Output Feedback Control for Stabilizable and Incompletely Observable Nonlinear Systems: Theory[1]

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Abstract

This paper introduces a new approach for output feedback stabilization of SISO systems which, unlike most of the techniques found in the literature, does not use high-gain observers and control input saturation to achieve separation between the state feedback and observer designs. Rather, we show that by using nonlinear observers, together with a projection algorithm, the same kind of separation principle is achieved for a larger class of systems, namely stabilizable and incompletely observable plants. Furthermore, this new approach avoids using knowledge of the inverse of the observability mapping, which is needed by most techniques in the literature when controlling general stabilizable systems.

1 Introduction

The area of nonlinear output feedback control has received much attention after the publication of the work [1], in which the authors developed a systematic strategy for the output feedback control of input-output linearizable systems with full relative degree, which employed two basic tools: an high-gain observer to estimate the derivatives of the outputs (and hence the system states in transformed coordinates), and control input saturation to isolate the peaking phenomenon of the observer from the system states. Essentially the same approach has later been applied in a number of papers by various researchers (see, e.g., [2, 3, 4, 5]) to solve different problems in output feedback control. In most of the papers found in the literature, (see, e.g., [1, 2, 3, 4]) the authors consider input-output feedback linearizable systems with either full relative degree or minimum phase zero dynamics. The work in [6] showed that for nonminimum phase systems the problem can be solved by extending the system with a chain of integrators at the input side. However, the results contained there are local. In [7], by putting together this idea with the approach found in [1], the authors were able to show how to solve the output feedback stabilization problem for general stabilizable and uniformly completely observable systems, provided that the inverse of the observability mapping is explicitly known. The recent work in [5] unifies all these approaches to prove a separation principle for a very general class of nonlinear systems. It appears that the largest class of nonlinear SISO systems for which the output feedback stabilization problem has been solved is that of locally stabilizable and completely observable systems. Moreover, when dealing with systems which are not feedback linearizable, the works [6, 7, 5] require the explicit knowledge of the inverse of the observability mapping, thus somewhat restricting the variety of problems to which their algorithm can be applied.

The objective of this paper is to relax the two restrictions above, by developing a new output feedback strategy for nonlinear SISO locally or globally stabilizable systems which are only observable on regions of the state space. Furthermore, for the implementation of our controller, the inverse of the observability mapping is not needed. These two features are achieved by means of a nonlinear observer instead of the standard high-gain observer found in the literature, and of a new projection algorithm which *eliminates* the peaking phenomenon in the observer states, thus avoiding the need to use control input saturation. To the best of our knowledge, this work, besides introducing a new methodology for output feedback control design, enlarges the class of SISO systems considered in the literature of the field so far. Owing to space limitations, the results presented in this paper are stated without proof.

2 Problem Formulation and Assumptions

Consider the following dynamical system,

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n, u, y \in \mathbb{R}$, f and h are known smooth functions, and $f(0,0) = 0$. Our control objective is

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to construct a stabilizing controller for (1) without the availability of the system states x. In order to do so, we need an observability assumption. Define

$$y_e \triangleq \begin{bmatrix} y \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \mathcal{H}\left(x, u, \dots, u^{(n_u-1)}\right)$$
$$\triangleq \begin{bmatrix} h(x,u) \\ \varphi_1(x,u,u^{(1)}) \\ \vdots \\ \varphi_{n-1}\left(x,u,\dots,u^{(n_u-1)}\right) \end{bmatrix} \quad (2)$$

($y^{(n-1)}$ is the $n-1$-th derivative) where

$$\varphi_1(x,u,u^{(1)}) = \frac{\partial h}{\partial x} f(x,u) + \frac{\partial h}{\partial u} u^{(1)}$$
$$\vdots$$
$$\varphi_{n-1}\left(x,u,\dots,u^{(n_u-1)}\right) = \frac{\partial \varphi_{n-2}}{\partial x} f(x,u) +$$
$$+ \quad \frac{\partial \varphi_{n-2}}{\partial u} u^{(1)} + \dots + \frac{\partial \varphi_{n-2}}{\partial u^{(n_u-2)}} u^{(n_u-1)}$$

where $0 \leq n_u \leq n$ ($n_u = 0$ indicates that there is no dependence on u). In the most general case, $\varphi_i = \varphi_i(x,u,\dots,u^{(i)})$, $i = 1,2,\dots,n-1$. In some cases, however, we may have that $\varphi_i = \varphi_i(x,u)$ for all $i = 1,\dots,r-1$ and some integer $r > 1$. This happens in particular when system (1) has a well-defined relative degree r. Here, we do not require the system to be input-output feedback linearizable, and hence to possess a well-defined relative degree. In the case of systems with well-defined relative degree, $n_u = 0$ corresponds to having $r \geq n$, while $n_u = n$ corresponds to having $r = 0$. Now, we are ready to state our first assumption.

