MANEUVER REGULATION, TRANSVERSE FEEDBACK LINEARIZATION, AND ZERO DYNAMICS

Chris Nielsen [*,1] **Manfredi Maggiore** [*,1]

[*] *Department of Electrical and Computer Engineering, University of Toronto, Toronto, ON, Canada.*

{nielsen,maggiore}@control.utoronto.ca

Abstract: This paper presents necessary and sufficient conditions for the local linearization of dynamics transverse to an embedded submanifold of the state space. The main focus is on output maneuver regulation where stabilizing transverse dynamics is a key requirement.

Keywords: Maneuver regulation, path following, feedback linearization, zero dynamics, non-square systems, output stabilization.

1. INTRODUCTION

The maneuver regulation (or path following) problem entails designing a smooth feedback making the trajectories of a system follow a prespecified path, or *maneuver*. Unlike a tracking controller, a maneuver regulation controller drives the trajectories of a system to a maneuver *up to time-reparameterization*. This difference is crucial in robotics and aereospace applications where the system dynamics impose constraints on the time parameterization of feasible maneuvers.

This paper continues a research program initiated in Nielsen and Maggiore (2004a,b) and presents an approach to solving maneuver regulation problems inspired by the work of Banaszuk and Hauser (1995). There, the authors consider periodic maneuvers in the state space and present necessary and sufficient conditions for feedback linearization of the associated transverse dynamics. Feedback linearization is a natural framework for maneuver regulation, as evidenced by the body of work on path following which employs this approach (see for example Altafini (2002), Altafini (2003), Gillespie et al. (2001), Bolzern et al. (2001), Hauser and Hindman (1997), Coelho and Nunes (2003)). In all these papers, the maneuver regulation problem is converted to an input output feedback linearization problem with respect to a *suitable* output. This motivates our interest in establishing a general framework for doing this.

More specifically, the work presented here and in Nielsen and Maggiore (2004a,b) investigates systems with outputs and extends the results of Banaszuk and Hauser (1995) to the case of non-periodic maneuvers defined in the output space (rather than periodic maneuvers in the state space). This work treats the maneuver regulation problem as an output stabilization problem. We solve an output stabilization problem which, under appropriate conditions, also solves a maneuver regulation problem. The main challenge here lies in finding conditions for feedback linearization of dynamics transverse to an embedded submanifold of the state space whose dimension is not

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restricted to be one. A natural way to study this problem is to use the notion of zero dynamics.

In Nielsen and Maggiore (2004a,b), we presented necessary and sufficient conditions for global transverse feedback linearization (TFL). Here, we focus on deriving necessary and sufficient conditions for local TFL (Theorems 3-4), as well as a sufficient condition for a system *not to be* transversely feedback linearizable (Corollary 3). We also show (Lemmas 4 and 5) that, in the case of state maneuvers, our results recover the results of Banaszuk and Hauser (1995).

The following notation is used throughout the paper. We denote by $\Phi_t^v(x)$ the flow of a smooth vector field v. We let $\mathrm{col}(x_1,\dots,x_k) := [x_1 \ \dots \ x_n]^\top$ and, given two column vectors a and b, we let $\mathrm{col}(a,b) := [a^\top \ b^\top]^\top$. Given a smooth distribution D, we let $\mathrm{inv}\,(D)$ be its involutive closure (the smallest involutive distribution containing D) and $D^\perp$ be its annihilator. For brevity, the term *submanifold* is used in place of *embedded submanifold of* $\mathbb{R}^n$ throughout.

2. PROBLEM FORMULATION

Consider the smooth dynamical system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

defined on $\mathbb{R}^n$, with $h : \mathbb{R}^n \to \mathbb{R}^p$ ($p \geq 2$) of class C^r ($r \geq 1$), and $u \in \mathbb{R}$. Given a smooth parameterized curve $\sigma : \mathbb{D} \to \mathbb{R}^p$, where $\mathbb{D}$ is either $\mathbb{R}$ or S^1, the maneuver regulation problem entails finding a smooth control $u(x)$ driving the output of the system to the set $\sigma(\mathbb{D})$ and making sure that the curve is traversed in one direction. When $\mathbb{D} = S^1$, $\sigma(\mathbb{D})$ is a periodic curve. Banaszuk and Hauser (1995) provide a solution to this problem in the special case when $\mathbb{D} = S^1$ and $h(x) = x$. Notice that one particular instance of maneuver regulation is the case when a controller is designed to make $y(t)$ asymptotically track a specific time *parameterization* of the curve $\sigma(t)$ (Hauser and Hindman, 1995). Thus asymptotic tracking and maneuver regulation are closely related problems. In some cases, however, it may be undesirable or even impossible to pose a maneuver regulation problem as one of tracking (consider, for instance, the problem of maneuvering a wheeled vehicle with bounded translational speed by means of steering). We impose geometric restrictions on the class of curves $\sigma(\cdot)$.

