A Class of Rendezvous Controllers for Underactuated Thrust-Propelled Rigid Bodies

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***Abstract*— A framework is presented for rendezvous in a network of n underactuated thrust-propelled rigid bodies. A nested loop control structure is proposed whereby an outer loop consensus controller for a double integrator system provides reference signals to an inner loop thrust direction stabilizer. Both outer loop and inner loop controllers can be chosen from a wealth of solutions available in the literature, and it is shown that their combination solves the rendezvous problem almost globally for the rigid body network. We illustrate the theory by combining sample feedbacks, and we show robustness to noise by means of numerical simulations.**

I. Introduction

This paper investigates the rendezvous problem for a network of n underactuated vehicles. Each vehicle is modeled as a rigid body to which thrust is applied along a single body axis while a torque can be applied along any axis. The configuration space of each vehicle is $\mathsf{SE}(3)$ and the model takes into account the rigid body dynamics. Examples of vehicles in this class include vertical take-off and landing (VTOL) aircraft and quadrotor helicopters. The control objective is to design a locally distributed controller making the vehicles converge to one another and move in synchrony. Each vehicle can measure relative positions, relative velocities, and relative attitudes, but cannot measure any absolute quantities except for its own angular velocity and the gravity vector in its body frame. Measurement of body frame angular velocity is common in literature [1], [2], and in applications it can be obtained using a rate gyroscope mounted on the vehicle. Measurement of the gravity vector can be obtained using an accelerometer.

Synchronization and coordination in multi-agent systems have been researched heavily in recent years. Several authors have focused specifically on attitude synchronization. In [3], the authors adopted a passivity approach for a kinematic model, while in [2] a potential shaping approach was used for a dynamic model to achieve an almost global result. This potential shaping approach requires an angular velocity dissipation term that drives the angular velocity to zero. In [4], [5], [6], the solution requires that each vehicle measures its own attitude with respect to an inertial frame.

Many authors have considered kinematic vehicle models in which the velocity along a heading direction and the vehicles' angular velocities are directly controlled using only relative measurements with respect to neighboring vehicles. In $\mathsf{SE}(2)$ this corresponds to synchronization/coordination in a system of unicycles as discussed in [7], [8], [9]. The case of $\mathsf{SE}(3)$ is considered in [10], [11]. In [1], the authors present a control solution based on a variational formulation of mechanics to address synchronization in $\mathsf{SE}(3)$ for fully actuated dynamic vehicle models.

Finally, in [12], [13] the authors address synchronization in $\mathsf{SE}(3)$ for underactuated dynamic vehicle models. These approaches consider relative measurements between neighboring vehicles but are not computed locally, that is, on board with respect to the vehicle's own body frame. As such, each vehicle requires inertial measurements of its own state.

To the best of our knowledge, no result in literature solves the rendezvous problem for a set of underactuated dynamic models ensuring almost global asymptotic stability of the rendezvous manifold using locally distributed measurements. To attain this result, building on recent results [14], [15], we develop a nested loop design methodology, reducing the rigid body rendezvous problem to a consensus problem for double integrators. In the proposed framework, an outer loop consensus controller for double integrators provides reference signals for an inner loop thrust direction stabilizer. If the double integrator controller achieves consensus for a certain class of communication graphs, the nested loop controller proposed in this paper achieves almost global rendezvous in the network of rigid bodies for the same class of graphs.

II. Preliminaries and notation

In this section we summarize the notation and review preliminary notions and stability definitions used throughout the paper. We let $v \cdot w$ denote the Euclidean inner product between vectors $v,\ w \in \mathbb{R}^3$, and by $\|v\| := (v \cdot v)^{1/2}$ the Euclidean norm of v. We let $\{e_1, e_2, e_3\}$ denote the natural basis of $\mathbb{R}^3$. We denote

$$v^\times = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}.$$

With this definition, one has that $v^\times w = v \times w$. Vice versa, given a skew-symmetric matrix $M = -M^\top \in \mathbb{R}^{3\times3}$, we denote $M_\times := [M_{32} \quad M_{13} \quad M_{21}]^\top$. We denote $\mathsf{SO}(3) := \{R \in \mathbb{R}^{3\times3} : R^{-1} = R^\top, \det(R) = 1\}$. If Γ is a closed subset of a Riemannian manifold $\mathcal{X}$, and $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a distance metric on $\mathcal{X}$, we denote by $\|\chi\|_\Gamma := \inf_{\psi\in\Gamma} d(\chi, \psi)$ the point-to-set distance of $\chi \in \mathcal{X}$ to Γ. If $\epsilon > 0$, we let $B_\epsilon(\Gamma) := \{\chi \in \mathcal{X} : \|\chi\|_\Gamma < \epsilon\}$. By $\mathcal{N}(\Gamma)$ we

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denote a generic neighborhood of Γ in $\mathcal{X}$. We will let S^1 denote the set of real numbers modulo 2π and if A and B are two sets, we denote by $A\backslash B$ the set-theoretic difference of A and B. Finally, if $I = \{i_1, \ldots, i_n\}$ is an index set, the ordered list of elements $(x_{i_1}, \ldots, x_{i_n})$ is denoted as $(x_j)_{j \in I}$.