Assumption A1. System (1) is observable over the set $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$ containing the origin, i.e., the mapping

$$y_e = \mathcal{H}\left(x, u, \dots, u^{(n_u-1)}\right) \quad (3)$$

is invertible with respect to x and its inverse is smooth, for all $x \in \mathcal{X}$, $[u, u^{(1)}, \dots, u^{(n_u-1)}]^\top \in \mathcal{U}$.

Remark 1: In the existing literature, an assumption similar to A1 can be found in [6] and [7]. It is worth stressing, however, that in that work the authors adopt a global observability assumption, i.e., the set $\mathcal{X} \times \mathcal{U}$ is taken to be $\mathbb{R}^n \times \mathbb{R}^{n_u}$. In many practical applications the system under consideration may be observable in some subset of $\mathbb{R}^n \times \mathbb{R}^{n_u}$ only, thus preventing the use of most of the output feedback techniques found in the literature, including the ones found in [6], [7], and [5].

Next, augment the system with n_u integrators on the input side, which corresponds to using a compensator of order n_u. System (1) can be rewritten as follows,

$$\begin{aligned} \dot{x} &= f(x, z_1) \\ \dot{z}_1 &= z_2, \dots, \dot{z}_{n_u} = v \end{aligned} \quad (4)$$

Define the extended state variable $\chi = [x^\top, z^\top]^\top \in \mathbb{R}^{n+n_u}$, and the associated *extended system*

$$\begin{aligned} \dot{\chi} &= f_e(\chi) + g_e v \\ y &= h_e(\chi) \end{aligned} \quad (5)$$

where $f_e(\chi) = [f^\top(x,z_1),\ z_2, \dots, z_{n_u}, 0]^\top$, $g_e = [0,\ \dots,\ 1]^\top$, and $h_e(\chi) = h(x, z_1)$.

Assumption A2. The origin of (1) is locally stabilizable (stabilizable) by a static function of x, i.e., there exists a smooth function $\bar{u}(x)$ such that the origin is an asymptotically stable (globally asymptotically stable) equilibrium point of $\dot{x} = f(x, \bar{u}(x))$.

Remark 2: Assumption A2 implies that the origin of the extended system (5) is locally stabilizable (stabilizable) by a function of χ as well. A proof of the local stabilizability property for (5) may be found, e.g., in [8], while its global counterpart is a well known consequence of the integrator backstepping lemma (see, e.g., Theorem 9.2.3 in [9]). Therefore we conclude that for the extended system (5) there exists a smooth control $\bar{v}(\chi)$ such that its origin is asymptotically stable under closed loop control. Let $\mathcal{D}$ be the domain of attraction of the origin of (5), and notice that, when A2 holds globally, $\mathcal{D} = \mathbb{R}^n \times \mathbb{R}^{(n_u)}$.

3 Nonlinear Observer: Its Need and Stability Analysis

Assumption A2 allows us to design a stabilizing state feedback control $v = \phi(x, z)$. In order to perform output feedback control x should be replaced by its estimate. Researchers who have addressed this problem (e.g., [6], [7]) relied on the explicit knowledge of the inverse of the mapping $\mathcal{H}$ in (3)

$$x = \mathcal{H}^{-1}(y_e, z_1, \dots, z_{n_u})$$

so that estimation of the first $n-1$ derivatives of y (the vector y_e) provides an estimate of x

$$\hat{x} = \mathcal{H}^{-1}(\hat{y}_e, z_1, \dots, z_{n_u})$$

since the vector z, being the state of the controller, is known. Next, to estimate the derivatives of y, they employed an high-gain observer. Both the works [6] and [7] (the latter dealing with the larger class of stabilizable systems) rely on the knowledge of $\mathcal{H}^{-1}$ to prove closed loop stability. In addition to this, the recent work [5] proves that a separation principle holds for a quite general class of nonlinear systems which includes (1) provided that $\mathcal{H}^{-1}$ is explicitly known and that the system is uniformly completely observable. In order to develop a practical output feedback control algorithm, however, $\mathcal{H}^{-1}$ cannot be assumed to be explicitly known. Hence, rather than designing an high-gain observer to estimate y_e and using $\mathcal{H}^{-1}(\cdot,\cdot)$ to get x, the approach adopted here is to estimate x directly using a nonlinear observer for system (1) and using the fact that the z-states are known. In other words, we can regard our problem as that of building a reduced order observer for the closed loop system states[1]. The observer has the form

