A Smooth Distributed Feedback for Formation Control of Unicycles

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***Abstract*— We present a local and distributed control law with connected, undirected sensor graph to drive a group of kinematic unicycles to a desired line formation with aligned heading directions, and assuming initial heading angles lie in a common half-plane. We present a controller solving the line formation control problem, combining a bounded translational consensus controller with an attitude synchronizer. We present simulation results that suggest that the proposed approach also works in the case of more general formations.**

I. INTRODUCTION

In this paper, we present a solution to the formation control problem for kinematic unicycles with connected, undirected sensor graphs. The objective of the formation control problem is to drive a group of unicycles from any initial configuration, with headings lying in a common half-plane, to a desired formation. In our notion of formation, in steady state, agents form a fixed geometric pattern in which the fixed inter-agent displacements are defined a priori modulo roto-translations. The type of formations considered in this paper are limited to lines, with prescribed spacing of the robots. One important requirement is that the feedbacks are local and distributed, which means that the feedback of any unicycle depends only on relative displacements and angles between itself and its neighbors, quantities that can be measured on-board each robot using cameras. The graph is assumed to be static and undirected, and the feedbacks are time invariant and do not rely on inertial measurements or communicated information from neighboring robots. In this paper we present a new approach to address the problem of formation control in which we attach to each unicycle body frame an offset vector. The problem of stabilizing the desired formation is reduced to that of achieving consensus on these offset vectors and the unicycle attitudes. Although we only consider line formations, we present simulation results that suggest that the proposed approach also works in the case of more general formations. Proving stability of non-line formations is the subject of ongoing research. To the best of our knowledge there is no result in the literature that solves the line formation control problem for connected, undirected graphs with static, local and distributed, time invariant feedbacks without the requirement of communication.

Many authors have studied the relative equilibria that arise for a system of kinematic unicycles under local and distributed feedbacks. In [1], [2], the authors use a Lie group formulation to show that the possible relative equilibria are characterized by either parallel motion or motion on a common circle. In [3], the authors show that the relative equilibria for three dimensional kinematic vehicles correspond to parallel motion or motion along circles or helices, and in [4], feedbacks are presented to stabilize each one of these relative equilibria. The authors do this by imposing consensus on the so-called twist vectors for each robot but the sensing graph is assumed to be all-to-all. To avoid all-to-all sensing, the authors present a solution using dynamic variables, communicated between neighboring robots.

Much of the literature on formation control considers single or double-integrator models or fully-actuated unicycles. The dominant approach for single-integrator formation control is distance-based [5], [6], [7], where a desired scalar distance between neighboring robots specifies the formation for infinitesimally rigid graphs. Other approaches define formations in terms of relative angles between neighboring robots instead of distances, [8], [9], or in terms of a complex Laplacian, [10], [11], [12]. Formation flocking of double-integrators is considered in [13], [14], while formation control of fully-actuated unicycles is considered in [15], [16], [17].

Most relevant to this paper is the literature on kinematic unicycles. The paper [18] discusses feasibility conditions to achieve various formations of kinematic unicycles including necessary and sufficient conditions for stabilizing a line formation. Time-dependent solutions are presented in each case. In [19], the authors consider formations of two agents where the follower robot is controlled to stay at a desired distance from the leader with no specification on their relative angles. The result yields global convergence for graphs containing a directed spanning tree. In [20], a group of robots is considered in which at least one follower robot can see a leader that follows a desired path. The control presented attains the desired formation about the leader in finite time. However, the formation is not rotation invariant and the control is not local and distributed. Also in [21], a follower-leader approach is considered. The analysis transforms the unicycle model into a system of double integrators through a dynamic feedback linearization. The desired formation is attained for graphs containing a spanning tree but each follower robot requires access to the acceleration of the leader through communication. In [22] each unicycle estimates its own position using a dynamic extension with communication. The unicycles use these estimated states to attain the desired formation globally. The rotational control however is time-dependent and oscillatory. Finally, a result for dynamic unicycles is found in [23], driving them to a

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common circle with desired spacing.

II. Line Formation Control Problem

The notation and unicycle model in this paper are taken from [24]. We use interchangeably the notation $v = [v_1 \ \cdots \ v_n]^\top$ or $(v_1, \ldots, v_n)$ for a column vector in $\mathbb{R}^n$. Let $\{e_1, e_2\}$ denote the natural basis of $\mathbb{R}^2$, $\mathsf{SO}(2) := \{M \in \mathbb{R}^{2\times 2} : M^{-1} = M^\top, \det(M) = 1\}$ and let $\mathbb{S}^1$ denote the unit circle. If $I = \{i_1, \ldots, i_n\}$ is an index set, the ordered list of elements $(x_{i_1}, \ldots, x_{i_n})$ is denoted by $(x_j)_{j\in I}$.

