Hybrid Stabilization of Rotational Motion in Mechanical Systems with Degree of Underactuation One

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Abstract: This extended abstract addresses the orbital stabilization problem for mechanical systems with underactuation degree one, with a focus on stabilizing rotational motions. A common approach to generating closed orbits in such systems is through virtual holonomic constraints (VHCs), which reduce the system dynamics to a lower-dimensional manifold. Building on recent work on event-based VHCs for oscillation stabilization, we propose a new hybrid control framework to stabilize rotations with a guaranteed basin of attraction.

Keywords: robot control, nonlinear control systems, motion control

1. INTRODUCTION

Consider the steady pace of a human walking in a straight line, or a gymnast performing continuous giants around a bar to maintain rotation speed. These are both examples of repetitive motions governed by the asymptotic stability of closed orbits.

The challenge of enforcing such repetitive behaviour in mechanical systems is known as the *orbital stabilization problem*, and it lies at the heart of many motion control problems in robotics.

For underactuated robots, orbital stabilization is quite challenging because the geometry of the desired orbit and the feasible time parametrizations are tightly coupled, making viable orbit selection a nontrivial task. Moreover, asymptotic tracking is generally not feasible in this setting, as the system lacks enough control authority to follow a prescribed trajectory. Instead, the right framework is set stabilization, where one aims to stabilize the orbit as a subset of the state space, without enforcing a particular time evolution.

To address the OSP for underactuated systems, researchers have proposed using *virtual holonomic constraints* (VHCs)—relations between configuration variables that can be imposed via feedback (see, e.g., Shiriaev et al. (2005); Freidovich et al. (2008); Shiriaev et al. (2010)). The effectiveness of VHCs in underactuated robot control is perhaps most evident in bipedal locomotion, particularly in work by Grizzle and collaborators (e.g., Grizzle et al. (2001); Plestan et al. (2003); Grizzle et al. (2003); Westervelt et al. (2003, 2007)) which views stable walking as a hybrid limit cycle. This cycle is akin to a hybrid rotation, since the motion evolves monotonically in a phase variable. In this context, foot-ground impacts provide the key stabilizing mechanism. For robots without impacts, however, while VHCs help generate closed orbits, they cannot alone ensure stabilization because the dynamics restricted to the constraint manifold lack control inputs.

In recent work (Navarro and Maggiore (2024)) we developed event-based VHCs to locally asymptotically stabilize oscillations for mechanical systems without impact. In Navarro and Maggiore (2025), we improved the controller in Navarro and Maggiore (2024) to get a guaranteed basin of attraction for the target oscillation.

Contributions of this extended abstract: We extend the theory developed in Navarro and Maggiore (2024) to address the problem of stabilizing rotations. Specifically, we introduce a new class of event-based VHCs and develop a hybrid controller stabilizing rotations with a guaranteed basin of attraction. Under certain conditions reviewed in Section 2, VHCs generate a continuum of rotations, each uniquely characterized by a frequency parameter ω. Such rotations exist for all $\omega > \omega_{\min}$. Our controller is guaranteed to stabilize any such rotation with $\omega > \omega_{\min}$.

Notation: We let $\mathsf{B}_r^m := \{x \in \mathbb{R}^m \mid x^\top x < r^2\}$ be the open ball of radius r in $\mathbb{R}^m$ and $\overline{\mathsf{B}}_r^m$ denote its closure. We denote by $\mathrm{sat}_\varepsilon : \mathbb{R} \to \mathbb{R}$ the saturation function with uniform saturation level $\varepsilon > 0$. For $T > 0$ and $x \in \mathbb{R}$, we let $[x]_T$ the x modulo T. We denote by $[\mathbb{R}]_T$ the set of real numbers modulo T and let $\pi_T(x) := [x]_T$.