Assumption 1. The curve $\sigma : \mathbb{D} \to \mathbb{R}^p$ enjoys the following properties

(i) σ is C^r, ($r \geq 1$)
(ii) σ is regular, i.e., $\|\dot{\sigma}\| \neq 0$
(iii) $\sigma : \mathbb{D} \to \sigma(\mathbb{D})$ is injective (when $\mathbb{D} = S^1$ we instead require σ to be a Jordan curve)
(iv) σ is proper, i.e. for any compact $K \subset \mathbb{R}^p$, $\sigma^{-1}(K)$ is compact (automatically satisfied when $\mathbb{D} = S^1$)

Assumption 1 guarantees that $\sigma(\mathbb{D})$ is a submanifold of $\mathbb{R}^p$ with dimension 1.

Assumption 2. There exists a C^1 map $\gamma : \mathbb{R}^p \to \mathbb{R}^{p-1}$ such that 0 is a regular value of γ and $\sigma(\mathbb{D}) = \gamma^{-1}(0)$. Moreover, the *lift* of $\gamma^{-1}(0)$ to $\mathbb{R}^n$, $\Gamma := (\gamma \circ h)^{-1}(0)$, is a submanifold of $\mathbb{R}^n$.

A sufficient condition for

$$\Gamma = \{x : \gamma_1(h(x)) = \ldots = \gamma_{p-1}(h(x)) = 0\} \tag{2}$$

to be a submanifold of $\mathbb{R}^n$ is that h be *transversal* to $\gamma^{-1}(0)$, i.e., (Abraham and Robbin, 1967; Guillemin and Pollack, 1974)

$$(\forall x \in \Gamma) \ \ \mathrm{Im}\,(dh)_x + T_{h(x)}\gamma^{-1}(0) = \mathbb{R}^p.$$

The codimension of Γ is equal to the codimension of $\gamma^{-1}(0)$. Therefore $\dim \Gamma = n - p + 1$ (Consolini and Tosques, 2003). The problem of maneuvering y to $\gamma^{-1}(0)$ is thus equivalent to maneuvering x to Γ and can be cast as an output stabilization problem for the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y' &= (\gamma \circ h)(x). \end{aligned} \tag{3}$$

In general one may be able to maneuver x to the subset of Γ which can be made invariant by a suitable choice of the control input. Accordingly, let Γ^* be the largest controlled invariant submanifold of Γ under (1) and let $n^* = \dim \Gamma^*$ ($n^* \leq \dim \Gamma = n - p + 1$). Further, let u^* be a smooth feedback rendering Γ^* invariant and define $f^* := (f + gu^*)|_{\Gamma^*}$.

Assumption 3. Γ^* is a closed connected submanifold (with $n^* \geq 1$) and the following conditions hold

(i) $(\exists\, \epsilon > 0)(\forall\, x \in \Gamma^*) \ \ \|L_{f^*}h(x)\| > \epsilon$.
(ii) $f^* : \Gamma^* \to T\Gamma^*$ is complete

In Banaszuk and Hauser (1995), $\Gamma^* = \Gamma = \sigma(S^1)$, and it is assumed that $f(x) \neq 0$ on Γ^*. Thus in that work Assumption 3 is automatically satisfied (the completeness of f^* follows from the periodicity of $\sigma(S^1)$).

We first focus on the well-definiteness part of the assumption. In order to derive conditions guaranteeing that Γ^* is well defined, associate with each constraint in (2) the single input, single output system $\{f, g, \gamma_i \circ h\}$ where $i \in \{1, \ldots, p-1\}$ and a corresponding zero dynamics manifold Γ_i^*.

Lemma 1. If $\bigcap_k \Gamma_k^*$ is a closed, controlled invariant submanifold, then Γ^* exists and it is given by $\Gamma^* = \bigcap_k \Gamma_k^*$.

Proof : ($\subset$) Choose any point $x \in \Gamma^*$. Since $\Gamma^* \subset \Gamma$,

$$(\forall k \in \{1, \ldots, p-1\}) \;\; \gamma_k(h(x)) = 0.$$

This, together with the fact that, by definition, Γ^* is locally invariant around x, implies that

$$(\forall k \in \{1, \ldots, p-1\}) \;\; x \in \Gamma_k^*$$

or $x \in \bigcap_k \Gamma_k^*$.

($\supset$) Since $\bigcap_k \Gamma_k^*$ is controlled invariant and output zeroing, and since $\Gamma^* \subset \bigcap_k \Gamma_k^*$, one has that, by the maximality of Γ^*, $\Gamma^* = \bigcap_k \Gamma_k^*$. ■

Let r_i be the relative degree of system $\{f, g, \gamma_i \circ h\}$ and define $\mathcal{H}_i : x \mapsto \mathrm{col}(\gamma_i \circ h(x), L_f(\gamma_i \circ h(x)), \ldots, L_f^{r_i - 1}(\gamma_i \circ h)(x))$. If $\{r_1, \ldots, r_{p-1}\}$ is well-defined (uniform), one has that each Γ_i^* is globally defined and given by $\Gamma_i^* = \mathcal{H}_i^{-1}(0)$. Even if $\bigcap_k \Gamma_k^*$ is nonempty, it may not be a submanifold. A sufficient condition for the intersection $\Gamma_i^* \cap \Gamma_j^*$, $i \neq j$, to be a submanifold is that (Guillemin and Pollack, 1974)

$$(\forall x \in \Gamma_i^* \cap \Gamma_j^*) \;\; T_x\Gamma_i^* + T_x\Gamma_j^* = \mathbb{R}^n$$

or, equivalently, $\ker (d\mathcal{H}_i)_x + \ker (d\mathcal{H}_j)_x = \mathbb{R}^n$. Using the fact that $T_x(\Gamma_i^* \cap \Gamma_j^*) = T_x\Gamma_i^* \cap T_x\Gamma_j^*$ one easily arrives at the following result.