Consider a group of n kinematic unicycles. Let $\mathcal{I} = \{i_x, i_y\}$ be an inertial frame in two-dimensional space and consider the i-th unicycle. Fix a body frame $\mathcal{B}_i = \{b_{ix}, b_{iy}\}$ to the unicycle, where b_{ix} is the heading axis, and denote by $x_i \in \mathbb{R}^2$ the position of the unicycle in the coordinates of frame $\mathcal{I}$. The unicycle's attitude is represented by a rotation matrix R_i whose columns are the coordinate representations of b_{ix} and b_{iy} in frame $\mathcal{I}$. Letting $\theta_i \in \mathbb{S}^1$ be the angle between vectors i_x and b_{ix}, we have

$$R_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}.$$

The angular speed of robot i is denoted by ω_i. The equations of motion for the unicycles is as follows,

$$\dot{x}_i = u_i R_i e_1 \tag{1}$$

$$\dot{\theta}_i = \omega_i, \quad i = 1, \ldots, n. \tag{2}$$

Its control inputs are the linear speed u_i and angular speed ω_i. Let $x := (x_i)_{i\in\{1,\ldots,n\}}$ and $\theta := (\theta_i)_{i\in\{1,\ldots,n\}}$. The relative displacement of robot j with respect to robot i is $x_{ij} := x_j - x_i$ while the relative angles are given by $\theta_{ij} = \theta_j - \theta_i$. The frame of robot j with respect to frame i is defined by $R^i_j := R_i^\top R_j$ which is a function of θ_{ij}. If $v \in \mathbb{R}^2$ is the coordinate representation of a vector in frame $\mathcal{I}$, then we denote by $v^i := R_i^{-1} v$ the coordinate representation of v in body frame $\mathcal{B}_i$.

We define the *undirected sensor graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where each node in the node set $\mathcal{V}$ represents a robot, and an edge in the edge set $\mathcal{E}$ between node i and node j indicates that robot i can sense robot j and vice versa. We assume that $\mathcal{G}$ has no self-loops and is time-invariant. Given a node i, its set of neighbors $\mathcal{N}_i$ represents the set of vehicles that robot i can sense. If $j \in \mathcal{N}_i$, then we say that robot j is a *neighbour* of robot i. If this is the case, then robot i can sense the relative displacement of robot j in its own body frame, i.e., the quantity x^i_{ij} as well as the relative angle θ_{ij} between their body frames. Define the vector $y_i := (x_{ij}, \theta_{ij})_{j\in\mathcal{N}_i}$. The relative displacements and angles available to robot i are contained in the vector $y^i_i := (x^i_{ij}, \theta_{ij})_{j\in\mathcal{N}_i}$. A *local and distributed feedback* (u_i, ω_i) for robot i is a locally Lipschitz function of y^i_i.

The objective of the formation control problem considered in this paper is to develop local and distributed feedbacks to drive a group of unicycles, each modelled by (1), (2), to a desired formation. In our notion of formation, in steady state, agents form a fixed geometric pattern in which the fixed inter-agent displacements are defined a priori modulo roto-translations. A *parallel formation* is a configuration in which all unicycles have the same attitude with respect to the inertial frame $\mathcal{I}$, i.e., $\mathcal{B}_i = \mathcal{B}_1$, $\forall i \in \{2, \ldots, n\}$, and the relative displacement vector d_{i1} between unicycle 1 and unicycle $i \in \{2, \ldots, n\}$ is fixed with respect to this common body frame. A parallel formation can therefore be uniquely defined with respect to unicycle 1 by a constant vector $r := (d^1_{1i})_{i\in\{2,\ldots,n\}} \in \mathbb{R}^{2(n-1)}$. Notice that by this definition, a parallel formation is invariant under translations or rotations of the formation. An example of a parallel formation is illustrated in Figure 1(a). The type of parallel formations we will consider in this paper are limited to lines, along which the spacing of the robots can be specified. In particular, a *parallel line formation* is a parallel formation $r \in \mathbb{R}^{2(n-1)}$ in which $d^1_{1i} \cdot e_1 = 0$ for all $i \in \{2, \ldots, n\}$. An example of a parallel line formation is illustrated in Figure 1(b).

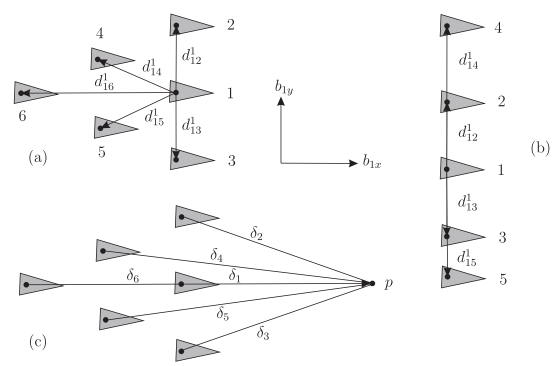

Fig. 1: Parallel formation represented in terms of: (a) relative displacement vectors d^1_{i1} represented in frame $\mathcal{B}_1$ and (c) offset vectors δ_i to a common point p in front of all unicycles. (b) shows an example of a parallel line formation