The following stability definitions are taken from [16]. Let $\Sigma : \dot{\chi} = f(\chi)$ be a smooth dynamical system with state space a Riemannian manifold $\mathcal{X}$, and let $\phi(t, \chi_0)$ denote its local phase flow. Let $\Gamma \subset \mathcal{X}$ be a closed set which is positively invariant for Σ, i.e., such that for all $\chi_0 \in \Gamma$, $\phi(t, \chi_0) \in \Gamma$ for all $t \geq 0$ for which $\phi(t, \chi_0)$ is defined.

Definition 1: The set Γ is *stable* for Σ if for any $\epsilon > 0$ there exists a neighborhood $\mathcal{N}(\Gamma) \subset \mathcal{X}$ such that, for all $\chi_0 \in \mathcal{N}(\Gamma)$ and all $t \geq 0$ for which $\phi(t, \chi_0)$ is defined, $\phi(t, \chi_0) \in B_\epsilon(\Gamma)$. Γ is *attractive* for Σ if there exists neighborhood $\mathcal{N}(\Gamma) \subset \mathcal{X}$ such that $\lim_{t\to\infty} \|\phi(t, \chi_0)\|_\Gamma = 0$ for all $\chi_0 \in \mathcal{N}(\Gamma)$. The *domain of attraction of* Γ is the set $\{\chi_0 \in \mathcal{X} : \lim_{t\to\infty} \|\phi(t, \chi_0)\|_\Gamma = 0\}$. Γ is *globally attractive* for Σ if it is attractive with domain of attraction $\mathcal{X}$. Γ is *locally asymptotically stable (LAS)* (or just asymptotically stable) for Σ if it is stable and attractive. Γ is *globally asymptotically stable (GAS)* for Σ if it is stable and globally attractive. Γ is *almost-globally asymptotically stable (AGAS)* for Σ if the set Γ is LAS for Σ with domain of attraction $\mathcal{X}\backslash N$ where $N \subset \mathcal{X}$ is a set of Lebesgue measure zero.

Definition 2: Let $\Gamma_1 \subset \Gamma_2$ be two closed subsets of $\mathcal{X}$ which are positively invariant for Σ. Γ_1 is *globally asymptotically stable relative to* Γ_2 if it is LAS when initial conditions are restricted to lie in Γ_2, and its domain of attraction contains Γ_2. Γ_2 is *locally stable near* Γ_1 if for all $\chi \in \Gamma_1$, for all $c > 0$, and all $\epsilon > 0$, there exists $\delta > 0$ such that for all $\chi_0 \in B_\delta(\Gamma_1)$ and all $t > 0$, if $\phi([0,t], \chi_0) \subset B_c(\chi)$ then $\phi([0,t], \chi_0) \subset B_\epsilon(\Gamma_2)$. Γ_2 is *locally attractive near* Γ_1 if there exists a neighborhood $\mathcal{N}(\Gamma_1)$ such that, for all $\chi_0 \in \mathcal{N}(\Gamma_1)$, $\|\phi(t, \chi_0)\|_{\Gamma_2} \to 0$ as $t \to \infty$.

We present now a reduction theorem that will be instrumental to derive our main result

Theorem 1 (Reduction Theorem [17]): Let Γ_1 and Γ_2, $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$, be two closed sets that are positively invariant for Σ, and suppose Γ_1 is compact. Consider the following conditions: (i) Γ_1 is LAS relative to Γ_2; (i)' Γ_1 is GAS relative to Γ_2; (ii) Γ_2 is locally stable near Γ_1; (iii) Γ_2 is locally attractive near Γ_1; (iii)' Γ_2 is globally attractive; (iv) all trajectories of Σ are bounded.

Then, the following implications hold: (i) $\wedge$ (ii) $\implies$ Γ_1 is stable. (i) $\wedge$ (ii) $\wedge$ (iii) $\iff$ Γ_1 is LAS. (i)' $\wedge$ (ii) $\wedge$ (iii)' $\wedge$ (iv) $\implies$ Γ_1 is GAS. $\square$

III. MODELING

We consider a model of n identical thrust-propelled vehicles. Each vehicle i is an underactuated mechanical system with six degrees of freedom and four scalar control inputs. This system is underactuated because one cannot independently assign linear and angular accelerations. This is depicted in Figure 1, where $\mathcal{I}$ and $\mathcal{B}_i = \{b_{i1}, b_{i2}, b_{i3}\}$ represent, respectively, the inertial and body frame. The vehicle is propelled by a thrust vector directed opposite to b_{i3}, which represents the propulsion direction. The thrust vector has, therefore, constant direction in the body frame, but its magnitude, u_i, can be freely controlled. The vehicle incorporates some actuation mechanism inducing a torque τ_i about the three body axes that can be freely assigned.

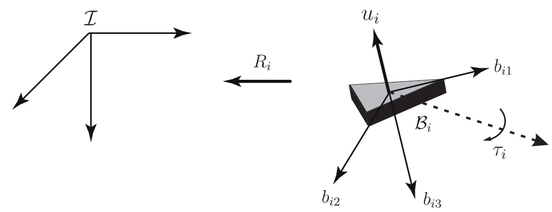

Fig. 1. Vehicle class under consideration.

