A Smooth Distributed Feedback for Formation Control of Unicycles

Ashton Roza, Manfredi Maggiore, Luca Scardovi

***Abstract*—This paper investigates a formation control problem in which a group of kinematic unicycles is made to converge to a desired formation with parallel heading angles and come to a stop. A control law is presented which solves this problem for almost all initial conditions in any given compact set. The proposed control law is local and distributed, meaning that each unicycle is only required to sense its relative displacement measured in its own body frame, and the relative heading angle with respect to each of its neighbours. No communication between the unicycles is required. The sensing graph is assumed to be connected, undirected and time-invariant. The idea used to solve the above formation control problem is to rigidly attach to the body frame of each unicycle an appropriate fixed offset vector. Stabilizing the desired formation amounts to achieving consensus of the endpoints of the offset vectors, and simultaneously synchronizing the unicycles' heading angles. A control law achieving this goal is constructed by combining a bounded translational consensus controller with an attitude synchronizer. As a special case, the proposed solution solves the full unicycle synchronization problem, in which the unicycle positions are made to converge to each other, while the unicycle headings are made to align.**

I. Introduction

This paper investigates a formation control problem for kinematic unicycles. For almost all initial conditions in any given compact set, the objective is to drive the unicycles to a *parallel formation*, i.e., one in which the unicycles' headings are all parallel, and their relative displacement vectors take on appropriate values corresponding to a desired geometric pattern. Moreover, it is required that the unicycles come to a stop when they meet the formation requirement. The asymptotic roto-translation of the formation with respect to a fixed frame depends on the initial conditions of the unicycles.

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The authors are with the Department of Electrical and Computer Engineering, University of Toronto, 10 King's College Road, Toronto, ON, M5S 3G4, Canada. ashton.roza@mail.utoronto.ca, maggiore@ece.utoronto.ca, scardovi@scg.utoronto.ca

Note on Previous Publication. A preliminary version of this work has appeared in the conference paper [1]. With respect to the preliminary version, we present, in this submission, a significant amount of new material. Firstly, the control problem in [1] studied formation control for the special class of *line formations* and the unicycle heading angles were assumed to start in a common half circle. In this paper, the result is extended to general formations where the fixed, relative spacing between unicycles can be chosen arbitrarily and the stability result is almost semiglobal. Secondly, compared to [1], this paper provides a more detailed discussion of the control solution in Section IV-D and of the simulation results in Section V. Finally, the proof presented in this paper is complete whereas in [1], some details were excluded due to space limitations. Naturally, there are strong similarities in the overlapping material between the two papers, and the wording in parts of the introduction, the notation, the unicycle modelling, and parts of the proof closely resembles, or is at times identical to, analogous wording in the conference paper [1].

The utility of static formations is manifest in problems where vehicles are required to distribute themselves over a terrain in order, for instance, to form an antenna array or a sensor network. More generally, the problem investigated in this paper is a conceptual gateway to the problem of inducing collective motions in formations.

A key requirement of the problem investigated is that the unicycles can only sense their relative heading angles and relative positions with respect to their neighbours in the sensor graph, which is assumed to be undirected and connected. Moreover, the unicycles cannot communicate with each other. A feedback meeting these sensing requirements is called *local and distributed*, and from a practical standpoint it has the beneficial property that it can be implemented on each unicycle using information deduced from a fixed on-board camera. As we shall see in a moment, the problem investigated in this paper, although basic, is essentially open, the sensing restriction being the key challenge.

Previous work: The majority of the literature on formation control focuses on single and double-integrator robot models. A dominant approach for single-integrator formation control is distance-based [2], [3], [4], where it is required that the distances between robots take on desired values. Often in this setting, the feedbacks are deduced from the gradient of a potential function whose minimum specifies the desired formation modulo roto-translation. This approach requires the sensing graph to be infinitesimally rigid. Other approaches define formations in terms of relative angles between neighbouring robots, instead of distances, [5], [6], or in terms of a complex Laplacian, [7], [8], [9]. In this latter case, formations are defined modulo scaling and roto-translations. Finally, formation flocking of double-integrators is considered in [10], where the authors stabilize a formation and make sure that all robots in the formation achieve a common final velocity. See also [11].

A formation controller for single integrator robots can be turned into a controller for kinematic unicycles if one considers a point at a positive distance d in front of each unicycle. These points behave like single integrators under an appropriate choice of feedback, and can be driven to a desired formation using the techniques above. However, although the points converge to a formation, the unicycles themselves do not. Choosing a small value of d reduces this error, but requires large control inputs.