For a given parallel line formation $r \in \mathbb{R}^{2(n-1)}$, we define the *line formation manifold* as,

$$\begin{aligned}\Gamma(r) := \big\{ &(x_i, \theta_i)_{i\in\{1,\ldots,n\}} \in \mathbb{R}^{2n} \times \mathbb{T}^n : \\ &x_i = x_1 + d_{1i}, \ \theta_{1i} = 0, \ \forall\, i \in \{2, \ldots, n\} \big\}.\end{aligned} \tag{3}$$

Let $S_\sigma := \big\{ (\theta_i)_{i\in\{1,\ldots,n\}} \in \mathbb{T}^n : |\theta_{ij}| < \sigma, \ \forall\, i, j \big\}$, where $\sigma \in (0, \pi]$, be the set where all vehicle angles lie in the same half plane on an open arc of σ radians. We are now ready to define the line formation control problem in which we stabilize any parallel line formation from any initial condition where the unicycle headings lie in a common half plane.

Line Formation Control Problem: Consider system (1)-(2) with connected, undirected sensor graph $\mathcal{G}$. For any parallel line formation $r \in \mathbb{R}^{2(n-1)}$ and any $\sigma \in (0, \pi)$, find local and distributed feedbacks $(u_i, \omega_i)_{i\in\{1,\ldots,n\}}$ that asymptotically stabilize the line formation manifold $\Gamma(r)$ with domain of attraction containing $\mathbb{R}^{2n} \times S_\sigma$. △

Although in this paper we consider parallel line formations, we believe our approach works for general parallel formations, as is suggested by the simulation results presented at the end of this paper.

III. Solution of the Line Formation Control Problem

In this section we present a solution to the line formation control problem defined in the previous section. From now on we will refer to a parallel formation simply as a formation with the understanding that the angles are aligned. We can represent a formation in an alternative way from that discussed in the previous section. Consider any point p that lies in front of the desired formation as shown in Figure 1(c). We add to each unicycle an offset (or lookahead) vector $\delta_i := \alpha_i R_i e_1 + \beta_i R_i e_2$ in its own body frame as the vector $p - x_i$, which starts from unicycle i and ends at the lookahead point p. In this set-up, $\alpha_i > 0$ is the component parallel to the heading axis of unicycle i and $\beta_i \in \mathbb{R}$ lies along the perpendicular axis. Notice that when the unicycles are in formation, $\delta_i = \alpha_i R_i e_1 + \beta_i R_i e_2 = \alpha_i R_1 e_1 + \beta_i R_1 e_2$ since the orientations of all the frames are equal. When the unicycles have not yet reached formation, the offset vectors attached to each body will not coincide, and as unicycle i moves and rotates, the vector δ_i fixed to its body frame will correspondingly move as well. Its endpoint represented in inertial frame $\mathcal{I}$ is denoted $\hat{x}_i := x_i + \delta_i$. The desired formation in Figure 1(c) is achieved when $\theta_{ij} = 0$ and $\hat{x}_{ij} = \hat{x}_j - \hat{x}_i = 0$ for all $i, j \in \{1, \ldots, n\}$. That is, the problem of formation stabilization has been reduced to that of consensus of the quantities $(\hat{x}_i, \theta_i)_{i \in \{1,\ldots,n\}}$. Recall that the type of formations we are considering in this paper are limited to lines, along which the spacing of the robots can be specified. In particular, these are formations in which $\alpha_i = \alpha_j =: \alpha$ for all $i, j \in \{1, \ldots, n\}$, while there is no restriction on $(\beta_i)_{i \in \{1,\ldots,n\}}$. In an initial design stage, one must design an appropriate set of offset vectors $(\delta_i)_{i \in \{1,\ldots,n\}}$ corresponding to a desired parallel line formation $r \in \mathbb{R}^{2(n-1)}$. The offset vectors $(\delta_i)_{i \in \{1,\ldots,n\}}$ are not unique (for example, α can be any value greater than zero). Unicycle i must store the quantities $(\beta_j)_{j \in \mathcal{N}_i}$ of its neighbors in memory.

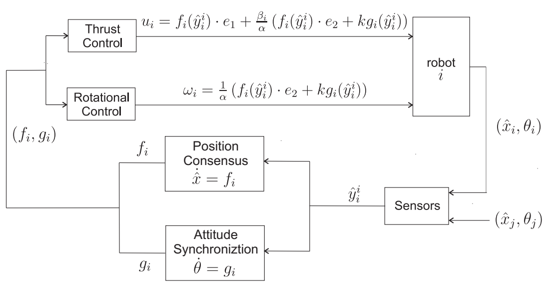

Fig. 2: Block diagram of the formation control system for robot i.