2. PRELIMINARIES

In this extended abstract we consider an underactuated mechanical system modelled as

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla_q U(q) = B(q)u, \tag{1}$$

where $q = (q_1, \ldots, q_n) \in \mathsf{Q}$ are the configuration variables, $u \in \mathbb{R}^{n-1}$ is the control vector, $D(q)$ is the symmetric and positive definite inertia matrix, $C(q,\dot{q})$ is the Coriolis matrix, $U : \mathsf{Q} \to \mathbb{R}$ is the potential function and $B : \mathsf{Q} \to$

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$\mathbb{R}^{n\times(n-1)}$ is the input matrix, assumed to have full rank $n-1$. We assume that these functions are C^1 and that the input matrix B has a nonvanishing C^1 left annihilator $B^\perp : \mathsf{Q} \to \mathbb{R}^{1\times n}$ (i.e., $B^\perp \neq 0$ and $B^\perp B = 0$). Note that the mechanical systems under consideration have n DOF and $n-1$ actuators, and therefore have underactuation degree one. We assume that the configuration space Q is a generalized cylinder. The state space of the mechanical system is the tangent bundle $T\mathsf{Q} := \{(q,\dot q) \mid q \in \mathsf{Q}, \dot q \in T_q\mathsf{Q}\}$.

In formulating the problem to be investigated, we begin with a C^2 function $h : \mathsf{Q} \to \mathbb{R}^{n-1}$ with the property that system (1) with output $y = h(q)$ has a well-defined vector relative degree $(2,\dots,2)$ everywhere on the level set $h^{-1}(0)$. The level set $h^{-1}(0)$ is called a (regular) *virtual holonomic constraint* (VHC), and we assume here that this level set is a simple closed curve. The zero dynamics manifold associated with the output $y = h(q)$ is

$$\Gamma = \{(q,\dot q) \in T\mathsf{Q} \mid h(q) = 0, dh_q \dot q = 0\}. \tag{2}$$

Since $\mathrm{Ker}(dh_q) = T_q h^{-1}(0)$, Γ is the tangent bundle of $h^{-1}(0)$. In the VHC theory, Γ is called the *constraint manifold* associated with the VHC $h(q) = 0$. The relative degree assumption on h implies the existence of a C^1 feedback (e.g., designed using IO feedback linearization) that asymptotically stabilizes Γ.

The dynamics on the constraint manifold Γ are characterized by the *zero dynamics vector field*. In order to determine the zero dynamics vector field, let $\tilde\sigma : \mathbb{R} \to \mathsf{Q}$ be a C^2 T-periodic regular parametrization of the closed curve $h^{-1}(0)$, and let $\sigma : [\mathbb{R}]_T \to \mathsf{Q}$ defined via the identity $\sigma([x]_T) := \tilde\sigma(x)$. Then $\sigma : [\mathbb{R}]_T \to \mathsf{Q}$ is a diffeomorphism onto its image, $h^{-1}(0)$.

Using this parametrization, we may express the constraint manifold Γ in (2) as follows

$$\begin{aligned}\Gamma &= \{(q,\dot q) \in T\mathsf{Q} \mid q = \sigma(\theta), \dot q = \sigma'(\theta)\dot\theta, (\theta,\dot\theta) \in T[\mathbb{R}]_T\} \\ &= T\sigma(T[\mathbb{R}]_T),\end{aligned} \tag{3}$$

where $T\sigma : T[\mathbb{R}]_T \to T\mathsf{Q}$ is the tangent map of σ, defined as $T\sigma(\theta,\dot\theta) := (\sigma(\theta), \sigma'(\theta)\dot\theta)$.

Since σ is a diffeomorphism onto its image, so is $T\sigma$, and this means that the dynamics on the constraint manifold Γ are entirely described by the two states $(\theta,\dot\theta) \in T[\mathbb{R}]_T$.