Corollary 1. Γ^* is a globally well defined closed submanifold if there exists a point $x_0 \in \Gamma$ around which each system $\{f, g, \gamma_i \circ h\}$, $i \in \{1 \ldots p-1\}$ has a uniform relative degree r_i and, if $p > 2$, the following conditions are satisfied for $k = 1, \ldots, p-2$.

(i) For $k = 1, \ldots, p-2$,

$$\left(\forall x \in \bigcap_{j=1}^{k+1} \Gamma_j^*\right) \;\; H_x^k + \ker(d\mathcal{H}_{k+1})_x = \mathbb{R}^n,$$

where H_x^k is defined recursively as

$$\begin{aligned} H_x^1 &:= \ker(d\mathcal{H}_1)_x, & k = 1 \\ H_x^k &:= H_x^{k-1} \cap \ker(d\mathcal{H}_k)_x, & k > 1. \end{aligned}$$

(ii) Letting $u_k^* := -\frac{L_f^{r_k}(\gamma_k \circ h)}{L_g L_f^{r_k - 1}(\gamma_k \circ h)}$, $1 \leq k \leq p-1$,

$$(u_1^*)\big|_{\bigcap_i \Gamma_i^*} = \cdots = (u_{p-1}^*)\big|_{\bigcap_i \Gamma_i^*}.$$

In this case, $n^* = n - \sum_{i=1}^{p-1} r_i$.

Remark 1. Rather than using transversality to derive the sufficient conditions of Corollary 1, one can employ a slight modification of the zero dynamics algorithm of Isidori and Moog (1988) (see also Isidori (1995)) or the constrained dynamics algorithm presented in Nijmeijer and van der Schaft (1990). In both cases a feasible initial condition for the algorithm should be defined to be any point $x_0 \in \Gamma^*$ such that $f(x_0) + g(x_0)u_0 \in T_{x_0}\Gamma^*$ for some real number u_0. If the sufficient conditions of Corollary 1 are not satisfied, the zero dynamics algorithm may still find a *locally* maximal controlled invariant submanifold of Γ.

The condition, in Assumption 3, that $\|L_{f^*}h(x)\| > \epsilon$ on Γ^* implies that there are no equilibria on Γ^* and that, whenever $x \in \Gamma^*$, $\|\dot{y}\| = \|L_{f^*}h(x)\| > \epsilon$. This condition ensures that the output of (1) traverses the curve $\sigma(\mathbb{D})$.

We are now ready to formulate the main problems investigated in this paper. The following are a direct generalization of analogous statements found in Banaszuk and Hauser (1995).

Problem 1: *Find, if possible, a single coordinate transformation $T : x \mapsto (z, \xi) \in \Gamma^* \times \mathbb{R}^{n-n^*}$ valid in a neighborhood $\mathcal{N}$ of Γ^* such that in (z, ξ) coordinates*

(i) $\Gamma^* = \{(z, \xi) \in \Gamma^* \times \mathbb{R}^{n-n^*} : \xi = 0\}$
(ii) The dynamics of system (1) *take the form*

$$\begin{aligned} \dot{z} &= f_0(z, \xi) \\ \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_{n-n^*-1} &= \xi_{n-n^*} \\ \dot{\xi}_{n-n^*} &= b(z, \xi) + a(z, \xi)u \end{aligned} \tag{4}$$

where $a(z, \xi) \neq 0$ in $\mathcal{N}$.

The following is the local version of Problem 1.

Problem 2: *For some $x^0 \in \Gamma^*$, find, if possible, a transformation $T^0 : x \mapsto (z^0, \xi^0) \in \Gamma^* \times \mathbb{R}^{n-n^*}$ valid in a neighborhood U^0 of $x^0 \in \Gamma^*$ such that in (z^0, ξ^0) coordinates properties (i) and (ii) of Problem 1 are satisfied in U^0.*

Remark 2. It is clear that if one can solve Problem 1 or 2, then the smooth feedback

$$u = -\frac{1}{a(z, \xi)}(b(z, \xi) + K\xi). \tag{5}$$

achieves local stabilization to Γ^* and traversal of $\sigma(\mathbb{D})$ in output coordinates. However, (5) does not prevent the closed-loop system from exhibiting finite escape time (i.e., the entire $\sigma(\mathbb{D})$ is traversed in finite time), even though the vector field of the closed-loop system is complete on Γ^*. A similar problem is encountered in feedback linearization when stabilizing a minimum phase system in normal form. There are various ways to modify (5) to avoid finite escape time. Discussing them is beyond the scope of this paper.