Now we will present a class of feedbacks to stabilize the desired line formation. We will follow a three-step procedure. First we define a consensus controller for single-integrators with the property of boundedness. We next define a rotational control law to synchronize angles. Finally, we combine these two controllers to design the unicycle control inputs u_i and ω_i. Define the vectors $\hat{y}_i := (\hat{x}_{ij}, \theta_{ij})_{j \in \mathcal{N}_i}$ and $\hat{y}_i^i := (\hat{x}_{ij}^i, \theta_{ij})_{j \in \mathcal{N}_i}$. Consider the system of single integrators,

$$\dot{\hat{x}}_i = f_i(\hat{y}_i), \; i = 1, \ldots, n \tag{4}$$

where $f_i(\hat{y}_i)$ is a *bounded integrator consensus controller* of the form,

$$f_i(\hat{y}_i) = \sum_{j \in \mathcal{N}_i} a_{ij} \frac{f(\|\hat{x}_{ij}\|)}{\|\hat{x}_{ij}\|} \hat{x}_{ij} \tag{5}$$

with $a_{ij} = a_{ji} > 0$ and Lipschitz continuous interaction function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following assumptions,

A1: $sf(s) > 0$ when $s \neq 0$ and $f(0) = 0$.

A2: $\sup |f(s)| < \infty$.

Note that the component $(\hat{x}_{ij}/\|\hat{x}_{ij}\|) f(\|\hat{x}_{ij}\|)$ of feedback (5) is well-defined at $\hat{x}_{ij} = 0$. Its value will be zero since $f(\|\hat{x}_{ij}\|) = 0$ when $\hat{x}_{ij} = 0$ by assumption A1. It is easy to show that $f_i(\hat{y}_i)$ is Lipschitz continuous for all $i \in \{1, \ldots, n\}$. As an example of a suitable bounded interaction function, one could use $f(s) = \tanh(s)$.

Proposition 1: Consider system (4) with feedback (5) and connected, undirected sensor graph $\mathcal{G}$. The consensus set $\{\hat{x} \in \mathbb{R}^{2n} : \hat{x}_i = \hat{x}_j, \; \forall i, j \in \{1, \ldots, n\}\}$ is globally asymptotically stable.

Proof: The input f_i in (5) for unicycle i points into the convex hull formed by its neighbours. By [25] the group of unicycles for system (4) achieves global consensus. ∎

Proposition 2: System (4) with feedback (5) is gradient with the following positive definite storage function,

$$V_t = \frac{1}{2} \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i} a_{ij} \int_0^{\|\hat{x}_{ij}\|} f(s) \mathrm{d}s, \tag{6}$$

assuming the sensor graph $\mathcal{G}$ is connected and undirected.

Proof: It follows from assumption A1 that $\int_0^{\|\hat{x}_{ij}\|} f(s) \mathrm{d}s$ attains a minimum when $\hat{x}_{ij} = 0$. Therefore V_t attains a minimum when $\hat{x}_{ij} = 0$ for all $i, j \in \{1, \ldots, n\}$ and therefore V_t is positive definite. To show V_t is a valid storage function, it needs to be shown that $(\partial / \partial \hat{x}_i) V_t = -f_i^\top$. The proof has been omitted due to space limitations. ∎

Proposition 3: Consider system (4) with feedback (5) and connected, undirected sensor graph $\mathcal{G}$. The following two properties hold,

(i) $R_i^\top f_i(\hat{y}_i) = f_i(\hat{y}_i^i)$ for all $i \in \{1, \ldots, n\}$.

(ii) $\{\hat{x} \in \mathbb{R}^{2n} : f_i = 0, \; \forall i \in \{1, \ldots, n\}\} = \{\hat{x} \in \mathbb{R}^{2n} : \hat{x}_i = \hat{x}_j, \; \forall i, j \in \{1, \ldots, n\}\}$.

Proof: To show (i) we use the fact that $\|\hat{x}_{ij}\| = \|\hat{x}_{ij}^i\|$. Then,

$$R_i^\top f_i(\hat{x}_{ij}) = \sum_{j \in \mathcal{N}_i} a_{ij} \frac{f(\|\hat{x}_{ij}^i\|)}{\|\hat{x}_{ij}^i\|} R_i^\top \hat{x}_{ij} = f_i(\hat{y}_i^i).$$

To show (ii), assume $f_i = 0$ for all $i \in \{1, \ldots, n\}$. Then system (4) is at a fixed point. But by Proposition 1, the set $\{\hat{x} \in \mathbb{R}^{2n} : \hat{x}_i = \hat{x}_j, \; \forall i, j \in \{1, \ldots, n\}\}$ is globally asymptotically stable and therefore contains all fixed points. Conversely, if $\hat{x}_{ij} = 0$ for all $i \in \{1, \ldots, n\}$ then it follows by definition that $f_i = 0$ for all $i \in \{1, \ldots, n\}$. ∎

Analogous to the translational case, consider the system of unicycle heading angles,

$$\dot{\theta}_i = g_i(\hat{y}_i),\ i = 1, \dots, n \tag{7}$$

where $g_i(\hat{y}_i)$ is an *attitude synchronizer* of the form,

$$g_i(\hat{y}_i) = g_i(\hat{y}_i^i) = \sum_{j\in\mathcal{N}_i} b_{ij}\sin(\theta_{ij}) \tag{8}$$

with $b_{ij} = b_{ji} > 0$. This is the well-known Kuramoto model for attitude synchronization of angles on $\mathbb{S}^1$ [26] with zero natural frequencies.