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, z_1) + \left[\frac{\partial \mathcal{H}(\hat{x}, z)}{\partial \hat{x}}\right]^{-1} \mathcal{E}^{-1} L\left[y(t) - \hat{y}(t)\right] \\ \hat{y}(t) &= h(\hat{x}, z_1) \end{aligned} \tag{6}$$

where L is a $n \times 1$ vector, $\mathcal{E} = \operatorname{diag}\left[\rho, \rho^2, \ldots, \rho^n\right]$, and $\rho \in (0, 1]$ is a fixed design constant.

Notice that (6) does not require any knowledge of $\mathcal{H}^{-1}$ and has the advantage of operating in x-coordinates. The observability assumption A1 implies that the Jacobian of the mapping $\mathcal{H}$ with respect to x is invertible, and hence the inverse of $\partial \mathcal{H}(\hat{x}, z)/\partial \hat{x}$ in (6) is well defined.

Theorem 1 *Consider system (4) and assume A1 is satisfied for $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{U} = \mathbb{R}^{n_u}$, the state χ belongs to a compact invariant set Ω, and that $|v(t)| \leq M$ for all $t \geq 0$, with M a positive constant. Choose L such that $A_c - LC_c$, where (A_c, B_c, C_c) is the controllable/observable canonical realization, is Hurwitz.*

Under these conditions and using observer (6), the following two properties hold

(i) Asymptotic stability of the estimation error: There exists $\bar{\rho}$, $0 < \bar{\rho} \leq 1$, such that for all $\rho \in (0, \bar{\rho})$, $\hat{x} \to x$ as $t \to +\infty$.

(ii) Arbitrarily fast rate of convergence: For each positive T, ϵ, there exists ρ^*, $0 < \rho^* \leq 1$, such that for all $\rho \in (0, \rho^*]$, $\|\hat{x} - x\| \leq \epsilon \; \forall t \geq T$.

Remark 3: It can be proved that $\tilde{\xi}(t)$ may exhibit peaking, and the size of the peak grows larger as ρ decreases and the convergence rate is made faster. This phenomenon and its implications on output feedback control has been studied in the seminal work [1]. The analysis in that paper shows that a way to isolate the peaking of the observer estimates from the system states is to saturate the control input outside of the compact set of interest. The same idea has then been adopted in several other works in the output feedback control literature (see, e.g., [1, 2, 7, 3, 4, 5]). Rather than following this approach, in the next section we will present a new technique for isolating the peaking phenomenon which allows for the use of the weaker Assumption A1.

Remark 4: It is interesting to note that in y_e-coordinates the nonlinear observer (6) is identical to the standard high-gain observer found in the nonlinear output feedback control literature (see, e.g., [6, 1, 2, 7, 3, 4, 5]). Our observer, however, has the advantage of avoiding the knowledge of the inverse of the mapping $\mathcal{H}$, as well as working in x coordinates, directly.

4 Output Feedback Stabilizing Control

Consider system (5), by using assumption A2 and Remark 2 we conclude that there exists a smooth stabilizing control $v = \phi(x, z) = \phi(\chi)$ which makes the origin of (5) an asymptotically stable equilibrium point with domain of attraction $\mathcal{D}$. By the converse Lyapunov theorem found in [10], there exists a continuously differentiable function V defined on $\mathcal{D}$ satisfying, for all $\chi \in \mathcal{D}$,

$$\alpha_1(\|\chi\|) \leq V(\chi) \leq \alpha_2(\|\chi\|) \tag{7}$$

$$\lim_{\chi \to \partial \mathcal{D}} \alpha_1(\|\chi\|) = \infty \tag{8}$$

$$\frac{\partial V}{\partial \chi}\left(f_e(\chi) + g_e\, v\right) \leq -\alpha_3(\|\chi\|) \tag{9}$$