Remark 3. If Assumtion 3(i) does not hold, then we have that the path is not traversed and the path following problem cannot be solved in this manner. In these cases, solving Problems 1 and 2 results in the solution to an output stabilization problem for system (3).

In Nielsen and Maggiore (2004a,b) we provided necessary and sufficient conditions to solve Problem 1. In this paper we present necessary and sufficient conditions to solve Problem 2 (Theorem 3), as well as a sufficient condition for Problems 1 and 2 to be *unsolvable*. We also show that in the special case when $\mathbb{D} = S^1$ and $y = x$ in (1) our conditions are equivalent to those presented in Banaszuk and Hauser (1995).

3. SOLUTION TO PROBLEM 1

For the sake of illustration, we begin our discussion by summarizing, without proof, the main results in Nielsen and Maggiore (2004a,b).

Theorem 1. Problem 1 is solvable if and only if there exists a function $\alpha : \mathbb{R}^n \to \mathbb{R}$ such that

(1) $\Gamma^* \subset \{x \in \mathbb{R}^n : \alpha(x) = 0\}$
(2) α yields a uniform relative degree $n-n^*$ over Γ^*.

The conditions in Theorem 1, although rather intuitive, are difficult to check in practice. In what follows we present sufficient conditions for the existence of a solution to Problem 1 which are easier to check.

Corollary 2. If one of the constraints in (2), $\gamma_{\bar{k}} \circ h$, yields a relative degree $n - n^*$ then Problem 1 is solved by setting $\alpha = \gamma_{\bar{k}} \circ h$.

Remark 4. The smooth feedback

$$u^* := \frac{-L_f^{n-n^*}\alpha}{L_g L_f^{n-n^*-1}\alpha}$$

makes Γ^* an invariant submanifold of (1).

Lemma 2. If there exists a function $\alpha : \mathbb{R}^n \to \mathbb{R}$ which satisfies the conditions of Theorem 1, then for each $x \in \Gamma^*$

$$T_x\Gamma^* + \mathrm{span}\{g, \dots, ad_f^{n-n^*-1}g\}(x) = \mathbb{R}^n. \quad (6)$$

Remark 5. Condition (6) is a generalization of the notion of transverse linear controllability to the case of controlled invariant submanifolds of any dimension. It is useful in deriving checkable sufficient conditions for the existence of a solution to Problem 1. The notion of transverse linear controllability was originally introduced in Nam and Arapostathis (1992) and later used in Banaszuk and Hauser (1995) for transverse feedback linearization. In both papers, $n^* = 1, \mathbb{D} = S^1$, and $T_x\Gamma^* = \mathrm{span}\{f^*(x)\}$.

Theorem 2. Problem 1 is solvable if

(1) Γ^* is parallelizable ($T\Gamma^* \cong \Gamma^* \times \mathbb{R}^{n^*}$)
(2) $T_x\Gamma^* + \mathrm{span}\{g \dots ad_f^{n-n^*-1}g\}(x) = \mathbb{R}^n$ on Γ^*
(3) The distribution $\mathrm{span}\{g \dots ad_f^{n-n^*-2}g\}$ is involutive.

4. SOLUTION TO PROBLEM 2

The following is an obvious result in the light of Theorem 1.

Theorem 3. Problem 2 is solvable if and only if there exists a function $\alpha : \mathbb{R}^n \to \mathbb{R}$ defined in a neighborhood U^0 of some $x^0 \in \Gamma^*$ such that

(1) $\Gamma^* \cap U^0 \subset \{x \in U^0 : \alpha(x) = 0\}$
(2) α yields a relative degree $n - n^*$ at x^0.

Proof : ($\Rightarrow$) Let $\alpha = \xi_1^0$, conditions (1) and (2) follow.

($\Leftarrow$) Let $\xi_1^0 = \alpha(x)$. A partial coordinate transformation on U^0 is given by

$$\xi_k^0 = L_f^{k-1}\alpha,\ k \in \{1 \dots n - n^*\}.$$

We seek n^* more independent functions to complete the transformation and yield the correct form. This can always be done (Isidori, 1995, Propostion 4.1.3). From the proof of Theorem 1 we have that the zero dynamics of the resulting normal form coincide, on U^0, with Γ^*. ■

Lemma 3. If there exists a function $\alpha : \mathbb{R}^n \to \mathbb{R}$ which satisfies the conditions of Theorem 3, then

$$T_{x^0}\Gamma^* + \mathrm{span}\{g, \dots, ad_f^{n-n^*-1}g\}(x^0) = \mathbb{R}^n.$$

Proof : The proof is almost identical to the proof of Lemma 2, which is available in Nielsen and Maggiore (2004b). We report it here for the sake of illustration.