The rigid body equations of motion are given by

$$\dot{x_i} = v_i,\ m\dot{v_i} = mge_3 - u_i R_i e_3 = mge_3 + T_i, \quad (1)$$

$$\dot{R}_i = R_i\,\omega_i^\times,\ J\dot{\omega}_i = \tau_i - \omega_i \times J\omega_i, \quad (2)$$

where $i = 1, 2, \ldots, n$. In the above, $x_i \in \mathbb{R}^3$ is the position of vehicle i expressed in frame $\mathcal{I}$, $v_i \in \mathbb{R}^3$ is the linear velocity of vehicle i expressed in frame $\mathcal{I}$, $R_i \in \mathsf{SO}(3)$ is the attitude of vehicle i with respect to the inertial frame, and $\omega_i \in \mathbb{R}^3$ is the angular velocity of vehicle i expressed in body frame $\mathcal{B}_i$. Further, the thrust axis of vehicle i in the inertial frame is denoted by $q_i = R_i e_3$. The remaining variables are the vehicle mass m, the symmetric inertia matrix J expressed in body frame, and finally the thrust vector $T_i = -R_i e_3 u_i$. The state vector for vehicle i is given by $\chi_i := (x_i, v_i, R_i, \omega_i) \in \mathcal{X}_i$ where $\mathcal{X}_i := \mathbb{R}^3 \times \mathbb{R}^3 \times \mathsf{SO}(3) \times \mathbb{R}^3$ and the configuration space of each vehicle is $\mathsf{SE}(3)$.

Denote $x := (x_1, \ldots, x_n)$, $v := (v_1, \ldots, v_n)$, $R := (R_1, \ldots, R_n)$, $\omega := (\omega_1, \ldots, \omega_n)$ where n is the number of vehicles in the configuration. The state vector is given by $\chi := (x, v, R, \omega) \in \mathcal{X}$ where $\mathcal{X} := \mathbb{R}^{3n} \times \mathbb{R}^{3n} \times \mathsf{SO}(3)^n \times \mathbb{R}^{3n}$ is the state space for the n-vehicle system.

An example of a vehicle that falls into our modeling framework is the quadrotor aerial vehicle [6], [18].

The information exchange between vehicles is modelled by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$. Each node $n_i \in \mathcal{V}$ of $\mathcal{G}$ represents a rigid body and $\mathcal{E}$ is the set of edges specified from some node i to another node j indicating that vehicle i can measure its state relative to vehicle j (sensing convention). The graph $\mathcal{G}$ is said to be an *undirected graph* if $\forall i, j \in \mathcal{V}$, $(i,j) \in \mathcal{E}$ implies $(j,i) \in \mathcal{E}$. It is otherwise called a *directed graph*. A is an $n \times n$ adjacency matrix corresponding to the graph $\mathcal{G}$ where the entry $a_{ij} \neq 0$ if $(i,j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. A path between two nodes n_1, n_l is a sequence of nodes $\{n_1, n_2, \ldots, n_l\}$ such that n_i, n_{i+1} is an edge for $i = 1, \ldots l-1$. We denote N_i as the set of vehicles $j \in \mathcal{V}$ such that there is an edge from vehicle i to vehicle j i.e. $(i,j) \in \mathcal{E}$.

IV. Rendezvous Control Problem

The communication topology, and the absence of a global reference frame, constrain the available information to each vehicle. Any feedback control law and computation procedure must therefore rely on only "measurable variables". This leads to the notion of a locally distributed function. For vehicles i and j, we define the relative displacement $x_{ij} := x_j - x_i$ of vehicle i with respect to vehicle j. Similarly, we define the relative velocity $v_{ij} := v_j - v_i$, relative attitude $R_{ij} := R_i^T R_j$, relative thrust axes $q_{ij} = q_j - q_i$ and relative angular velocity $\omega_{ij} := R_{ij}\omega_j - \omega_i$. The collection of relative displacements and velocities of vehicle i with respect to its neighbors in the sensor graph is defined as $\hat{y}_i := \big(x_{ij}, v_{ij}\big)_{j \in N_i}$. On the other hand, the collection of the same quantities projected onto body frame $\mathcal{B}_i$ is defined as $y_i := \big(R_i^\top x_{ij}, R_i^\top v_{ij}\big)_{j \in N_i}$.

In this paper, we denote an inertial vector $r \in \mathbb{R}^3$ with respect to the reference frame $\mathcal{B}_i$ by r^i, that is, $R_i^\top r = r^i$. We use bold characters to denote state-dependent reference values: $\mathbf{q}_i$ is a thrust direction reference and $\boldsymbol{\omega}_i$ is an angular velocity reference for vehicle i. The tilde mark on top of a symbol denotes an error between an actual state and its corresponding reference: $\tilde{\mathbf{q}}_i = R_i^\top \mathbf{q}_i$ is a thrust direction error measured with respect to e_3 and $\tilde{\boldsymbol{\omega}}_i = \omega_i - \boldsymbol{\omega}_i$ is an angular velocity error.

Definition 3 (Locally Distributed Function): A smooth function is a *locally distributed function* for vehicle i if its arguments include exclusively

1) The collection y_i of relative positions and velocities of vehicle i with respect to its neighbors in the sensor graph, projected on the body frame $\mathcal{B}_i$;
2) Relative attitudes $(R_{ij})_{j \in N_i}$ and relative angular velocities $\big(\omega_{ij}\big)_{j \in N_i}$;
3) the angular velocity ω_i in the body frame $\mathcal{B}_i$;
4) the values $\big(u_j, \dot{u}_j\big)_{j \in N_i}$.