Using the parametrization of Γ via $(\theta,\dot\theta) \in T[\mathbb{R}]_T$, the zero dynamics vector field is given by the second-order ODE

$$\ddot\theta = \Psi_1(\theta) + \Psi_2(\theta)\dot\theta^2 \tag{4}$$

with state space $T[\mathbb{R}]_T$, where

$$\begin{aligned}\Psi_1(\theta) &= -\left.\frac{B^\perp \nabla_q U}{B^\perp D\sigma'(\theta)}\right|_{q=\sigma(\theta)} \\ \Psi_2(\theta) &= -\left.\frac{B^\perp D\sigma'' + B^\perp C\sigma'(\theta)}{B^\perp D\sigma'(\theta)}\right|_{\substack{q=\sigma(\theta)\\ \dot q=\sigma'(\theta)}}.\end{aligned} \tag{5}$$

We call system (4) the **constrained dynamics** and its significance is this. Any solution $(q(t),\dot q(t))$ of (1) entirely contained in Γ has the form $(q(t),\dot q(t)) = T\sigma(\theta(t),\dot\theta(t))$, where $(\theta(t),\dot\theta(t))$ is a solution of the constrained dynamics (4).

Recall that the functions Ψ_i in (5) are defined on $[\mathbb{R}]_T$. Using $\tilde\sigma : \mathbb{R} \to \mathsf{Q}$ in place of $\sigma : [\mathbb{R}]_T \to \mathsf{Q}$ in (5) we obtain functions $\tilde\Psi_i : \mathbb{R} \to \mathbb{R}$ satisfying the identity $\tilde\Psi_i(x) := \Psi_i([x]_T)$. Using these functions, we define the *virtual mass and virtual potential* functions, $\tilde M, \tilde V : \mathbb{R} \to \mathbb{R}$, as

$$\begin{aligned}\tilde M(x) &= \exp\left(-2\int_0^x \tilde\Psi_2(\tau)d\tau\right) \\ \tilde V(x) &= -\int_0^x \tilde\Psi_1(\tau)\tilde M(\tau)d\tau.\end{aligned} \tag{6}$$

If, and only if (see Mohammadi et al. (2017)), $\tilde M$ and $\tilde V$ are T-periodic, then the constrained dynamics (4) represent a one-DOF mechanical system with Lagrangian $L(\theta,\dot\theta) = (1/2)M(\theta)\dot\theta^2 - V(\theta)$ and total energy

$$E(\theta,\dot\theta) = \frac{1}{2}M(\theta)\dot\theta^2 + V(\theta), \tag{7}$$

where $M, V : [\mathbb{R}]_T \to \mathbb{R}$ are the unique C^1 functions such that $M([x]_T) = \tilde M(x)$ and $V([x]_T) = \tilde V(x)$.

Assuming $\tilde M$ and $\tilde V$ are periodic, the energy $E(\theta,\dot\theta)$ is conserved along solutions of (4). Using this fact, and letting $V_{\max} := \max_{\theta\in[\mathbb{R}]_T} V(\theta)$, we have that for each initial condition $(\theta(0),\dot\theta(0)) = ([0]_T,\omega)$ such that $|\omega| > \sqrt{2V_{\max}}$, the corresponding orbit of (4) is $\{(\theta,\dot\theta) \mid \dot\theta = \varphi_\omega(\theta)\}$, where

$$\varphi_\omega(\theta) := \mathrm{sign}(\omega)\sqrt{(\omega^2 - 2V(\theta))/M(\theta)}.$$

This orbit is closed because it is expressed as the graph of a function over $[\mathbb{R}]_T$, and it is a rotation since along it, $|\dot\theta(t)|$ is bounded away from zero and the phase variable $\theta(t)$ covers its entire domain of definition, $[\mathbb{R}]_T$. To represent this rotation in $(q,\dot q)$ coordinates, we make use of the diffeomorphism $T\sigma : T[\mathbb{R}]_T \to T\mathsf{Q}$ and define

$$\mathsf{O}(\omega) := \left\{(q,\dot q) = T\sigma(\theta,\dot\theta) \;\middle|\; \theta \in [\mathbb{R}]_T,\ \dot\theta = \varphi_\omega(\theta)\right\}. \tag{8}$$

To summarize the construction so far, we started with a regular VHC $h(q) = 0$, henceforth referred to as the *nominal* VHC, such that $h^{-1}(0)$ is a Jordan curve. We parametrized the curve $h^{-1}(0)$ obtaining the map $\sigma : [\mathbb{R}]_T \to \mathsf{Q}$. We constructed functions $\tilde M, \tilde V$ defined in (6), reliant on $\tilde\Psi_1, \tilde\Psi_2$. Under the assumption that $\tilde M$ and $\tilde V$ are T-periodic, we determined that the constraint manifold Γ contains rotations $\mathsf{O}(\omega)$ given in (8), for any $|\omega| > \sqrt{2V_{\max}}$.

