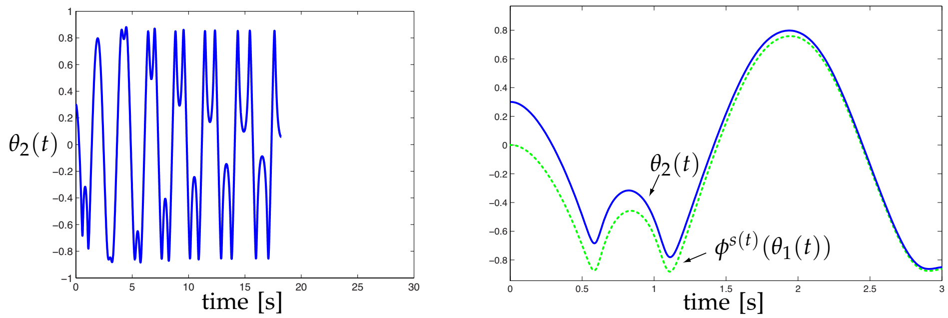

Fig. 5. On the left-hand side, angle $\theta_2(t)$ of the pendubot. On the right-hand side, plots of $\theta_2(t)$ (solid) and $\phi^{s(t)}(\theta_1(t))$ (dashed).

controller asymptotically stabilizes the manifold $\bar{\Gamma} = \{\theta_2 = \phi^s(\theta_1),\ \dot{\theta}_2 = (\partial_{\theta_1}\phi^s)\dot{\theta}_1 + (\partial_s\phi^s)\dot{s}\}$. In order to meet the swing-up requirement, we will design a smooth feedback $v(\bar{q}, \dot{\bar{q}})$ so as to stabilize the set $\bar{\gamma} = \{(\bar{q}, \dot{\bar{q}}) \in \bar{\Gamma} : E(\theta_1, \dot{\theta}_1) = 0, s = \dot{s} = 0\}$. This set is the homoclinic orbit on the plane $s = \dot{s} = 0$ bounding the shaded region in Figure 3. Pick $v(\bar{q}, \dot{\bar{q}}) = K_1 E(\theta_1, \dot{\theta}_1)\dot{\theta}_1 + K_2 s + K_3 \dot{s}$, with $K_1 = -0.01$, $K_2 = -0.5$, $K_3 = -3$. With this choice, we find that the characteristic multipliers of the linearization of the reduced dynamics along $\bar{\gamma}$ are $\{0.4884, 0.3704, 2.57 \cdot 10^{-4}\}$, so that $\bar{\gamma}$ is asymptotically stable for the reduced dynamics on $\bar{\Gamma}$. Figure 4 shows the curve $(\theta_1(t), \dot{\theta}_1(t))$ when the pendulum is initialized in a neighborhood of the low-high equilibrium, illustrating that $(\theta_1(t), \dot{\theta}_1(t))$ approaches the homoclinic orbit of Figure 3, so that the swing-up requirement is met. The left-hand side of Figure 5 illustrates that during transient the angle $\theta_2(t)$ is bounded inside the interval $(-\pi/2, \pi/2)$, so that the boundedness requirement is met. This latter property is a consequence of the enforcement of the VHC $\theta_2 = \phi^s(\theta_1)$, shown on the right-hand side of Figure 5. Our simulations suggest that the domain of attraction of $\bar{\gamma}$ contains a neighborhood of the low-high equilibrium. The simulations also illustrate the benefit of enforcing the virtual constraint $\theta_2 = \phi^s(\theta_1)$ while simultaneously stabilizing the closed orbit $\bar{\gamma} \subset \bar{\Gamma}$. Namely, that if the pendulum state is initialized in a neighborhood of the low-high equilibrium, and thus near $\bar{\Gamma}$, and if $s(0)$, $\dot{s}(0)$ are small, then $\theta_2(t) - \phi^{s(t)}(\theta_1(t))$ and $s(t)$ remain small and converge to zero. This in particular implies that $\theta_2(t)$ remains bounded in $(-\pi/2, \pi/2)$, i.e., the second link of the pendulum does not fall over during swing-up. This property would not be guaranteed by an approach that purely stabilizes $\bar{\gamma}$ without enforcing the virtual constraint.

VI. Conclusions

We have investigated *virtual holonomic constraints* for Euler-Lagrange systems with n degrees-of-freedom and $n-1$ controls. We have given conditions under which a virtual holonomic constraint is feasible. We have provided sufficient conditions under which the dynamics on the constraint manifold correspond to an Euler-Lagrange system.

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