The most relevant literature to the work in this paper concerns kinematic unicycles. The papers [12], [13] show that

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the only possible relative equilibria for unicycles with local and distributed control laws correspond to either parallel or circular motions. In [14], the problem of full synchronization is considered in which both positions and attitudes of the unicycles are synchronized using a discontinuous distributed control. The communication graph is allowed to be time-dependent and assumed to be initially connected. In [15], a discontinuous controller is presented that stabilizes formations with synchronized heading directions, but unicycles require a common sense of direction. In [16], the authors discuss feasibility conditions to achieve various formations of kinematic unicycles. Time-dependent solutions are presented in each case. For general geometric patterns, unicycles require a common sense of direction. Similarly, the solution in [17] is time dependent and requires measurement of a common direction in addition to the velocity input of a neighbouring unicycle, which can only be obtained if the unicycles communicate with each other. The special case of full synchronization can, however, be achieved without a common sense of direction. In [18], the authors present a local and distributed solution for formation control using a leader-follower approach with a hierarchical graph structure. In [19], a group of robots is considered in which at least one follower robot can see a leader that follows a desired path. The feedback law presented in the paper attains the desired formation about the leader in finite time. However, the formation is not invariant under rotations, and the control law is not local and distributed. In [20], a leader-follower approach is considered. The analysis transforms the unicycle model into a system of double integrators through dynamic feedback linearization. The desired formation is attained for graphs containing a spanning tree, but each follower robot requires access to the acceleration of the leader through communication. In [21], each unicycle estimates its own position using dynamic extension, requiring communication among unicycles. The unicycles use these estimated states to attain the desired formation globally. The rotational control is time-dependent and oscillatory. Finally, the work in [22] presents a local and distributed control law making kinematic and dynamic unicycles converge to a common circle with arbitrary desired ordering and spacing on the circle.

Contributions of this paper: To the best of our knowledge, the problem of stabilizing static parallel formations by means of smooth, local and distributed feedbacks is to date open. This paper presents an almost semiglobal solution to this problem, i.e., a solution making the unicycles achieve the desired objective for almost all initial conditions in any given compact subset of their collective state space. This solution is presented in Theorem 2.

The idea we employ to solve the formation control problem is to attach to the body frame of each unicycle a fixed offset vector in such a way that the problem of stabilizing a parallel formation turns into that of achieving consensus of the end points of the offset vectors, while simultaneously synchronizing the heading angles. Accordingly, we combine a uniformly bounded, globally convergent consensus controller for kinematic integrators with an almost global consensus controller for kinematic integrators on the n-torus. This latter controller, a crucial component of our development, was recently developed by Mallada-Freeman-Tang in [23].

A special case of the setup investigated in this paper is when the formation is a point, in which case our solution achieves full synchronization of the unicycles, a problem of note in its own right. In this special case, our solution is to be compared to the one in [14], which relies on discontinuous control, but considers a more general class of sensor graphs than ours.

The results presented in this paper are an enhancement of preliminary work we presented in the conference paper [1]. In [1], formations were limited to lie on a line, and it was assumed that the unicycle headings would be initialized in a common half-plane. In this paper, we extend the ideas in [1] to general parallel formations and the final result is almost semiglobal.

Organization of the paper: In Section II, we review basic notions of set stability. In Section III, we formulate the formation control problem. The solution to the formation control problem is presented in Section IV. The main theorem of the paper is Theorem 2, whose proof is presented in Section VI. In Section V, we test the proposed feedbacks via numerical simulation. Finally, in Section VII, we end the paper with some concluding remarks.

Notation: Throughout the paper, we identify column vectors $v = [v_1 \ \cdots \ v_n]^\top$ in $\mathbb{R}^n$ with n-tuples $(v_1, \ldots, v_n)$ in $(\mathbb{R})^n$. If v, w are vectors in $\mathbb{R}^2$, we denote by $v \cdot w := v^\top w$ their Euclidean inner product. We denote by $\{e_1, e_2\}$ the natural basis of $\mathbb{R}^2$, by $\mathbf{1}$ the vector of ones in $\mathbb{R}^n$, by $\mathsf{SO}(2)$ the set $\mathsf{SO}(2) := \{M \in \mathbb{R}^{2\times 2} : M^{-1} = M^\top, \det(M) = 1\}$, and by $\mathbb{S}^1$ the unit circle, which we identify with the set of real numbers modulo 2π. By $\mathbb{S}^n$ we denote the n-dimensional unit sphere, and by $\mathbb{T}^n$ the n-torus $\mathbb{T}^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ (n times). We denote $\mathbf{n} := \{1, \ldots, n\}$ and $\mathbf{k}\colon\mathbf{n} := \{k, \ldots, n\}$.