The goal of this extended abstract is to design a controller that simultaneously stabilizes a rotation $\mathsf{O}(\omega)$, while also ensuring that if the state $(q,\dot q)$ is initialized close to Γ, then during transient the state remains close to Γ. In what follows we will be more precise in formulating the problem, but first we need to present the control methodology we employ.

3. DESIGN OF HYBRID ORBITAL STABILIZER

Starting from a nominal VHC $h(q) = 0$ with parametric representation $q = \sigma(\theta)$, $\sigma : [\mathbb{R}]_T \to \mathsf{Q}$, we want to define a VHC family $\{h^a(q) = 0\}_{a\in\mathsf{A}}$ with corresponding parametric representations $\{q = \sigma^a(q)\}_{a\in\mathsf{A}}$ such that $h^0(q) \equiv h(q)$. In other words, the nominal VHC should be the family member corresponding to $a = 0$. In this extended abstract, we let the parameter set A be the closed ball of radius r in $\mathbb{R}^m$, $\mathsf{A} = \overline{\mathsf{B}}^m_r := \{a \in \mathbb{R}^m \mid a^\top a \le r^2\}$.

To define $h^a(q)$ and $\sigma^a(\theta)$, we adopt a technique introduced in Navarro and Maggiore (2024). It is shown in Navarro and Maggiore (2024) that it is always possible, in a neighborhood of the curve $h^{-1}(0)$, to express the VHC $h(q)=0$ as the graph of a function via a coordinate transformation. Specifically, there exists an open set $\mathsf{U}\subset\mathsf{Q}$ containing $h^{-1}(0)$ and a smooth function $\theta^\star:\mathsf{U}\to[\mathbb{R}]_T$ such that $\theta^\star|_{h^{-1}(0)}=\sigma^{-1}$ and the map $N:\mathsf{U}\to[\mathbb{R}]_T\times\mathbb{R}^{n-1}$ given by $q\mapsto(\theta,e)=(\theta^\star(q),h(q))$ is a diffeomorphism onto its image. In (θ,e) coordinates, the nominal VHC is expressed as $e=0$. This is the graph of a function (the zero function), and we can employ the idea presented earlier to embed this VHC in the family $\{e=\phi^a(\theta)\}_{a\in\mathsf{A}}$ with associated parametric representations $\{\theta\mapsto(\theta,\phi^a(\theta))\}_{a\in\mathsf{A}}$. We pull these representations back to q coordinates using the diffeomorphism N and obtain

$$h^a(q):=h(q)-\phi^a\circ\theta^\star(q), \tag{9a}$$

$$\sigma^a(\theta):=N^{-1}(\theta,\phi^a(\theta)). \tag{9b}$$

The above expressions give a VHC family $\{h^a(q)=0\}_{a\in\mathsf{A}}$ and corresponding parametrizations $\{q=\sigma^a(\theta)\}_{a\in A}$, constructed starting from a family of functions $\{\phi^a\}_{a\in\mathsf{A}}$.

In this extended abstract, we consider functions ϕ^a satisfying this assumption:

Assumption 1. Fix $r>0$, and let $\mathsf{A}=\overline{\mathsf{B}}^m_r$, $m\in\mathbb{Z}_{>0}$, and $\delta\in(0,T/2)$. The function family $\{\phi^a:[\mathbb{R}]_T\to\mathbb{R}^{n-1}\}_{a\in\mathsf{A}}$ enjoys these properties:

- *differentiability:* for each $a\in\mathsf{A}$, $\phi^a:[\mathbb{R}]_T\to\mathbb{R}^{n-1}$ is C^2 and for each $\theta\in[\mathbb{R}]_T$, $a\mapsto\phi^a(\theta)$ is C^1;
- *geometry:* $a=0_m\implies\phi^a(\theta)\equiv 0_{n-1}$ and

$$(\forall a\in\mathsf{A})(\forall x\in[0,\delta])\ \phi^a([x]_T)=0_{n-1}. \tag{10}$$

△

We now present the high-level controller structure we employ in this extended abstract.