Proposition 4 ([25]): Consider system (7) with feedback (8) and connected, undirected sensor graph $\mathcal{G}$. the set $\{\theta \in \mathbb{T}^n : \theta_i = \theta_j,\ \forall i, j \in \{1, \dots, n\}\}$ is asymptotically stable with domain of attraction containing S_π.

Proposition 5: System (7) with feedback (8) is gradient with the following positive definite storage function,

$$V_r = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} b_{ij}(1-\cos(\theta_{ij})), \tag{9}$$

assuming the sensor graph $\mathcal{G}$ is connected and undirected.

We combine $f_i(\hat{y}_i^i)$ and $g_i(\hat{y}_i^i)$ to construct the following feedbacks,

$$\begin{aligned} u_i &= f_i(\hat{y}_i^i)\cdot e_1 + \beta_i\omega_i, \\ \omega_i &= \frac{1}{\alpha}\left(f_i(\hat{y}_i^i)\cdot e_2 + k g_i(\hat{y}_i^i)\right),\ i = 1, \dots, n, \end{aligned} \tag{10}$$

where $k > 0$ scales the contribution of the attitude synchronizer $g_i(\hat{y}_i^i)$. The feedbacks in (10) depend on the new translational quantities $(\hat{x}_i)_{i\in\{1,\dots,n\}}$.

Remark 1: For unicycle i, the feedback in (10) requires the formation parameters $(\beta_j)_{j\in\mathcal{N}_i}$ of its neighbors be stored in memory. The unicycle feedbacks are not identical and the final formation is not invariant to a relabelling of the agents. The choice of α has no effect on the final formation but will affect the transient behaviour of the system.

The feedbacks in (10) are local and distributed. This follows because u_i and ω_i are functions of the vector $\hat{y}_i^i = (\hat{x}_{ij}^i, \theta_{ij})_{j\in\mathcal{N}_i}$ in which,

$$\begin{aligned} \hat{x}_{ij}^i &= R_i^\top(x_j + \alpha R_j e_1 + \beta_j R_j e_2 - x_i - \alpha R_i e_1 - \beta_i R_i e_2) \\ &= x_{ij}^i + \alpha R_j^i e_1 + \beta_j R_j^i e_2 - \alpha e_1 - \beta_i e_2, \end{aligned}$$

is a function of y_i^i and $(\beta_j)_{j\in\mathcal{N}_i}$ and therefore so is $\hat{y}_i^i$.

The block diagram in Figure 2 summarizes the design of feedbacks $(u_i, \omega_i)_{i\in\{1,\dots,n\}}$. From its sensors, unicycle i obtains the vector $\hat{y}_i^i$ of measurable quantities relative to its neighbors. These are the relative displacements $\hat{x}_{ij}$ between the end points of the offset vectors fixed to each body frame and relative angles θ_{ij}. The relative quantities can be measured locally in unicycle i's body frame using, for example, on-board cameras. The position consensus block assigns the feedback f_i as a bounded integrator consensus controller in (5) satisfying assumptions A1 and A2 and the attitude synchronization block assigns the feedback g_i as the Kuramoto model in (8). The thrust control and rotational control blocks then combine the feedbacks f_i and g_i to design the unicycle control inputs u_i and ω_i as in (10). The result below states that for sufficiently large k, the feedbacks in (10) solve the line formation control problem.

Theorem 1: Consider a collection of n unicycles satisfying (1), (2) with connected, undirected sensor graph $\mathcal{G}$, a parallel line formation $r \in \mathbb{R}^{2(n-1)}$ and a corresponding set of offset vectors $(\delta_i)_{i\in\{1,\dots,n\}}$. Consider any translational interaction function $f(s)$ satisfying assumptions A1 and A2 and any parameters $a_{ij} = a_{ji} > 0$, $b_{ij} = b_{ji} > 0$ for $i \in \{1, \dots, n\}, j \in \mathcal{N}_i$. Then for any $\sigma \in (0, \pi)$, there exists $k^\star > 0$ such that for all $k > k^\star$ feedback (10) asymptotically stabilizes the formation manifold $\Gamma(r)$ with domain of attraction containing $\mathbb{R}^{2n} \times S_\sigma$.