The existence of α implies that one can locally transform the system dynamics into the form (4) and specifically that $\Gamma^* \cap U^0$ is locally controlled invariant. Let $f^* := (f + gu^*)|_{\Gamma^*}$, with u^* defined as in Remark 4. Then, for all $x \in \Gamma^* \cap U^0$, $f^*(x) \in T_x\Gamma^*$. Also, $\mathrm{span}\{f^*\}$ is a one dimensional, hence involutive, distribution. These facts imply that, on $\Gamma^* \cap U^0$,

$$\mathrm{span}\{f^*\}^\perp = (T_x\Gamma^*)^\perp + \mathrm{span}\{d\phi_2 \dots d\phi_{n^*}\}.$$

By Assumption 3, $f^* \neq 0$. Define the map $t \mapsto \Phi_t^{f^*}(x^0)$ and its inverse $\phi_1 : \Gamma^* \cap U^0 \to \phi_1(\Gamma^*)$. The map $\Phi_t^{f^*}(x^0)$ is a diffeomorphism of $\phi_1(\Gamma^* \cap U^0)$ onto $\Gamma^* \cap U^0$. By construction $L_{f^*}\phi_1 = 1$ on $\Gamma^* \cap U^0$, implying that $d\phi_1(x^0) \notin \operatorname{span}\{f(x^0)\}^\perp$ and thus that

$$(\mathbb{R}^n)^* = \operatorname{span}\{f^*(x^0)\}^\perp \oplus \operatorname{span}\{d\phi_1(x^0)\}$$

or, equivalently,

$$\mathbb{R}^n = T_{x^0}\Gamma^* \oplus \left(\operatorname{span}\{d\phi_1\ d\phi_2 \dots d\phi_{n^*}\}(x^0)\right)^\perp. \tag{7}$$

Consider a set of linearly independent vectors $\{v_1, \dots, v_{n^*}\}$ spanning $T_{x^0}\Gamma^*$, let

$$V = \begin{bmatrix} v_1 & \cdots & v_{n^*} \end{bmatrix},$$

and define a matrix S as follows (Banaszuk and Hauser, 1995)

$$S(x^0) = \begin{bmatrix} d\phi_1 \\ d\phi_2 \\ \vdots \\ d\phi_{n^*} \\ dL_f^{n-n^*-1}\alpha \\ \vdots \\ \alpha \end{bmatrix} \begin{bmatrix} V & g & \cdots & ad_f^{n-n^*-1}g \end{bmatrix}$$
$$= \begin{bmatrix} L_V\phi(x^0) & * \\ 0 & \Delta(x^0) \end{bmatrix}$$

(all vector fields and 1-forms are evaluated at x^0) where $\{L_V\phi\}_{ij} = L_{v_j}\phi_i$, $i, j = 1, \dots, n^*$, and $\Delta \in \mathbb{R}^{n-n^* \times n-n^*}$ is upper triangular with non-zero diagonal (this follows from condition (2) in Theorem 1). It is clear that if the matrix $L_V\phi(x^0)$ is nonsingular then S is nonsingular as well, implying that $\operatorname{Im}([V\ g(x)\ \cdots\ ad_f^{n-n^*-1}g(x)]) = T_x\Gamma^* + \operatorname{span}\{g, \dots, ad_f^{n-n^*-1}g\}(x) = \mathbb{R}^n$ and the proof is complete. To prove that $L_V\phi(x^0)$ is nonsingular, we use the fact that the product of two matrices AB is full rank if and only if $\operatorname{Im} B \cap \ker A = 0$. In this case we must show that

$$\operatorname{Im} V \cap \ker\left(\operatorname{col}(d\phi_1(x^0), \dots, d\phi_{n^*}(x^0))\right) = 0$$

or, equivalently,

$$T_{x^0}\Gamma^* \cap \ker\left(\operatorname{col}(d\phi_1(x^0), \dots, d\phi_{n^*}(x^0))\right) = 0,$$

and this follows directly from (7). ■

Let

$$D = \operatorname{span}\{g \dots ad_f^{n-n^*-2}g\}. \tag{8}$$

Theorem 2 proves that involutivity of D, together with transverse linear controllability, are sufficient conditions for the existence of a function α satisfying conditions (1) and (2) in Theorem 1 and hence solving Problem 1. When the involutive closure of D, $\operatorname{inv}(D)$, is regular at $x^0 \in \Gamma^*$, the next result provides necessary and sufficient conditions to solve Problem 2. These conditions are easier to check than those in Theorem 3.

Theorem 4. Assume that $\operatorname{inv}(D)$ is regular at $x^0 \in \Gamma^*$. Then Problem 2 is solvable if and only if

(1) $T_{x^0}\Gamma^* + \operatorname{span}\{g, \dots, ad_f^{n-n^*-1}g\}(x^0) = \mathbb{R}^n$
(2) $ad_f^{n-n^*-1}g(x^0) \notin \operatorname{inv}(D)(x^0)$.

Proof : ($\Rightarrow$) Assume that the conditions of Theorem 3 hold. By Lemma 3, (1) holds. By definition of relative degree, in a neighborhood $\Gamma^* \cap U^0$, $d\alpha \in D^\perp$ and $L_{ad_f^{n-n^*-1}g}\alpha \neq 0$. Recall that $d\alpha \in D^\perp$ implies $d\alpha \in (\operatorname{inv} D)^\perp$. Since $L_{ad_f^{n-n^*-1}g}\alpha \neq 0$, one has that $ad_f^{n-n^*-1}g \notin \operatorname{span}\{d\alpha\}^\perp$ and thus also $ad_f^{n-n^*-1}g \notin \operatorname{inv}(D)$, showing that (2) holds.