Remark 1: Following from Definition 3, a locally distributed feedback for vehicle i can also be a function of $(\omega_j)_{j \in N_i}$. To see this notice that, since ω_{ij}, ω_i and R_{ij} are available to vehicle i, vehicle i can compute $\omega_j = R_{ij}^\top(\omega_i + \omega_{ij})$ for any $j \in N_i$.

Remark 2: By Definition 3, a function is locally distributed for vehicle i if it can be computed using relative quantities expressed locally in the vehicle's body frame. Practically, such quantities could be measured with on-board devices like cameras. The only absolute measurement that vehicle i needs is its own body referenced angular velocity ω_i. This quantity is commonly considered available in literature [1], [2], in applications using an on-board rate gyroscope and therefore can be justified. In our notion of a locally distributed function, vehicle i is allowed to receive two real numbers $(u_j, \dot{u}_j)$ from vehicle j. That is, vehicle j computes its own control input u_j and its derivative $\dot{u}_j(R_j^\top x_{jk}, R_j^\top v_{jk}, R_{jk}, \omega_j, u_j, u_k)_{k \in N_j}$ where the quantity u_k is the control input of neighboring vehicle k and sends these quantities to vehicle i. The computation of a locally distributed function, therefore, relies on communication between vehicles.

If a vector function of $\hat{y}_i$ represented in body frame i is a locally distributed function for vehicle i, that is it can be computed locally using relative quantities, then the first two derivatives of this vector along the vector field of (1)-(2) represented in body frame i are also locally distributed for vehicle i, as shown in the next lemma.

Lemma 1: If a smooth function $f(\hat{y}_i)$, has the property that there exists a locally distributed smooth function $f^i(y_i)$ such that $R_i^\top f(\hat{y}_i) = f^i(y_i)$ then there exist locally distributed smooth functions $\dot{f}^i((y_i, R_{ij}, u_j)_{j \in N_i}), \ddot{f}^i((y_i, R_{ij}, \omega_i, \omega_j, u_j, \dot{u}_j)_{j \in N_i})$ such that $R_i^\top \dot{f}((\hat{y}_i, R_i, R_j, u_j)_{j \in N_i}) = \dot{f}^i((y_i, R_{ij}, u_j)_{j \in N_i})$ and $R_i^\top \ddot{f}((\hat{y}_i, R_i, R_j, \omega_i, \omega_j, u_j, \dot{u}_j)_{j \in N_i}) = \ddot{f}^i((y_i, R_{ij}, \omega_i, \omega_j, u_j, \dot{u}_j)_{j \in N_i})$, where time derivatives are calculated using equations (1)-(2).

The proof of Lemma 1 has been omitted due to space limitations.

We are now ready to present the problem investigated in this paper.

Rendezvous Control Problem (RCP)**:** Given system (1), (2), design locally distributed feedback functions (u_i, τ_i) such that the rendezvous manifold

$$\Gamma := \{\chi \in \mathcal{X} : x_{ij} = v_{ij} = 0, \tilde{\mathbf{q}}_i = e_3, \tilde{\boldsymbol{\omega}}_i = 0 \ \forall i, j\} \quad (3)$$

is AGAS for the the closed loop system. The goal of RCP is to synchronize the vehicle positions and velocities. The conditions that $\tilde{\mathbf{q}}_i = e_3$ and $\tilde{\boldsymbol{\omega}}_i = 0$ impose that the desired vehicle thrust direction and angular velocity are achieved.

V. Control Architecture

In this section we present a class of feedback controllers solving RCP. The proposed control has two nested feedback loops for each vehicle, shown in the block diagram of Figure 2. The outer loop treats the vehicle as a double integrator (1) with the thrust control input vector $T_i = T_d(\hat{y}_i)$. The feedback $T_d(\hat{y}_i)$ is a consensus control law for double integrators. The inner loop is a thrust direction controller for the rotational subsystem (2) which orients the thrust vector T_i of the vehicle to match the desired thrust $T_d(\hat{y}_i)$ assigned by the outer loop consensus controller using the vehicle torque input τ_i.

The approach just described is inspired by the two-stage control architecture used in [15] for position control. A similar control architecture is also used in the work of Tayebi and collaborators [19], [20], [21] devoted to position tracking of individual aerial vehicles.

We now present *classes* of consensus and thrust direction controllers that can be combined to solve RCP.

Definition 4 (Double Integrator Consensus Control Class): A smooth function $T_d(\hat{y}_i)$ is a controller of class CC, written $T_d \in$ CC, if $T_d(\hat{y}_i)$ satisfies the following properties:

1) $\sup \|T_d(\hat{y}_i)\| < \infty$ and $\|T_d(\hat{y}_i)\| > 0, \ \forall \hat{y}_i$;
2) When $T_i = T_d(\hat{y}_i)$ in (1), the set $\big\{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : x_{ij} = 0, v_{ij} = 0, \ i, j \in \{1, \ldots, n\}\big\}$ is GAS for the point-mass system (1);

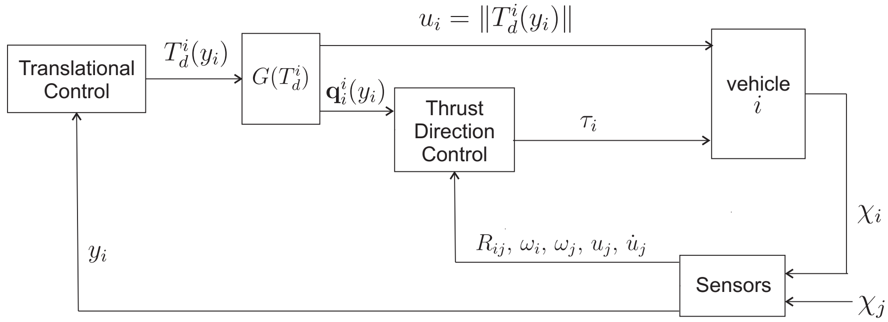

Fig. 2. Block diagram of the rendezvous control system for vehicle i. The outer loop assigns a desired thrust vector $T_d^i(y_i)$. The reference thrust direction $\mathbf{q}_i^i = T_d^i(y_i)/\|T_d^i(y_i)\|$ is then used by a thrust direction controller to assign the vehicle torque input τ_i.