Proof: Take an arbitrary connected, undirected graph $\mathcal{G}$, desired parallel line formation $r \in \mathbb{R}^{2(n-1)}$ and $\sigma \in (0, \pi)$. Define a set of offset vectors $(\delta_i)_{i\in\{1,\dots,n\}}$ using the procedure outlined in Section III. This yields the new translational quantities $(\hat{x}_i)_{i\in\{1,\dots,n\}}$. The set of quantities $(\hat{x}_i, \theta_i)_{i\in\{1,\dots,n\}}$ can be treated as new states. Computing the time derivative of $\hat{x}_i$ yields,

$$\begin{aligned} \dot{\hat{x}}_i &= u_i R_i e_1 + R_i \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix}(\alpha_i e_1 + \beta_i e_2) \\ &= u_i R_i e_1 + \alpha\omega_i R_i e_2 - \beta_i\omega_i R_i e_1 \\ &= (u_i - \beta_i\omega_i)R_i e_1 + \alpha\omega_i R_i e_2, \end{aligned}$$

and therefore the equations of motion of the $(\hat{x}_i, \theta_i)_{i\in\{1,\dots,n\}}$ states are given by,

$$\begin{aligned} \dot{\hat{x}}_i &= (u_i - \beta_i\omega_i)R_i e_1 + \alpha\omega_i R_i e_2 \\ \dot{\theta}_i &= \omega_i, \quad i = 1, \dots, n. \end{aligned} \tag{11}$$

Using Proposition 3(i) and the property that $g_i(\hat{y}_i^i) = g_i(\hat{y}_i)$ the feedbacks in (10) can equivalently be represented with inertial quantities $(\hat{y}_i)_{i\in\{1,\dots,n\}}$ as,

$$\begin{aligned} u_i &= f_i(\hat{y}_i)\cdot R_i e_1 + \beta_i\omega_i, \\ \omega_i &= \frac{1}{\alpha}\left(f_i(\hat{y}_i)\cdot R_i e_2 + k g_i(\hat{y}_i)\right),\ i = 1, \dots, n. \end{aligned} \tag{12}$$

Substituting (12) into (11) and using the fact that $f_i(\hat{y}_i) = (f(\hat{y}_i)\cdot R_i e_1)R_i e_1 + (f(\hat{y}_i)\cdot R_i e_2)R_i e_2$ for all $i \in \{1, \dots, n\}$ yields the closed-loop system,

$$\begin{aligned} \dot{\hat{x}}_i &= f_i(\hat{y}_i) + k g_i(\hat{y}_i)R_i e_2 \\ \dot{\theta}_i &= \frac{1}{\alpha}\left(f_i(\hat{y}_i)\cdot R_i e_2 + k g_i(\hat{y}_i)\right), \quad i = 1, \dots, n. \end{aligned} \tag{13}$$

Now we'll re-define the line formation manifold with respect to the new coordinates as,

$$\begin{aligned} \hat{\Gamma}(r) := \{&(\hat{x}_i, \theta_i)_{i\in\{1,\dots,n\}} \in \mathbb{R}^{2n}\times\mathbb{T}^n : \\ &\hat{x}_{ij} = 0,\ \theta_{ij} = 0,\ \forall\, i, j\}. \end{aligned} \tag{14}$$

On this set the desired formation r is achieved. To prove Theorem 1 we need to show that $\hat{\Gamma}(r)$ is asymptotically stable with domain of attraction containing $(\hat{x}_i, \theta_i)_{i\in\{1,\dots,n\}} \in \mathbb{R}^{2n} \times S_\sigma$ using the feedbacks in (12).

From Proposition 2 and Proposition 5 systems (4), (7) are gradient with positive definite storage functions $V_t(\hat{x})$ and $V_r(\theta)$ respectively. Combining these, we produce a positive definite Lyapunov function for the closed-loop system (13) given by $V = V_t + k\alpha V_r$. Using (13), the time derivative of V_t is given by,

$$\begin{aligned}\dot{V}_t &= \sum_{i=1}^{n} \frac{\partial}{\partial \hat{x}_i} V_t \cdot \dot{\hat{x}}_i = \sum_{i=1}^{n} -f_i \cdot (f_i + k g_i R_i e_2) \\ &= \sum_{i=1}^{n} \left(-\|f_i\|^2 - (f_i \cdot R_i e_2) k g_i \right)\end{aligned} \tag{15}$$

where the arguments of $f_i(\hat{y}_i)$ have been dropped for notational convenience. Analogously, the derivative of V_r is given by,

$$\begin{aligned}\dot{V}_r &= \sum_{i=1}^{n} \frac{\partial}{\partial \theta_i} V_r \cdot \dot{\theta}_i = \frac{1}{\alpha} \sum_{i=1}^{n} -g_i \cdot (f_i \cdot R_i e_2 + k g_i) \\ &= \frac{1}{\alpha} \sum_{i=1}^{n} \left(-(f_i \cdot R_i e_2) g_i - k g_i^2 \right).\end{aligned} \tag{16}$$

Combining (15) and (16), the time derivative of V can be computed as follows,

$$\begin{aligned}\dot{V} &= \dot{V}_t + k\alpha \dot{V}_r \\ &= \sum_{i=1}^{n} \left(-\|f_i\|^2 - 2(f_i \cdot R_i e_2)(k g_i) - (k g_i)^2 \right) \\ &= \sum_{i=1}^{n} \left(-\|f_i \cdot R_i e_1\|^2 - \|f_i \cdot R_i e_2 + k g_i\|^2 \right) \leq 0.\end{aligned}$$