Consider a target rotation $\mathsf{O}(\omega)$ with $|\omega|>\sqrt{2V_{\max}}$. In order to asymptotically stabilize $\mathsf{O}(\omega)$, we will design a hybrid controller $\mathcal{H}_{\rm rot}$ with state $(\mathsf{d},a)\in\{0,1\}\times\mathsf{A}$ in which $\mathsf{d}\in\{0,1\}$ is a toggle variable designed to disable multiple consecutive jump events, and $a\in\mathsf{A}$ identifies the currently active VHC in the family. This controller has the form [1]

$$\mathcal{H}_{\rm rot}:\begin{cases}(q,\dot q,\mathsf{d},a)\in\mathsf{C} & (\dot{\mathsf{d}},\dot a)=(0,0)\\ (q,\dot q,\mathsf{d},a)\in\mathsf{D} & (\mathsf{d},a)^+=G(q,\dot q,\mathsf{d},a)\\ & \tau=\tau^a(q,\dot q),\end{cases} \tag{11}$$

and the state space of the closed-loop system is

$$\mathsf{X}:=T\mathsf{Q}\times\{0,1\}\times\mathsf{A}.$$

The hybrid controller $\mathcal{H}_{\rm rot}$ produces a feedback controller $\tau=\tau^a(q,\dot q)$ enforcing the VHC $h^a(q)=0$. When the state $(q,\dot q,\mathsf{d},a)$ of the closed-loop system is in the *flow set* $\mathsf{C}\subset\mathsf{X}$, the controller state (d,a) is held constant. On the other hand, when the state $(q,\dot q,\mathsf{d},a)$ is in the *jump set* $\mathsf{D}\subset\mathsf{X}$, the controller updates its state (d,a) via the *jump map* G. In particular, the update of $a\in\mathsf{A}$ instantiates a new VHC. The idea is that through careful design of the sets C,D, and of the jump map G, the controller will drive $(q,\dot q)$ to the target orbit $\mathsf{O}(\omega)$ and the parameter a to zero, the parameter value corresponding to the nominal VHC.

[1] The definition of $\mathcal{H}_{\rm rot}$ in (11) follows the framework developed in the book Sanfelice (2021).

With each VHC $h^a(q)=0$, we associate a constraint manifold

$$\Gamma^a:=\{(q,\dot q)\mid h^a(q)=0,(dh^a)_q\dot q=0\}. \tag{12}$$

We define the *lifted constraint manifold*

$$\Gamma_{\rm lift}:=\{(q,\dot q,\mathsf{d},a)\in\mathsf{C}\mid(q,\dot q)\in\Gamma^a\}, \tag{13}$$

and the *lifted orbit* $\mathsf{O}_{\rm lift}(\omega)\subset\Gamma_{\rm lift}$ as

$$\mathsf{O}_{\rm lift}(\omega):=\{(q,\dot q,\mathsf{d},a)\in\mathsf{C}\mid(q,\dot q)\in\mathsf{O}(\omega),\ a=0,\\ \theta^\star(q)\in\mathsf{I}^{\mathsf{d}}\}, \tag{14}$$

where $\theta^\star(q)$ is a function presented in the next section that roughly speaking, projects a configuration q onto the curve $q=\sigma(\theta)$ and

$$\mathsf{I}^{\mathsf{d}}:=\begin{cases}\pi_T([0,T/2]) & \text{if } \mathsf{d}=0\\ \pi_T([T/2,T]) & \text{if } \mathsf{d}=1.\end{cases}$$

Note that the image of $\mathsf{O}_{\rm lift}(\omega)$ under the projection $(q,\dot q,\mathsf{d},a)\mapsto(q,\dot q)$ is $\mathsf{O}(\omega)$, and that on $\mathsf{O}_{\rm lift}(\omega)$ the controller state a is zero (i.e, the associated VHC is the nominal VHC) while the toggle variable d is set according to the value of $\theta^\star(q)$.