3) There exists $\epsilon > 0$ such that for any piecewise continuous function $\rho_i : \mathbb{R} \to \mathbb{R}^3$ such that $\sup \|\rho_i\| < \epsilon$ and $\rho_i(t) \to 0$ as $t \to \infty$, letting $T_i = T_d(\hat{y}_i) + \rho_i(t)$, the solutions of relative states $(x_{ij}, v_{ij}),\ \forall i, j$ of the system (1) are bounded;
4) There exists a locally distributed, smooth function $T_d^i(y_i)$ such that $R_i^\top T_d(\hat{y}_i) = T_d^i(y_i)$.

Definition 5 (Thrust Direction Control Class): A smooth function $\tau_d : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is a feedback of class TDC, written $\tau_d \in \mathsf{TDC}$, if the system,

$$\dot{\Gamma}_i = \Gamma_i \times \omega_i,\ J\dot{\omega}_i = \tau_d(\Gamma_i, \omega_i) - \omega_i \times J\omega_i \tag{4}$$

has an AGAS equilibrium point $(\Gamma_i, \omega_i) = (e_3, 0)$.

Examples of feedback functions in the classes CC and TDC are presented at the end of this section. Controllers in the CC and TDC classes can be combined to define the rendezvous controller class.

Definition 6 (Rendezvous Controller Class): Let $T_d \in \mathsf{CC}$, $\tau_d \in \mathsf{TDC}$. The feedback (u_i, τ_i) of class RC is defined as,

$$\begin{aligned} &u_i = \|T_d^i(y_i)\|, \\ &\tau_i = \tau_d(\tilde{\mathbf{q}}_i, \tilde{\boldsymbol{\omega}}_i) - \tilde{\boldsymbol{\omega}}_i \times J\tilde{\boldsymbol{\omega}}_i + \omega_i \times J\omega_i + J\dot{\boldsymbol{\omega}}_i, \end{aligned} \tag{5}$$

where,

$$\begin{aligned} &\mathbf{q}_i(\hat{y}_i) := \frac{T_d(\hat{y}_i)}{\|T_d(\hat{y}_i)\|} \\ &\tilde{\mathbf{q}}_i(y_i) := R_i^\top \mathbf{q}_i = \frac{R_i^\top T_d(\hat{y}_i)}{\|R_i^\top T_d(\hat{y}_i)\|} = \frac{T_d^i(y_i)}{\|T_d^i(y_i)\|} \\ &\boldsymbol{\omega}_i((y_i, R_{ij}, u_j)_{j\in N_i}) := \mathbf{q}_i^i(y_i) \times \dot{\mathbf{q}}_i^i((y_i, R_{ij}, u_j)_{j\in N_i}) \\ &\tilde{\boldsymbol{\omega}}_i((y_i, R_{ij}, \omega_i, u_j)_{j\in N_i}) := \omega_i - \boldsymbol{\omega}_i, \end{aligned}$$

and $\dot{\mathbf{q}}_i$, $\dot{\boldsymbol{\omega}}_i$ are the time derivatives of the maps $\mathbf{q}_i, \boldsymbol{\omega}_i$ along (1), (2) computed setting $u_i = \|T_d^i(y_i)\|$.

Remark 3: The vector $\tilde{\mathbf{q}}_i$ is the error between the desired thrust axis $\mathbf{q}_i$ and the actual thrust axis $q_i = R_i e_3$ with respect to the reference vector e_3. In other words, $\tilde{\mathbf{q}}_i = e_3$ if and only if $q_i = \mathbf{q}_i$. The term $\tilde{\boldsymbol{\omega}}_i$ is the angular velocity error. In conclusion, the objective of the torque feedback controller τ_i in (5) is to drive $(\tilde{\mathbf{q}}_i, \tilde{\boldsymbol{\omega}}_i)$ to $(e_3, 0)$.

Now we will present the error dynamics. It immediately follows from the definition $\boldsymbol{\omega}_i := R_i^\top \mathbf{q}_i \times R_i^\top \dot{\mathbf{q}}_i$ that $R_i^\top \dot{\mathbf{q}}_i$ is perpendicular to $\boldsymbol{\omega}_i$. Further, since $\mathbf{q}_i$ is a unit vector, the derivative $\dot{\mathbf{q}}_i$ is perpendicular to it and therefore $R_i^\top \dot{\mathbf{q}}_i$ is perpendicular to $R_i^\top \mathbf{q}_i$. It follows that $R_i^\top \dot{\mathbf{q}}_i = \boldsymbol{\omega}_i \times R_i^\top \mathbf{q}_i$. Then taking the derivative of $\tilde{\mathbf{q}}_i$,

$$\begin{aligned} \dot{\tilde{\mathbf{q}}}_i &= -\omega_i \times R_i^\top \mathbf{q}_i + R_i^\top \dot{\mathbf{q}}_i \\ &= -\omega_i \times R_i^\top \mathbf{q}_i + \boldsymbol{\omega}_i \times R_i^\top \mathbf{q}_i \\ &= R_i^\top \mathbf{q}_i \times (\omega_i - \boldsymbol{\omega}_i) = \tilde{\mathbf{q}}_i \times \tilde{\boldsymbol{\omega}}_i. \end{aligned}$$