Therefore $\dot{V}$ is less than or equal to zero with equality if and only if $f_i \cdot R_i e_1 = 0$ and $f_i \cdot R_i e_2 = -k g_i$ for all $i \in \{1, \ldots, n\}$. Together these conditions imply that points in the zero level set of $\dot{V}$, denoted E, satisfy,

$$f_i = -k g_i R_i e_2, \ \forall i \in \{1, \ldots, n\}. \tag{17}$$

Using LaSalle's invariance principle, since the sublevel sets of V are compact in relative coordinates $(\hat{x}_{1i}, \theta_{1i})_{i \in \{2,\ldots,n\}}$, the closed-loop solutions converge to the largest invariant set contained in E which we now characterize on the set $\mathbb{R}^{2n} \times S_\pi$, i.e., the set where all vehicle headings lie in the same half plane. By the definition of (5), it holds that $\sum_{i=1}^{n} f_i = 0$ and therefore, $-\sum_{i=1}^{n} g_i R_i e_2 = -\sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i} b_{ij} \sin(\theta_{ij}) R_i e_2 = 0$. For each edge $(i,j) \in \mathcal{E}$ there are two corresponding components in the previous summation, $b_{ij} \sin(\theta_{ij}) R_i e_2$ and $b_{ji} \sin(\theta_{ji}) R_j e_2$. This implies that,

$$\begin{aligned}-\sum_{i=1}^{n} g_i R_i e_2 &= -\sum_{(i,j) \in \mathcal{E}} b_{ij} \left(\sin(\theta_{ij}) R_i e_2 + \sin(\theta_{ji}) R_j e_2 \right) \\ &= \sum_{(i,j) \in \mathcal{E}} b_{ij} \sin(\theta_{ij}) \left(R_j e_2 - R_i e_2 \right) = 0,\end{aligned} \tag{18}$$

and it can be shown that this implies,

$$\sum_{(i,j) \in \mathcal{E}} 2 b_{ij} \sin(\theta_{ij}) \sin\left(\frac{\theta_j - \theta_i}{2} \right) \begin{bmatrix} -\cos\left(\frac{\theta_i + \theta_j}{2} \right) \\ \sin\left(\frac{\theta_i + \theta_j}{2} \right) \end{bmatrix} = 0. \tag{19}$$

Suppose all unicycle headings lie in a common half plane, i.e., $(\hat{x}_i, \theta_i)_{i \in \{1,\ldots,n\}} \in \mathbb{R}^{2n} \times S_\pi$. Then without loss of generality, we can assume that $\theta_i \in (-\pi/2, \pi/2)$ for all $i \in \{1, \ldots, n\}$. Both $\sin(\theta_{ij})$ and $\sin(\theta_{ij}/2)$ are less than zero for $\theta_{ij} \in (-\pi, 0)$, equal to zero for $\theta_{ij} = 0$ and greater than or equal to zero for $\theta_{ij} \in (0, \pi)$. Therefore $2 b_{ij} \sin(\theta_{ij}) \sin(\theta_{ij}/2) \geq 0$ for $\theta_{ij} \in (-\pi, \pi)$. On the other hand, the quantity $-\cos\left(\frac{\theta_i + \theta_j}{2} \right)$ is less than zero on the range $\theta_i + \theta_j \in (-\pi, \pi)$. Since by choice $\theta_i \in (-\pi/2, \pi/2)$, it holds that $\theta_{ij} \in (-\pi, \pi)$ and $\theta_i + \theta_j \in (-\pi, \pi)$ for all $i, j \in \{1, \ldots, n\}$ and therefore the first vector component in (19) is less than or equal to zero for all summands. Therefore, the only way the first vector component in (19) is zero is if $\theta_{ij} = 0$ for all $i, j \in \{1, \ldots, n\}$ which implies that $g_i = 0$ for all $i \in \{1, \ldots, n\}$ on the set E. By (17) this implies that $f_i = 0$ for all $i \in \{1, \ldots, n\}$ on E and therefore, by Proposition 3(ii), $\hat{x}_i = \hat{x}_j$ for all $i, j \in \{1, \ldots, n\}$. It follows that $E \subset \hat{\Gamma}$.

This proves that on the set $\mathbb{R}^{2n} \times S_\pi$, $\dot{V} \leq 0$ with equality only on $\hat{\Gamma}$. So if it can be shown that for any solution with initial conditions in $\mathbb{R}^{2n} \times S_\sigma$ there exists a time such that the solution enters and remains in the set $\mathbb{R}^{2n} \times S_\pi$, then by LaSalle's invariance principle, solutions with initial conditions in $\mathbb{R}^{2n} \times S_\sigma$ must converge to $\hat{\Gamma}$, proving attractivity. Stability of $\hat{\Gamma}$ follows directly from the fact that $\hat{\Gamma}$ lies at the minimum of the Lyapunov function V.