($\Leftarrow$) Assume $\operatorname{inv}(D)$ is regular at x^0 and conditions (1) and (2) hold. This part of the proof closely follows the idea of the proof of Theorem 2.3 in Banaszuk and Hauser (1995). Notice that $n - n^* - 2 \leq \dim(\operatorname{inv} D) \leq n - 1$. If $\dim(\operatorname{inv} D) = n - n^* - 2$ then essentially the same proof of Theorem 2 applies and we are done. Hence, we focus on the case $n - n^* - 1 \leq \dim(\operatorname{inv} D) \leq n-1$. As in the proof of Theorem 2, let $\{v_1, \dots, v_{n^*}\}$ be a set of vector fields defined on Γ^* such that $T_x\Gamma^* = \operatorname{span}\{v_1, \dots, v_{n^*}\}(x)$, and generate s-coordinates by flowing along the vector fields $v_1, \dots, v_{n^*}, ad_f^{n-n^*-1}g, \dots, g$ with times $s_1, \dots, s_n$, respectively. By condition (1), there exists a neighborhood U of x^0 such that the map F defined as

$$s \mapsto \Phi_{s_n}^g \circ \dots \circ \Phi_{s_{n^*+1}}^{ad_f^{n-n^*-1}g} \circ \Phi_{s_{n^*}}^{v_{n^*}} \circ \dots \circ \Phi_{s_1}^{v_1}(x^0),$$

is a diffeomorphism of $F^{-1}(U)$ onto U. Define the set

$$M := \{x \in U : s_{n^*+2}(x) = \dots = s_n(x) = 0\}$$

which is a submanifold of U containing $\Gamma^* \cap U$ of dimension $n^* + 1$. The submanifold M is the set of points reachable from Γ^* by flowing along $ad_f^{n-n^*-1}g$. Since, by assumption, $\operatorname{inv}(D)$ is regular at x^0, it follows that $\operatorname{inv}(D)$ generates a foliation by integral submanifolds, $\mathcal{S}$, in a neighborhood of x^0 which, without loss of generality, we can take to be U. Let S_x denote a leaf of the foliation passing through $x \in \Gamma^* \cap U$. On $\Gamma^* \cap U$, $T_xM = T_x\Gamma^* + \operatorname{span}\{ad_f^{n-n^*-1}g\}$. By condition (1) $T_xM + D = \mathbb{R}^n$, implying that $T_xM + \operatorname{inv}(D) = \mathbb{R}^n$ or, equivalently, $T_xM + T_xS_x = \mathbb{R}^n$. This shows that, on $\Gamma^* \cap U$, M is transversal to $\mathcal{S}$ and $T_xM \cap T_xS_x = T_x\Gamma^* \cap \operatorname{inv}(D)(x)$ is a regular distribution. Let $\hat{n}$ be its dimension. Since we are considering the case $n - n^* - 1 \leq \dim(\operatorname{inv} D) \leq n - 1$, we have that $1 \leq \hat{n} \leq n^*$. Making, if needed, $M \cap U$ smaller, let $\{\hat{v}_1, \dots, \hat{v}_{\hat{n}}\}$ be a set of vector fields defined on $M \cap U$ spanning $T_xM \cap \operatorname{inv}(D)$ on $M \cap U$. Choose additional $n^* - \hat{n}$ vector fields $\{\hat{v}_{\hat{n}+1}, \dots, \hat{v}_{n^*}\}$ defined on $\Gamma^* \cap U$ such that $T_x\Gamma^* = \operatorname{span}\{\hat{v}_1, \dots, \hat{v}_{n^*}\}(x)\ \ \forall x \in \Gamma^* \cap U$. Then, by condition (1),

$$\{\hat{v}_1, \ldots, \hat{v}_{\hat{n}}, \hat{v}_{\hat{n}+1}, \ldots, \hat{v}_{n^*}, g, \ldots, ad_f^{n-n^*-1} g\}$$

is a set of independent vector fields on $\Gamma^* \cap U$. Moreover, $\mathrm{inv}(D) = \mathrm{span}\{\hat{v}_1, \ldots, \hat{v}_{\hat{n}}\} + D$. By making, if necessary, $M \cap U$ smaller we can assume that the vector fields

$$\{\hat{v}_1, \ldots, \hat{v}_{\hat{n}}, g, \ldots, ad_f^{n-n^*-1} g\}$$

are independent on $M \cap U$. The domain of definition of the vector fields involved in our construction is summarized as:

$$\begin{aligned} &\{\hat{v}_{\hat{n}+1}, \ldots, \hat{v}_{n^*}\} \text{ on } \Gamma^* \cap U \\ &\{\hat{v}_1, \ldots, \hat{v}_{\hat{n}}\} \text{ on } M \cap U \\ &\{g, \ldots, ad_f^{n-n^*-1} g\} \text{ on } U. \end{aligned}$$