In what follows, we will assume without loss of generality that the parameter ω identifying the target rotation $\mathsf{O}(\omega)$ is positive. For $\omega<0$ we can replace the VHC parametrization $\sigma(\theta)$ with $\sigma(-\theta)$ and proceed.

From the function $\theta^\star:\mathsf{U}\to[\mathbb{R}]_T$ defined earlier, we define the function

$$\dot\theta^\star:T\mathsf{U}\to T[\mathbb{R}]_T,\ \dot\theta^\star(q,\dot q):=d\theta^\star_q\,\dot q.$$

For $\mathsf{d}\in\{0,1\}$, we define the following subsets of $[\mathbb{R}]_T$:

$$\mathsf{I}^{\mathsf{d}}_{\mathsf{C}}=\begin{cases}\{[x]_T\in[\mathbb{R}]_T\mid x\in[-T/2+\delta,T/2]\} & \text{if } \mathsf{d}=0\\ \{[x]_T\in[\mathbb{R}]_T\mid x\in[\delta,T]\} & \text{if } \mathsf{d}=1\end{cases} \tag{15a}$$

$$\mathsf{I}^{\mathsf{d}}_{\mathsf{D}}=\begin{cases}\{[x]_T\in[\mathbb{R}]_T\mid x\in[T/2,T/2+\delta]\} & \text{if } \mathsf{d}=0\\ \{[x]_T\in[\mathbb{R}]_T\mid x\in[0,\delta]\} & \text{if } \mathsf{d}=1.\end{cases} \tag{15b}$$

The flow and jump sets of the controller $\mathcal{H}_{\rm rot}$ are defined as

$$\mathsf{C}=\{(q,\dot q,\mathsf{d},a)\mid q\in\mathsf{U},\ \theta^\star(q)\in\mathsf{I}^{\mathsf{d}}_{\mathsf{C}}\}\\ \mathsf{D}=\{(q,\dot q,\mathsf{d},a)\mid q\in\mathsf{U},\ \theta^\star(q)\in\mathsf{I}^{\mathsf{d}}_{\mathsf{D}}\}, \tag{16}$$

and the jump map $G(q,\dot q,\mathsf{d},a)$ is defined as

$$\begin{cases}(\mathsf{d}^+,a^+)=(1,a) & \text{if } \mathsf{d}=0\\ (\mathsf{d}^+,a^+)=(0,v) & \text{if } \mathsf{d}=1,\end{cases} \tag{17}$$

where $v\in\mathbb{R}^m$ is an *auxiliary feedback* (we omit its arguments for now) to be designed in Section 4.

Recall from Section 3 that the hybrid controller in (11) has state (d,a), where $d\in\{0,1\}$ is a toggle variable and $a\in\mathsf{A}=\overline{\mathsf{B}}^m_r$ is the parameter vector identifying which VHC in the family $\{h^a(q)=0\}_{a\in\mathsf{A}}$ is currently active.

4. MAIN RESULT

The orbital stabilization mechanism in the controller $\mathcal{H}_{\rm rot}$ is the update $a^+=v$ in (17), occurring when $\theta^\star(q(t))$ crosses the threshold $[0]_T$. The final and crucial step in our development is the design of the auxiliary feedback for v.

Since the auxiliary control v is updated when $\mathsf{d} = 1$ and $\theta^\star(q(t))$ crosses the threshold $[0]_T$ (see (16), (17)), we define the Poincaré section

$$\mathsf{P}_{\rm rot} := \{(\theta, \dot\theta, \mathsf{d}, a) \mid \theta = [0]_T, \mathsf{d} = 1, \dot\theta > 0\}$$

and design the auxiliary feedback to stabilize the associated Poincaré map. This feedback is then expressed in terms of the plant state by means of the tangent map $T\theta^\star$.