If $(\mathcal{X}, g)$ is a complete Riemannian manifold, $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is the associated Riemannian distance function on $\mathcal{X}$, and $\Gamma \subset \mathcal{X}$ is a closed subset of $\mathcal{X}$, then we denote by $\|\chi\|_\Gamma := \inf_{\psi\in\Gamma} d(\chi, \psi)$ the point-to-set distance of $\chi \in \mathcal{X}$ to Γ. If $\varepsilon > 0$, we let $B_\varepsilon(\Gamma) := \{\chi \in \mathcal{X} : \|\chi\|_\Gamma < \varepsilon\}$ and by $\mathcal{N}(\Gamma)$ we denote a neighborhood of Γ in $\mathcal{X}$. If $I = \{i_1, \ldots, i_n\}$ is an index set, the ordered list of elements $(x_{i_1}, \ldots, x_{i_n})$ is denoted by $(x_j)_{j\in I}$.

II. Preliminaries

We begin by reviewing the notion of asymptotic stability of a closed set. Consider a forward complete smooth dynamical system $\Sigma : \dot\chi = f(\chi)$ with state space a Riemannian manifold $\mathcal{X}$, and let $\phi(t, \chi_0)$ denote the local phase flow on $\mathcal{X}$ generated by Σ. Let $\Gamma \subset \mathcal{X}$ be a closed set that is positively invariant for Σ, i.e., such that, for all $\chi_0 \in \Gamma$ and all $t > 0$, $\phi(t, \chi_0) \in \Gamma$.

Definition 1. *The set Γ is stable for Σ if for any $\varepsilon > 0$, there exists a neighborhood $\mathcal{N}(\Gamma) \subset \mathcal{X}$ such that, for all $\chi_0 \in \mathcal{N}(\Gamma)$, $\phi(t, \chi_0) \in B_\varepsilon(\Gamma)$, for all $t > 0$ for which $\phi(t, \chi_0)$ is defined. The domain of attraction of Γ for system Σ is the set $D(\Gamma) := \{\chi_0 \in \mathcal{X} : \lim_{t\to\infty} \|\phi(t, \chi_0)\|_\Gamma = 0\}$. The set Γ is attractive for Σ if $D(\Gamma)$ is a neighborhood of Γ. The set Γ is globally attractive for Σ if $D(\Gamma) = \mathcal{X}$. The set Γ is locally* *asymptotically stable (LAS) for Σ if Γ is stable and attractive. The set Γ is globally asymptotically stable for Σ if Γ is stable and globally attractive.* $\triangle$

Next we present two variants of the notion of asymptotic stability of a closed set Γ: almost global and almost semiglobal asymptotic stability.

Definition 2. *The set Γ is* almost globally asymptotically stable *for Σ if Γ is asymptotically stable for Σ, and $\mathcal{X}\backslash D(\Gamma)$ has Lebesgue measure zero.* $\triangle$

Now consider a dynamical system $\Sigma(k) : \dot{\chi} = f(\chi, k)$ with state space a Riemannian manifold $\mathcal{X}$, where $k \in \mathbb{R}$ is a parameter (typically, a control gain), and f is a smooth vector field on $\mathcal{X}$. In what follows, $D_k(\Gamma)$ denotes the domain of attraction of a closed set Γ for system $\Sigma(k)$

Definition 3. *The set Γ is* semiglobally asymptotically stable with high-gain parameter *k for $\Sigma(k)$ if for each compact set K satisfying $\Gamma \subset K \subset \mathcal{X}$, there exists $k^\star > 0$ such that for all $k > k^\star$, Γ is asymptotically stable for $\Sigma(k)$ and $K \subset D_k(\Gamma)$.* $\triangle$

Definition 4. *The set Γ is* almost semiglobally asymptotically stable with high-gain parameter *k for $\Sigma(k)$ if there exists a set $N \subset \mathcal{X}\backslash\Gamma$ of Lebesgue measure zero such that for each compact subset K satisfying $\Gamma \subset K \subset (\mathcal{X}\backslash N)$, there exists $k^\star > 0$ such that for all $k > k^\star$, Γ is asymptotically stable for $\Sigma(k)$ and $K \subset D_k(\Gamma)$.* $\triangle$

The difference between global asymptotic stability and semiglobal asymptotic stability of a closed set Γ is that with the former, solutions converge to the set Γ from all initial conditions, while with the latter, the domain of attraction can be made arbitrarily large within the state-space with increasing choice of the control gain k. The difference between almost global asymptotic stability and almost semiglobal asymptotic stability is that with the former, solutions converge to the set Γ with domain of attraction $D(\Gamma)$ of full measure, while with the latter, the domain of attraction approaches full measure with increasing control gain k.