We use these vector fields to define the map $G : G^{-1}(U^0) \to U^0$ ($U^0 \subset U$ is a neighborhood of x^0),

$$\begin{aligned} p \mapsto &\Phi^g_{p_n} \circ \cdots \circ \Phi^{ad_f^{n-n^*-2}}_{p_{n^*+2}} \circ \Phi^{\hat{v}_{\hat{n}}}_{p_{n^*+1}} \circ \cdots \circ \Phi^{\hat{v}_1}_{p_{n^*-\hat{n}+2}} \\ &\circ \Phi^{ad_f^{n-n^*-1} g}_{p_{n^*-\hat{n}+1}} \circ \Phi^{\hat{v}_{n^*}}_{p_{n^*-\hat{n}}} \circ \cdots \circ \Phi^{\hat{v}_{\hat{n}+1}}_{p_1}(x^0). \end{aligned}$$

Let $P_1 = (p_1, \ldots, p_{n^*-\hat{n}})$, $P_2 = (p_{n^*-\hat{n}+1}, \ldots, p_{n^*+1})$, $P_3 = (p_{n^*+2}, \ldots, p_n)$, and define

$$\begin{aligned} G_1^{P_1}(x^0) &:= \Phi^{\hat{v}_{n^*}}_{p_{n^*-\hat{n}}} \circ \cdots \circ \Phi^{\hat{v}_{\hat{n}+1}}_{p_1}(x^0) \\ G_2^{P_2}(x^1) &:= \Phi^{\hat{v}_{\hat{n}}}_{p_{n^*+1}} \circ \cdots \circ \Phi^{\hat{v}_1}_{p_{n^*-\hat{n}+2}} \circ \Phi^{ad_f^{n-n^*-1} g}_{p_{n^*-\hat{n}+1}}(x^1) \\ G_3^{P_3}(x^2) &:= \Phi^g_{p_n} \circ \cdots \circ \Phi^{ad_f^{n-n^*-2}}_{p_{n^*+2}}(x^2), \end{aligned}$$

so that $G(p) = G_3^{P_3} \circ G_2^{P_2} \circ G_1^{P_1}(x^0)$. For a fixed x^0, $x^1 \in \Gamma^* \cap U^0$, and $x^2 \in M \cap U^0$, each $G_i^{P_i}$ is a diffeomorphism onto its image, thus G is a diffeomorphism onto U^0. This can be most easily seen by examining the order in which the various flows are composed. In particular, since $\hat{v}_{\hat{n}+1}, \ldots, \hat{v}_{n^*}$ are independent on Γ^*, the set of points reached by flowing along these vector fields is a submanifold, $\bar{S}$, of dimension $n^* - \hat{n}$, contained in Γ^*. Next, since $ad_f^{n-n^*-1} g, \hat{v}_1, \ldots, \hat{v}_{\hat{n}}$ are independent on $M \cap U^0$, the set of points reachable from $\bar{S}$ by flowing along these vector fields is precisely $M \cap U^0$. Thus $M \cap U^0 = \{x \in U^0 : P_3(x) = 0\}$ and $\Gamma^* \cap U^0 = \{x \in U^0 : p_{n^*-\hat{n}+1}(x) = 0, P_3(x) = 0\}$. Finally, the set of points reachable from $M \cap U^0$ by flowing along $g, \ldots, ad_f^{n-n^*-2} g$ is the entire U^0.

Choose $\alpha(x) = p_{n^*-\hat{n}+1}(x)$. Then $\Gamma^* \cap U^0 \subset \{x \in U^0 : \alpha(x) = 0\}$ and thus condition (1) in Theorem 3 is satisfied. By the involutivity of $\mathrm{inv}(D) = \mathrm{span}\{\hat{v}_1, \ldots, \hat{v}_{\hat{n}}\} + D$, the vector fields $ad_f^i g$, $i = 0, \ldots, n - n^* - 2$, in s-coordinates have the form (Su and Hunt, 1986, Lemma 4)

$$ad_f^i g = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \\ \vdots \\ * \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow \text{ row } n^* - \hat{n} + 2, \\ \\ \\ \end{matrix}$$

and thus, on U^0, $L_{ad_f^i g}\alpha = 0$, $i = 0, \ldots, n - n^* - 2$. It is also clear that $L_{ad_f^{n-n^*-1} g}\alpha \neq 0$ on U^0. Thus the assumptions of Theorem 3 are satisfied. ■

Corollary 3. If $\dim(\mathrm{inv}\, D) = n$, then Problems 1 and 2 are unsolvable.

Corollary 4. Assume that $\mathrm{inv}\, D$ is regular on Γ^* and that

(1) $T_x\Gamma^* + \mathrm{span}\{g, \ldots, ad_f^{n-n^*-1} g\}(x) = \mathbb{R}^n$ on Γ^*
(2) $ad_f^{n-n^*-1} g(x) \notin \mathrm{inv}(D)(x)$ on Γ^*.

Then there exists an open covering $\{U^{(i)}\}$ of Γ^* and a collection of transformations $\{T^{(i)}\}$, with $T^{(i)} : x \mapsto (z^{(i)}, \xi^{(i)}) \in \Gamma^* \cap U^{(i)} \times \mathbb{R}^{n-n^*}$ such that $\Gamma^* \cap U^{(i)} = \{\xi^{(i)} = 0\}$ and in $(z^{(i)}, \xi^{(i)})$ coordinates the systems has the form (4).