With τ_i from (5) we obtain error dynamics given by,

$$\dot{\tilde{\mathbf{q}}}_i = \tilde{\mathbf{q}}_i \times \tilde{\boldsymbol{\omega}}_i,\ J\dot{\tilde{\boldsymbol{\omega}}}_i = \tau_d(\tilde{\mathbf{q}}_i, \tilde{\boldsymbol{\omega}}_i) - \tilde{\boldsymbol{\omega}}_i \times J\tilde{\boldsymbol{\omega}}_i.$$

These dynamics have the same structure as the dynamics in (4). With $\tau = \tau_d \in \mathsf{TDC}$, we have that the equilibrium $(\tilde{\mathbf{q}}_i, \tilde{\boldsymbol{\omega}}_i) = (e_3, 0)$ is AGAS for the system above so that the thrust axis q_i tracks the reference $\mathbf{q}_i$. Therefore the torque feedback τ_i in (5) converts the thrust axis stabilizer τ_d into a thrust direction tracker.

Remark 4: Any feedback $(u_i, \tau_i) \in \mathsf{RC}$ is locally distributed for vehicle i. To see this, note that $\tilde{\mathbf{q}}_i = R_i^\top \mathbf{q}_i$ is locally distributed. By Lemma 1, $R_i^\top \dot{\mathbf{q}}_i$ is also locally distributed, so that $\boldsymbol{\omega}_i = R_i^\top \mathbf{q}_i \times R_i^\top \dot{\mathbf{q}}_i$ and hence $\tilde{\boldsymbol{\omega}}_i = \omega_i - \boldsymbol{\omega}_i$ are locally distributed as well. Finally, $\dot{\boldsymbol{\omega}}_i = \omega_i \times (R_i^\top \dot{\mathbf{q}}_i \times R_i^\top \mathbf{q}_i) + R_i^\top \mathbf{q}_i \times R_i^\top \ddot{\mathbf{q}}_i$ is locally distributed by the same arguments. In conclusion, τ_i is a locally distributed feedback for vehicle i.

Remark 5: On the rendezvous set in (3) we have that $\hat{y}_i = 0$ and therefore $\mathbf{q}_i(\hat{y}_i) = \mathbf{q}_i(0)$ is constant. It follows that $\dot{\mathbf{q}}_i = 0$ and that $\boldsymbol{\omega}_i = R_i^\top(\mathbf{q}_i \times \dot{\mathbf{q}}_i) = 0$. Therefore, on the rendezvous set, the angular velocity is driven to zero. The same is true in current approaches in literature [1], [2].

Theorem 2: Any feedback of class RC solves RCP for system (1), (2).

□

Proof: We have already shown in Remark 4 that any feedback of class RC is locally distributed for vehicle i. For analysis purposes, we consider the system under relative translational coordinates and absolute attitude coordinates, $(x_{ij}, v_{ij}, R_i, \omega_i)_{i,j\in\{1,\dots,n\}} \in \bar{\mathcal{X}} := \mathbb{R}^{3n^2} \times \mathbb{R}^{3n^2} \times \mathsf{SO}(3)^n \times \mathbb{R}^{3n}$ with the dynamics given by:

$$\begin{aligned} &\dot{x}_{ij} = v_{ij},\ \dot{v}_{ij} = R_i e_3 u_i - R_j e_3 u_j \\ &\dot{R}_i = R_i\, \omega_i^\times,\ J\dot{\omega}_i = \tau_i - \omega_i \times J\omega_i, \end{aligned} \tag{6}$$