where α_i, $i = 1, 2, 3$ are class $\mathcal{K}$ functions (see [11] for a definition), and $\partial \mathcal{D}$ stands for the boundary of the set $\mathcal{D}$. Define compact sets $\Omega_{c_1} \triangleq \{\chi \,|\, V \leq c_1\}$, $\Omega_{c_2} \triangleq \{\chi \,|\, V \leq c_2\}$, $\Omega^x_{c_2} \triangleq \{x \in \mathbb{R}^n \,|\, \chi \in \Omega_{c_2}\}$, $\Omega^z_{c_2} \triangleq \{z \in \mathbb{R}^{n_u} \,|\, \chi \in \Omega_{c_2}\}$ where $c_2 > c_1 > 0$. Next, the following assumption is needed.

Assumption A3. Assume c_2 can be selected so that the following conditions are satisfied:

1. $\mathcal{H}(\Omega^x_{c_2}, z) \subset C_\xi(z) \subset \mathcal{H}(\mathcal{X}, z)$, for all $z \in \Omega^z_{c_2}$, for some convex compact $C_\xi(z)$
2. $\Omega^z_{c_2} \subset \mathcal{U}$

Remark 5: This assumption represents a basic requirement for output feedback control. It is satisfied when there exists a sphere of dimension $n + n_u$, centered at the origin, which is contained in $\mathcal{X} \times \mathcal{U}$; this requirement is satisfied in most practical examples. On the other hand, Assumption A3 fails when, for example, the origin belongs to the boundary of $\mathcal{X} \times \mathcal{U}$, and thus there is no neighborhood centered at the origin and contained in $\mathcal{X} \times \mathcal{U}$.

[1] Throughout this section we assume A1 to hold globally, since we are interested in the ideal convergence properties of the state estimates. In the next section we will show how to modify the observer equation in order to achieve the same convergence properties when A1 holds over the set $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$.

4.1 Observer Estimates Projection

As we already pointed out in Remark 4, in order to isolate the peaking phenomenon from the system states, the approach generally adopted in several papers is to saturate the control input to prevent it from growing above a given threshold. This technique, however, does not eliminate the peak in the observer estimate and, hence, cannot be used to control general systems like the ones satisfying assumption A1, since even when the system state lies in the observable region $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^{n_u}$, the observer estimates may enter the unobservable domain where (6) is not well defined. It appears that in order to deal with systems that are not completely observable, one has to eliminate the peaking from the observer by guaranteeing its estimates to be confined in a prespecified compact set contained in $\mathcal{X}$.

A very common procedure used in the adaptive control literature (see [12]) to confine vectors of parameter estimates within a desired convex set is gradient projection. This idea cannot be directly applied to our problem, mainly because $\dot{\hat{x}}$ is not proportional to the gradient of the observer Lyapunov function and, thus, the projection cannot be guaranteed to preserve the convergence properties of the estimate. Inspired by this idea, however, we propose a way to modify the $\dot{\hat{x}}$ equation which confines $\hat{x}$ to within a prespecified compact set while preserving its convergence properties. Let

$$\xi = \mathcal{H}(x,z), \qquad \hat{\xi} = \mathcal{H}(\hat{x},z), \qquad \tilde{\xi} = \hat{\xi} - \xi \tag{10}$$

Next, *project*[2] the observer estimate as follows

$$\dot{\hat{x}}^P = \left[\frac{\partial \mathcal{H}}{\partial \hat{x}}\right]^{-1} \left\{ \mathcal{P}\left(\hat{\xi}, \dot{\hat{\xi}}, z, \dot{z}\right) - \frac{\partial \mathcal{H}}{\partial z}\dot{z} \right\}$$

$$\mathcal{P}(\hat{\xi}, \dot{\hat{\xi}}, z, \dot{z}) = \begin{cases} \dot{\hat{\xi}} - \Gamma \dfrac{N(\hat{\xi})\left(N(\hat{\xi},z)^\top \dot{\hat{\xi}} + N_z(\hat{\xi},z)^\top \dot{z}\right)}{N(\hat{\xi},z)^\top \Gamma N(\hat{\xi},z)} \\ \quad \text{if } N(\hat{\xi},z)^\top \dot{\hat{\xi}} + N_z(\hat{\xi},z)^\top \dot{z} \geq 0 \\ \quad \text{and } \hat{\xi} \in \partial C_\xi(z) \\ \dot{\hat{\xi}} \quad \text{otherwise} \end{cases} \tag{11}$$