It remains to show that there exists $k^\star > 0$ such that for all initial conditions $\chi_0 \in \mathbb{R}^{2n} \times S_\sigma$ there exists $t^\star$ such that choosing $k > k^\star$ implies solutions $\phi_k(t, \chi_0) \in \mathbb{R}^{2n} \times S_\pi$ for all $t > t^\star$. Consider just the rotational system from (13),

$$\dot{\theta}_i = \frac{1}{\alpha} \left(f_i(\hat{y}_i^i) \cdot e_2 + k g_i(\hat{y}_i^i) \right), \ i = 1, \ldots, n \tag{20}$$

where the term $f_i(\hat{y}_i^i) \cdot e_2$ is bounded from assumption A2 and therefore acts as a bounded perturbation on the nominal rotational gradient system with storage function V_r. By Proposition 4, for the nominal rotational system, the set $\{\theta \in \mathbb{T}^n : \theta_i = \theta_j, \ \forall i, j \in \{1, \ldots, n\}\}$ is asymptotically stable with domain of attraction containing S_π. Let U_1, U_2 be sublevel sets of V_r such that $U_1 \subset U_2 \subset S_\pi$. This property implies that there exists $k_0 > 0$ such that choosing $k > k_0$ implies U_2 is positively invariant for (20).

Now consider a solution $\phi_k(t, \bar{\theta})$ of (20) with initial angle $\bar{\theta} = (\theta_i(0))_{i \in \{1,\ldots,n\}} \in \bar{S}_\sigma$ where $\bar{S}_\sigma$ is the closure of S_σ and hence a compact set. To conclude the proof it is sufficient to show that there exists $k^\star > k_0$ such that for all $\bar{\theta} \in \bar{S}_\sigma$, there exists $t^\star > 0$ such that choosing $k > k^\star$ implies $\phi_k(t^\star, \bar{\theta}) \in U_2$ a positive invariant set. Take any $\bar{\theta} \in \bar{S}_\sigma$. For the nominal system, the solution $\phi_k(t, \bar{\theta})$ converges to the set $\{\theta \in \mathbb{T}^n : \theta_i = \theta_j, \ \forall i, j \in \{1, \ldots, n\}\}$. By [27] theorem 3.4, there exists $k_1 > k_0$, $t_1 > 0$, $\epsilon > 0$ such that for all $\theta \in B_\epsilon(\bar{\theta}) =: W_1$ choosing $k > k_1$ implies $\phi_k(t_1, \bar{\theta}) \in U_1$ and $\phi_k(t_1, \theta) \in U_2$ which by positive invariance of the set U_2 implies that $\phi_k(t, \theta) \in U_2$ for all $t > t_1$. Since $\bar{S}_\sigma$ is compact, it has a finite sub-covering of ϵ-neighborhoods $\{W_i\}_{i \in I}$ where I is a finite index set. Corresponding to each set W_i there exists $k_i > 0$ and $t_i > 0$ such that choosing

$k > k_i$ implies $\phi_k(t, W_i) \subset U_2$ for all $t > t_i$. Then choosing $k^\star > \max\{k_i\}_{i \in I}$ concludes the proof. ■

Simulation results are presented in Figure 3. In the left figure, a group of five unicycles stabilize a line formation specified by $(\alpha_i)_{i\in\{1,\ldots,5\}} = (5, 5, 5, 5, 5)$ and $(\beta_i)_{i\in\{1,\ldots,5\}} = (-10, -5, 0, 5, 10)$ while in the right figure they stabilize a triangular formation specified by $(\alpha_i)_{i\in\{1,\ldots,5\}} = (15, 10, 5, 10, 15)$ and $(\beta_i)_{i\in\{1,\ldots,5\}} = (-10, -5, 0, 5, 10)$. We have chosen random initial unicycle positions on a 40m $\times$ 40m area with random initial angles and a ring graph. In addition, we have chosen $f(s) = s$ as the translational interaction function. Notice that $f(s)$ does not meet assumption A2. Although Theorem 1 only considers parallel line formations, the simulations suggest that the feedback in (10) also works for general formations.

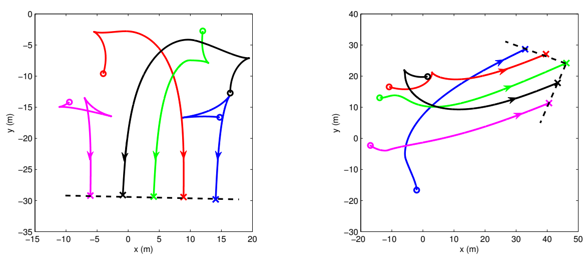

Fig. 3: Simulation for a line formation (left) and a triangular formation (right). Initial positions are indicated with $\circ$ and final positions are indicated with $\times$.

IV. Conclusion

We have presented a local and distributed control law for connected, undirected sensor graphs to drive a group of kinematic unicycles to a desired line formation with aligned heading directions. We assumed initial conditions in which the unicycle headings all lie in a common half plane. We presented a controller that solves the line formation control problem, combining a bounded translational consensus controller meeting certain assumptions with an attitude synchronizer. We presented simulation results that indicate that the proposed approach also works in the case of more general formations. In future work, we look to extend the present framework to solve the formation control problem for general formations over a larger set of initial conditions.

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