An important consideration that arises in the analysis is the stabilizability of the linearized Poincaré map, and for that we shall see that we require the row vector

$$b(\omega)^\top := -\frac{1}{\omega}\Big(\frac{\omega^2}{2}\partial_a \tilde M^a(x) + \partial_a \tilde V^a(x)\Big)\Big|_{(x,a)=(T,0_m)}. \quad (18)$$

to be nonzero. In the definition of $b(\omega)^\top$, the functions $\tilde M^a, \tilde V^a : \mathbb{R} \to \mathbb{R}$ are the virtual mass and potential of the VHC $h^a(q) = 0$.

Next, the main result of this extended abstract.

Theorem 2. Consider the hybrid controller constructed using the procedure above and assume that there exists a vector $L \in \mathbb{R}^m$ such that

$$\begin{aligned} \partial_a \tilde M^a(x) L \,\big|_{a=\varepsilon L, x=T} &\le 0 \text{ for small } |\varepsilon| \\ \partial_a \tilde V^a(x) L \,\big|_{a=0, x=T} &< 0. \end{aligned} \quad (19)$$

For each $\omega_1 > \sqrt{2V_{\max}}$ and each $\omega > \omega_1$, there exists $\varepsilon^\star(\omega_1, \omega) \in (0, r)$ such that for each $\varepsilon \in (0, \varepsilon^\star(\omega_1, \omega))$, the hybrid controller $\mathcal{H}_{\rm rot}$ given in (11), (16), and (17) with auxiliary feedback

$$v = v^\star_\varepsilon(\dot\theta^\star(q,\dot q) - \omega) := -L \,\mathrm{sat}_\varepsilon \left(\frac{\dot\theta^\star(q,\dot q) - \omega}{b(\omega)^\top L}\right) \quad (20)$$

renders the lifted orbit $\mathsf{O}_{\rm lift}(\omega)$ asymptotically stable for $\mathcal{H}_{\rm cl}$ with basin of attraction containing the set $\mathsf{O}_{\rm lift}([\omega_1, \infty))$.

The proof is omitted.

Remark 3. Recall $b(\omega)$ as defined in (18). The inequalities in (19) imply that $b(\omega)^\top L > 0$ for all $\omega > 0$ and $\varepsilon > 0$. In particular, under the assumption of Theorem 2 one has $b(\omega) \neq 0$ for every $\omega > 0$, and guarantees that the auxiliary feedback in (20) is well-defined. △

Remark 4. The set $\mathsf{O}_{\rm lift}([\omega_1, \infty))$ guaranteed to be contained in the interior of the basin of attraction of $\mathsf{O}_{\rm lift}(\omega)$ is the subset of states $(q, \dot q, \mathsf{d}, a)$ where the plant state $(q, \dot q)$ is initialized in the part of the constraint manifold Γ given by the union of rotations. Indeed, the parameter ω characterizing the target rotation can be chosen arbitrarily close to the lower bound $\sqrt{2V_{\max}}$ below which there are no rotations with positive ω in the nominal constraint manifold. On the other hand, the controller state is initialized at $a = 0_m$ and $\mathsf{d} = 0$ if $\theta^\star(q_0) \in \mathsf{I}^0$ and $\mathsf{d} = 1$ if $\theta^\star(q_0) \in \mathsf{I}^1$.

The closer ω_1 is chosen to the threshold $\sqrt{2V_{\max}}$, the smaller the upper bound $\varepsilon^\star$ for ε ensuring stability. Since ε is the saturation level of the auxiliary controller, a small ε generally causes slower convergence of the augmented state to the target orbit $\mathsf{O}_{\rm lift}(\omega)$. △

5. CONCLUSIONS

In this extended abstract, we extended the orbital stabilization framework introduced in Navarro and Maggiore (2025) by presenting a new hybrid controller that stabilizes rotations with a certified basin of attraction. Together with the results in Navarro and Maggiore (2024, 2025), this contribution forms a coherent framework for orbital stabilization of oscillations and rotations in mechanical control systems with underactuation degree one.

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