Define the sets $\Gamma_1 = \{\bar{\chi} \in \bar{\mathcal{X}} : x_{ij} = 0, v_{ij} = 0, \tilde{\mathbf{q}}_i = e_3, \tilde{\boldsymbol{\omega}}_i = 0 \ \forall i, j\}$, $\Gamma_2 = \left\{\bar{\chi} \in \bar{\mathcal{X}} : \tilde{\mathbf{q}}_i = e_3, \tilde{\boldsymbol{\omega}}_i = 0 \ \forall i\right\}$. The set Γ_1 corresponds to the rendezvous set in (3) since on Γ_1, $\mathbf{q}_i = \mathbf{q}_j$ and $\boldsymbol{\omega}_i = \boldsymbol{\omega}_j = 0$ by Remark 5 and hence $q_{ij} = 0, \omega_{ij} = 0$ for $i, j \in \{1, 2, \ldots, n\}$. The set Γ_2 is the set where all vehicles have the desired thrust vector $T_d(\hat{y}_i)$ and the problem of rigid body rendezvous is reduced to one of double integrator consensus. Γ_1 is compact for system (6) since $x_{ij} = v_{ij} = 0$, R_i lies on a compact set and $\omega_i = \boldsymbol{\omega}_i = 0$. It is obvious that $\Gamma_1 \subset \Gamma_2$. By the choice of τ_i in Definition 6, Γ_2 is AGAS provided that the closed-loop system has no finite escape times. Assume for the moment that this is the case, and let D be the domain of attraction of Γ_2. Then, D is positively invariant, and Γ_2 is GAS relative to D. Now, for all $\bar{\chi} \in \Gamma_2$, $T_i = T_d(\hat{y}_i)$ and the motion on Γ_2 is governed by $\dot{x}_i = v_i$, $\dot{v}_i = mge_3 + T_d(\hat{y}_i)$ in the translational system (1). Since $T_d \in \mathsf{CC}$, the equilibrium Γ_1 is therefore GAS respect to Γ_2. Given that Γ_2 is GAS relative to D and Γ_1 is GAS relative to Γ_2, we now wish to apply the reduction Theorem 1. To this end, we need to show that all solutions of the closed-loop system (6) originating in D have no finite escape times (implying that Γ_2 is GAS relative to D) and they are bounded. Let $\bar{\chi}(0) \in \bar{\mathcal{X}}$ be arbitrary, and let $\bar{\chi}(t)$ be the corresponding solution of the closed-loop system. The dynamics of the translational subsystem can be written as $\ddot{x}_{ij} = -T_d(\hat{y}_i) + T_d(\hat{y}_j) - (-\|T_d(\hat{y}_i)\| R_i e_3 - T_d(\hat{y}_i)) + (-\|T_d(\hat{y}_j)\| R_j e_3 - T_d(\hat{y}_j))$. Here, $R_i e_3$, $R_j e_3$ have unit norm and, by assumption, $\sup \|T_d(\hat{y}_i)\|, \sup \|T_d(\hat{y}_j)\| < \infty$, so $\ddot{x}_{ij}$ is bounded for all i, j and so the signal $(x_{ij}(t), v_{ij}(t))_{i,j \in \{1,\ldots,n\}}$ is defined for all $t \geq 0$.

The function $\boldsymbol{\omega}_i$ depends indirectly on $(x_{ij}, v_{ij}, R_i, R_j, u_j)_{j \in N_i}$. Since the first two arguments are defined for all $t \geq 0$, $R_i, R_j \in \mathsf{SO}(3)$, a compact set, and u_j is bounded, $\boldsymbol{\omega}_i(x_{ij}, v_{ij}, R_i, R_j, u_j)_{j \in N_i}$ is defined for all $t \geq 0$. Since $(q_i, \omega_i) = (\mathbf{q}_i, \boldsymbol{\omega}_i)$ is AGAS, $\omega_i - \boldsymbol{\omega}_i$ is bounded, implying that $\omega_i(t)$ is defined for all $t \geq 0$. In conclusion, $\bar{\chi}(t)$ is defined for all $t \geq 0$, so that the closed-loop system has no finite escape times, and Γ_2 is GAS relative to D. Now consider the third property in Definition 4, and let $\rho_i(t) = -\|T_d(\hat{y}_i)\| R_i e_3 - T_d(\hat{y}_i)$. By the global asymptotic stability of Γ_2 relative to D, $\rho_i(t) \to 0$. Since $T_d \in \mathsf{CC}$, this implies that the translational system (1) is bounded by the third property in Definition 4. Since $\boldsymbol{\omega}_i$ is a continuous function of $(x_{ij}, v_{ij}, R_i, R_j, u_j)_{j \in N_i}$, and the signals $(x_{ij}(t), v_{ij}(t), R_i(t), R_j(t), u_j(t))_{j \in N_i}$ are bounded, $\boldsymbol{\omega}_i(t)$ is bounded. Finally, the boundedness of $\tilde{\boldsymbol{\omega}}_i(t)$ and that of $\boldsymbol{\omega}_i(t)$ imply that $\omega_i(t)$ is bounded. Having shown that all solutions of the closed-loop system originating in D are bounded, by Theorem 1 we conclude that Γ_1 is GAS relative to D, proving that the feedback (5) solves RCP. ∎

A. Sample Feedback of Class CC

We now present a feedback of class CC for the point-mass system

$$\dot{x}_i = v_i, \ m\dot{v}_i = mge_3 + T_i.$$

To meet the requirements in part (i) of Definition 4 one can use the nested saturation controller derived in [22] where the component-wise saturation is replaced with a magnitude saturation

$$\begin{aligned} T_d(\hat{y}_i) =& \alpha w + \sum_{j=1}^{n} a_{ij} \left\{ x_{ij} \frac{\tanh(\|x_{ij}\|)}{\|x_{ij}\|} \right. \\ & \left. + \frac{v_{ij}}{b} \frac{\tanh(\|v_{ij}\|)}{\|v_{ij}\|} \right\} \end{aligned} \tag{7}$$

where $\alpha, a_{ij} \geq 0$, $b > 1$ satisfying $2\sum_{j=1}^{n} a_{ij} < \|\alpha w\|$ $\forall i$ and $a_{ij} = a_{ji}$. The vector w is an inertial vector available to all vehicles in their body frames and chosen such that $T_d \neq 0$ for every $\hat{y}_i$. For example, this could be the gravity vector obtained using an on-board device such as an accelerometer or magnetometer. Alternatively in space applications, w could be an inertial vector measured using a star sensor.

Proposition 1: Assuming an undirected and strongly connected[1] graph $\mathcal{G}$, the feedback in (7) is of class CC.