5. STATE MANEUVERS

In this section, we show that when $y = x$ in (1) and $\mathbb{D} = S^1$ the results obtained thus far are equivalent to the results presented in Banaszuk and Hauser (1995). See also Nijmeijer and Campion (1993). The conditions presented in (Banaszuk and Hauser, 1995, Theorem 2.1) for Problem 1 to be solvable are

(a) $\dim\left(\mathrm{span}\{f^*, g, \ldots, ad_{f^*}^{n-2} g\}\right) = n$ on Γ^*
(b) There exists a function $\alpha : \mathbb{R}^n \to \mathbb{R}$ such that
  (i) $d\alpha \neq 0$ on Γ^*.
  (ii) $\alpha = 0$ on Γ^*
  (iii) $L_{ad_{f^*}^i g}\alpha = 0$ near Γ^* for $i = 0 \ldots n - 3$.

Lemma 4. Conditions (a) and (b) above hold if and only if the conditions of Theorem 1 hold.

Proof : (⇒) Assume conditions (a) and (b) hold. Condition (b.ii) is the same as condition (1) in Theorem 1. Next, since f^* is by definition tangent to Γ^*, condition (b.ii) implies that $L_{f^*}\alpha = 0$ on Γ^*. This, together with condition (b.iii), implies that, on Γ^*, $\mathrm{span}\{d\alpha\}^\perp = \mathrm{span}\{f^*, g, \ldots, ad_{f^*}^{n-3} g\}$. By condition (a), necessarily $L_{ad_{f^*}^{n-3} g}\alpha \neq 0$ on Γ^*. This, together with condition (b.iii) shows that α yields a relative degree $n - 1$, which is precisely condition (2) in Theorem 1.

(⇐) Assume the conditions of Theorem 1 hold. Condition (a) holds by Lemma 2. Condition (b.ii) is identical to condition (1) in Theorem 1. Finally, since α yields a relative degree $n - 1$ (recall that here $n^* = 1$), conditions (b.i) and (b.iii) are satisfied. ■

As for Problem 2, consider the distribution D, in (8), with $n^* = 1$. The conditions presented in (Banaszuk and Hauser, 1995, Theorem 2.4) for Problem 2 to be solvable are

(a) $\dim\left(\text{span}\{f^*, g, \ldots, ad_{f^*}^{n-2} g\}\right) = n$ on Γ^*
(b) The distribution D is either
  (i) involutive or
  (ii) $\dim(\text{inv}\, D) = n-1$ in a neighborhood of Γ^* and $f^* \in \text{inv}\, D$ on Γ^*.

Lemma 5. Conditions (a) and (b) above hold if and only if the conditions of Corollary 4 hold.

Proof : ($\Rightarrow$) Assume (a) and (b) above hold. Then we just have to show that condition (2) of Corollary 4 holds. If D is involutive then (a) immediately gives (2). Otherwise, since $f^* \in \text{inv}\, D$ on Γ^*, condition (a) implies condition (2).

($\Leftarrow$) Obvious. ■

A key difference between the normal form presented in this paper (4) and the one presented in Banaszuk and Hauser (1995) lies in the structure given to the vector field f_0 in (4). In the case $n^* = 1$ the following procedure illustrates how to obtain the normal form presented in Banaszuk and Hauser (1995). Fix a point $x_0 \in \Gamma^*$ and define the map $t \mapsto \Phi_t^{f^*}(x^0)$ and its inverse $\varphi' : \Gamma^* \to \varphi'(\Gamma^*)$. Note that, by Assumption 3(ii), φ' is globally defined and that, when $\mathbb{D} = S^1$, $\varphi'(\Gamma^*) = S^1$. By construction $L_{f^*}\varphi' = 1$ on Γ^*. Let $z = \varphi'(x)$ and let $\xi_i = L_f^{i-1}\alpha$, $i = 1 \ldots n-1$. With this transformation, together with the feedback

$$u = \frac{-L_f^{n-1}\alpha + v}{L_g L_f^{n-2}\alpha}$$

one obtains

$$\begin{aligned}
\dot{z} &= 1 + f_1(z,\xi) + g_0(z,\xi)v \\
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_{n-2} &= \xi_{n-1} \\
\dot{\xi}_{n-1} &= v,
\end{aligned} \tag{9}$$

($f_1(z,0) = 0$) which is the normal form as presented in Banaszuk and Hauser (1995). It is interesting to note that the normal form of Banaszuk and Hauser (1995) is also valid when $\mathbb{D} = \mathbb{R}$ (in such a case, the domain of z is $\varphi'(\Gamma^*) = \mathbb{R}$ rather than S^1).

When $n^* > 1$, the normal form (9), could perhaps be generalized by finding a partial coordinate transformation $z = \varphi(x)$ yielding $\dot{z} = \text{col}(1, 0, \ldots, 0)$ on Γ^*. This is always possible locally. Doing so globally amounts to finding a *global* rectification for a vector field on a manifold.

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