where $\Gamma = (S\mathcal{E}')^{-1}(S\mathcal{E}')^{-1}$, $S = S^\top$ denotes the matrix square root of P, solution of the Lyapunov equation $P(A_c - LC_c) + (A_c - LC_c)^\top P = -I$, and $N(\hat{\xi}, z)$, $N_z(\hat{\xi}, z)$ are the normal vectors to the boundary of $C_\xi(z)$ with respect to ξ and z, respectively. The following lemma shows that (11) guarantees boundedness and preserves convergence for $\hat{x}$.

Lemma 1 *: If A3 holds and (11) is used:*

(i) Boundedness: $\hat{x}^P(t) \in \mathcal{H}^{-1}(C_\xi(z), z) \subset \mathcal{X}$ for all t, and for all $z \in \Omega^z_{c_2}$.

If, in addition, $x \in \Omega^x_{c_2}$ and the assumptions of Theorem 1 are satisfied, then the following is also true

(ii) Preservation of original convergence characteristics: properties (i) and (ii) established by Theorem 1 remain valid for $\hat{x}^P$.

Remark 6: In order to avoid the discontinuity in the right hand side of (11) introduced by $\mathcal{P}$, one can define $\mathcal{P}$ to be the smooth projection introduced in [13]. In this case, part (i) of Lemma 1 would have to be modified to $\hat{x}^P(t) \in \mathcal{H}^{-1}(\bar{C}_\xi(z), z)$, $\forall z \in \mathbb{R}^{n_u}$, where $\bar{C}_\xi(z) \supset C_\xi(z)$ is a convex set which can be made arbitrarily close to $C_\xi(z)$, and condition 1of A3 would have to be replaced by the following

$1'$. $\mathcal{H}(\Omega^x_{c_2}, z) \subset C_\xi(z) \subset \bar{C}_\xi(z) \subset \mathcal{H}(\mathcal{X}, z), \forall z \in \Omega^z_{c_2}$, where $C_\xi(z)$ and $\bar{C}_\xi(z)$ convex and compact

Remark 7: From Lemma 1 we conclude that (11) performs a projection for $\hat{x}$ over the compact set $\mathcal{H}^{-1}(C_\xi(z), z)$ which, in general, is unknown since we do not know $\mathcal{H}^{-1}$, and is generally *not* convex. It is interesting to note that applying a standard gradient projection for $\hat{x}$ over an arbitrary convex domain does not necessarily preserve the convergence result (ii) in Theorem 1.

4.2 Closed Loop Stability

To perform output feedback control we replace the state feedback law $v = \phi(x,z)$ with $\hat{v} = \phi(\hat{x}^P, z)$ which, by the smoothness of ϕ and the fact that $\hat{x}^P$ is guaranteed to belong to the compact set $\mathcal{H}^{-1}(C_\xi(z), z)$, is bounded provided that z is confined to within a compact set. Furthermore, the limit (8) guarantees that for any compact set $\mathcal{D}'$ contained in the region of attraction $\mathcal{D}$, one can choose a large enough c_1 so that $\mathcal{D}' \subset \Omega_{c_1} \subset \Omega_{c_2} \subset \mathcal{D}$. When the observability assumption A1 is satisfied globally, one can choose any compact $\mathcal{D}' \subset \mathcal{D}$; hence, if A1 and A2 hold globally, $\mathcal{D}'$ can be any compact set in $\mathbb{R}^n \times \mathbb{R}^{n_u}$. Taking in account the restriction on c_2 imposed by Assumption A3, choose $\mathcal{D}'$ to be an arbitrary compact set contained in Ω_{c_1}.

Let $\tilde{x} \triangleq \hat{x} - x$, and $\tilde{x}^P \triangleq \hat{x}^P - x$, and note that part (ii) of Lemma 1 applies and $\tilde{x}^P \to 0$ as $t \to \infty$ with arbitrarily fast rate, as long as $\chi \in \Omega_{c_2}$. Moreover, the smoothness of the control law implies that

$$\|\phi(\hat{x}, z) - \phi(x,z)\| \leq \bar{\gamma}\|\tilde{x}\| \tag{12}$$

for all $x, \hat{x} \in \mathcal{H}^{-1}(C_\xi(z), z)$, for all $z \in \Omega^z_{c_2}$ and some $\bar{\gamma} > 0$. Assume that $\hat{x}(0) \in \Omega^x_{c_2}$, and let A be a positive constant satisfying $\|\partial V / \partial \chi\| \leq A$ for all χ in Ω_{c_2} (its existence is guaranteed by V being continuously differentiable). Now, we can state the following lemma.