Proof: For the feedback (7), we have $\|T_d(\hat{y}_i)\| \neq 0$ since $2\sum_{j=1}^{n} a_{ij} < \|\alpha w\|$. Since the feedback is saturated, we also satisfy $\sup \|T_d(\hat{y}_i)\| < \infty$. The proof of (ii) employs similar arguments as in the case of component-wise saturation of [22] and makes use of the assumptions on the graph. The proof of (iii) has been omitted due to space limitations. As the last property (iv) we note that

$$\begin{aligned} R_i^\top T_d(\hat{y}_i) =& \alpha R_i^\top w + \sum_{j=1}^{n} a_{ij} \left\{ R_i^\top x_{ij} \frac{\tanh(\|R_i^\top x_{ij}\|)}{\|R_i^\top x_{ij}\|} \right. \\ & \left. + R_i^\top \frac{v_{ij}}{b} \frac{\tanh(\|R_i^\top v_{ij}\|)}{\|R_i^\top v_{ij}\|} \right\} = T_d^i(y_i) \end{aligned}$$

is a locally distributed function if $R_i^\top w$ is available for vehicle i with $w \in \mathbb{R}^3$. ∎

B. Sample Feedback of Class TDC

Now we need to define a thrust direction controller τ_d that achieves almost global stabilization of $(\Gamma_i, \omega) = (e_3, 0)$. Our modular design allows one to pick from a multitude of designs. We pick the controller presented in [23],

$$\tau_d(\Gamma_i, \omega_i) = k_q \left(e_3 \times \Gamma_i\right) - K_\omega \omega_i, \tag{8}$$

where k_q is positive and K_ω is a symmetric and positive definite matrix.

Proposition 2: Feedback (8) belongs to class TDC

Proof: The fact that (8) is of class TDC is proved in Theorem 2 of [23]. ∎

VI. Simulation Results

In this section, we present simulation results for a feedback of class RC for a group of five vehicles with an undirected graph, with non-adjacency matrix entries $a_{12} = a_{21} = a_{23} = a_{32} = a_{34} = a_{43} = a_{45} = a_{54} = 1$ and initial conditions shown in Table I.

[1]A graph is called *strongly connected* if any two distinct nodes of the graph can be connected via a path.

TABLE I

SIMULATION INITIAL CONDITIONS

Vehicle	x_0 (m)	v_0 (m/s)	R_0	ω_0 (rad/s)
1	$(10, 0, 0)$	$(1, 0, 0)$	up	$(0, 0, 0)$
2	$(0, 10, 0)$	$(0, 1, 0)$	side	$(0, 0, 0)$
3	$(0, 0, 0)$	$(0, 0, 0)$	down	$(0, 0, 0)$
4	$(0, -10, 0)$	$(0, -1, 0)$	side	$(0, 0, 0)$
5	$(-10, 0, 0)$	$(-1, 0, 0)$	up	$(0, 0, 0)$

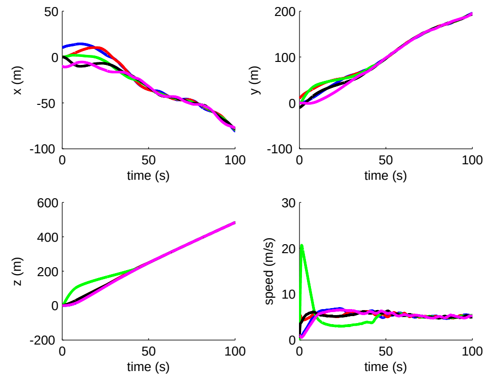

Fig. 3. Rendezvous control simulation in the presence of disturbances.

For the initial rotations R_0, up(right), side(ways) and (upside)down. The vehicle masses are chosen to be $m = 2\,\mathrm{Kg}$. The inertia matrix J is diagonal, with diagonal elements equal to $1.2416\,\mathrm{Kg{\cdot}m^2}$ as in [18]. All simulations use $T_d \in \mathsf{CC}$ given in (7).

The gains in (8) are chosen as $k_q = 50$, $K_\omega = 10I$. Figure 3 shows the simulation when disturbances are present. The disturbance includes an additive random noise on the applied force and torque with maximum magnitude of $0.5\,\mathrm{N}$ and $0.5\,\mathrm{N{\cdot}m}$, respectively. We also include random additive noise on the attitude (yaw, pitch and roll) and on the angular velocity measurements with maximum magnitudes of $0.5\,\mathrm{rad}$ and $0.5\,\mathrm{rad/s}$, respectively, and include a constant bias of $0.1\,\mathrm{rad/s}$ added to the measurement of angular velocity. The translational plots show the vehicle positions on the inertial frame x, y, z axes, and the linear speed given by $\sqrt{v_1^2 + v_2^2 + v_3^2}$.

In Figure 3, the vehicles converge to a small neighbourhood of one another and have a steady state speed of about $5m/s$.

VII. CONCLUSIONS

We have presented a class of feedbacks and a control architecture solving the rendezvous control problem for underactuated thrust-propelled vehicles. The control is static and yields almost global asymptotic stability of the rendezvous configuration. The final control assumes each vehicle can measure only relative quantities, its own angular velocity and its neighbors' thrust inputs and their derivatives. We have not explicitly addressed issues of robustness against unmodelled uncertainties and measurement errors but simulation results were presented that include the effect of disturbances on the force and torque inputs.

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