III. Formation Control Problem

We begin by modelling a group of n kinematic unicycles. We fix an orthogonal frame $\mathcal{I} = \{i_x, i_y\}$ in $\mathbb{R}^2$, and attach to unicycle i an orthogonal body frame $\mathcal{B}_i = \{b_{ix}, b_{iy}\}$ in such a way that b_{ix} is the heading axis of the unicycle. We pick the frames so that their y-axes result from the counterclockwise rotation of their x-axes by angle $\pi/2$. We denote by $x_i \in \mathbb{R}^2$ the position of unicycle i in the coordinates of frame $\mathcal{I}$. The unicycle's attitude is represented by a rotation matrix R_i whose columns are the coordinate representations of b_{ix} and b_{iy} in frame $\mathcal{I}$. Letting $\theta_i \in \mathbb{S}^1$ be the angle between vectors i_x and b_{ix}, we have

$$R_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}.$$

With these conventions, the model of unicycle i is

$$\dot{x}_i = u_i R_i e_1 \tag{1}$$

$$\dot{\theta}_i = \omega_i, \quad i \in \mathbf{n}, \tag{2}$$

where the pair (u_i, ω_i), the linear and angular speeds of unicycle i, is the control input. We let $x := (x_i)_{i\in\mathbf{n}}$ and $\theta := (\theta_i)_{i\in\mathbf{n}}$.

The relative displacement of robot j with respect to robot i is $x_{ij} := x_j - x_i$ while the relative angles are given by $\theta_{ij} = \theta_j - \theta_i$. The rotation of robot j with respect to frame i is defined by $R_j^i := (R_i)^{-1} R_j$, and it is a function of θ_{ij}. If $v \in \mathbb{R}^2$ is the coordinate representation of a vector in frame $\mathcal{I}$, then we denote by $v^i := R_i^{-1} v$ the coordinate representation of v in body frame $\mathcal{B}_i$.

We define the *undirected sensor graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where each node in the node set $\mathcal{V}$ represents a robot, and an edge in the edge set $\mathcal{E}$ between node i and node j indicates that robot i can sense robot j and vice versa. We assume that $\mathcal{G}$ has no self-loops and is time-invariant. Given a node i, its set of neighbours $\mathcal{N}_i$ represents the set of vehicles that robot i can sense. If $j \in \mathcal{N}_i$, then we say that robot j is a *neighbour* of robot i. If this is the case, then robot i can sense the relative displacement of robot j in its own body frame, i.e., the quantity x_{ij}^i, as well as the relative heading angle θ_{ij} between unicycles i and j.

We now define the notion of local and distributed feedback. Define vectors $y_i := (x_{ij})_{j\in\mathcal{N}_i}$, $y_i^i := (x_{ij}^i)_{j\in\mathcal{N}_i}$, and $\varphi_i := (\theta_{ij})_{j\in\mathcal{N}_i}$. The relative displacements and angles available to robot i are contained in the vector (y_i^i, φ_i). A *local and distributed feedback* for robot i is a locally Lipschitz function $(y_i^i, \varphi_i) \mapsto (u_i, \omega_i)$. Relative positions and angles can be practically measured using cameras fixed on-board each robot. In particular, robot i can use its camera to observe tags (e.g., AprilTags) attached to the body frames of its neighbors to accurately compute, using detection software, its relative position and orientation.

The objective of the formation control problem considered in this paper is to design local and distributed feedbacks to drive a group of unicycles, each modelled by (1), (2), to a desired formation. A formation is a geometric pattern defined modulo roto-translation by means of desired inter-agent displacements. In this paper we consider *parallel formations*, i.e., formations in which the unicycles' headings are parallel to each other: $\theta_{ij} = 0$ for all $i, j \in \mathbf{n}$.