[2] The projection defined in (11) is discontinuous in the variable $\hat{\xi}$, therefore raising the issue of the existence and uniqueness of its solutions. We refer the reader to Remark 6, were this issue is addressed and a solution is proposed.

Lemma 2 *Suppose that the initial condition $\chi(0)$ is contained in $\mathcal{D}' \subset \Omega_{c_1}$, define the set $\Omega_\epsilon \stackrel{\triangle}{=} \{\chi : V(\chi) \leq d_\epsilon\}$, where $d_\epsilon = \alpha_2 \circ \alpha_3^{-1}(\mu \, A \, \bar{\gamma} \, \epsilon)$, and choose $\epsilon > 0$ and $\mu > 1$ such that $d_\epsilon < c_1$. Then, there exists a positive scalar $\rho^*, 0 < \rho^* \leq 1$, such that, for all $\rho \in (0, \rho^*]$, the closed loop system trajectories remain confined in Ω_{c_2}, the set $\Omega_\epsilon \subset \Omega_{c_1}$ is positively invariant, and is reached in finite time.*

Lemma 2 proves that the all the trajectories starting in Ω_{c_1} will remain confined within Ω_{c_2} and converge to an arbitrarily small neighborhood of the origin in finite time. Now, in order to complete the stability analysis, it remains to show that the origin of the output feedback closed loop system is asymptotically stable, so that if Ω_ϵ is small enough all the closed loop system trajectories converge to it.

Lemma 3 *There exists a positive scalar ϵ^* such that for all $\epsilon \in (0, \epsilon^*]$ all the trajectories starting inside the compact set $\Delta_\epsilon \stackrel{\triangle}{=} \{[\chi^\top, \tilde{x}^\top]^\top \,|\, V \leq d_\epsilon \text{ and } \|\tilde{x}\| \leq \epsilon\}$ converge asymptotically to the origin.*

We are now ready to state the following closed loop stability theorem.

Theorem 2 *For the closed loop system (5), (6), (11), satisfying assumptions A1, A2, and A3, the control law $\hat{v} = \phi(\hat{x}^P, z)$, guarantees that there exists a scalar $\rho^*, 0 < \rho^* \leq 1$, such that, for all $\rho \in (0, \rho^*]$, the set $\mathcal{D}' \times \Omega^x_{c_2}$ is contained in the region of attraction of the origin $(x = 0, z = 0, \hat{x} = 0)$.*

Remark 8: Theorem 2 proves regional stability of the closed-loop system, since given an observability subspace $\mathcal{X} \times \mathcal{U}$, and provided condition 1 of A3 is satisfied, the control law $\hat{v}$, together with (6) and (11), make the compact set $\mathcal{D}' \times \Omega^x_{c_2}$ a basin of attraction for the origin of the closed loop system. The size of the region of attraction $\mathcal{D}' \times \Omega^x_{c_2}$ depends on the size of the set $\mathcal{X} \times \mathcal{U}$ (see condition (4)). If A1 is satisfied globally (as in [6, 7, 5]), or $\mathcal{X} \times \mathcal{U}$ is large enough, then Theorem 2 guarantees that the domain of attraction $\mathcal{D}$ of the closed loop system under state feedback is recovered by the output feedback controller, in that $\mathcal{D}'$ can be chosen to be any arbitrary compact set contained in $\mathcal{D}$. If, besides being completely uniformly observable, system (1) is also stabilizable (and, therefore, A2 holds globally), then $\mathcal{D} = \mathbb{R}^n \times \mathbb{R}^{n_u}$ and the result of Theorem 2 becomes semi-global, in that Ω_{c_1} and Ω_{c_2} can be chosen arbitrarily large, thus achieving the same property of the controller found in [7].

Remark 9: Analogous to the result in [1, 5], Theorem 2 proves a separation principle for nonlinear systems: given a stabilizing state feedback controller, the performance of the output feedback controller recovers the one under state feedback provided that the parameter ρ is chosen small enough.

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