To precisely define a parallel formation, we draw the unicycles in formation with parallel headings, as in Figure 1, and label the unicycles $\{1, \ldots, n\}$. The roto-translation of the formation in the drawing is irrelevant. The labelling of the unicycles is done solely for the purpose of defining the formation, and does not imply any attribution of priority to the unicycles. We let $\bar{x}_i$ denote the position of unicycle i in the drawing, and $d_{1i}^1 \in \mathbb{R}^2$ denote the displacement of unicycle i relative to unicycle 1, measured in the frame of unicycle 1: $d_{1i}^1 := R_1^{-1}(\bar{x}_i - \bar{x}_1)$. The labelling of unicycles is chosen such that unicycle 1 is at the front of the formation, i.e., it is such that $d_{1i}^1 \cdot e_1 \leq 0$ for all $i \in \mathbf{2\!:\!n}$.

We collect the above relative displacements in a vector $d := (d_{1i}^1)_{i\in\mathbf{2:n}}$. We will say that the vector d is a *parallel formation*, and the set

$$\mathsf{F} := \{d = (d_{1i}^1)_{i\in\mathbf{2:n}} \in \mathbb{R}^{2(n-1)} : d_{1i}^1 \cdot e_1 \leq 0, i \in \mathbf{2:n}\}$$

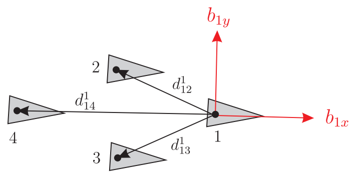

Fig. 1: Parallel formation represented in terms of fixed relative displacement vectors d_{1i}^1, $i \in \mathbf{2}\!:\!\mathbf{n}$ represented in frame $\mathcal{B}_1$.

represents the set of all parallel formations of n unicycles. A formation specification $d \in \mathsf{F}$ specifies a formation modulo roto-translation because it is defined in terms of relative displacements d_{1i}^1 expressed in the coordinates of frame $\mathcal{B}_1$.

For a given parallel formation $d \in \mathsf{F}$, we define the *formation manifold* $\Gamma(d)$ as,

$$\begin{aligned}\Gamma(d) := \big\{&(x_i,\theta_i)_{i\in\mathbf{n}} \in \mathbb{R}^{2n}\times\mathbb{T}^n :\\ &x_{1i} = R_1 d_{1i}^1,\ \theta_{1i}=0,\ i \in \mathbf{2}\!:\!\mathbf{n}\big\}.\end{aligned} \tag{3}$$

The formation manifold $\Gamma(d)$ is the subset of the unicycles' collective state space on which the unicycles have parallel headings, and their relative displacements meet the formation specification. Now the problem investigated in this paper.

Formation Control Problem: Consider the collection of n unicycles in (1)-(2), and an undirected, connected sensor graph $\mathcal{G}$. For any parallel formation $d \in \mathsf{F}$, find local and distributed feedbacks $(u_i^\star, \omega_i^\star) : (y_i^i, \varphi_i) \mapsto (u_i, \omega_i)$ rendering the formation manifold $\Gamma(d)$ almost semiglobally asymptotically stable and such that $(u_i^\star, \omega_i^\star)|_{\Gamma(d)} = (0,0)$. △

The requirement $(u_i^\star, \omega_i^\star)|_{\Gamma(d)} = (0,0)$ means that the unicycles do not move when they are in formation.

IV. Solution of the Formation Control Problem

In this section we present a class of feedbacks solving the formation control problem stated above. Our solution combines uniformly bounded consensus controllers for single-integrators with Kuramoto-like consensus controllers on $\mathbb{T}^n$ developed recently by Mallada-Freeman-Tang in [23]. In Section IV-A, we define a uniformly bounded consensus controller for single-integrators. In Section IV-B, we review the rotation control law by Mallada-Freeman-Tang in [23]. Finally, in Section IV-C, we combine these two controllers to design the unicycle control inputs u_i and ω_i.

A. Single Integrator Consensus

Consider n single integrators on $\mathbb{R}^2$,

$$\dot z_i = v_i,\ i \in \mathbf{n}, \tag{4}$$

where $z_i \in \mathbb{R}^2$ and sensor graph as in Section III. A feedback

$$v_i = f_i((z_{ij})_{j\in\mathcal{N}_i}) := \sum_{j\in\mathcal{N}_i} a_{ij}\frac{f(\|z_{ij}\|)}{\|z_{ij}\|} z_{ij}, \tag{5}$$

where $z_{ij} := z_j - z_i$, is a *bounded integrator consensus controller* if $a_{ij} = a_{ji} > 0$ and $f : \mathbb{R} \to \mathbb{R}$, the *interaction function*, is a locally Lipschitz function satisfying:

A1: $sf(s) > 0$ for all $s \neq 0$, $f(0) = 0$, and there exist $c_1, c_2 > 0$ such that $|f(s)| > c_1$ for all $|s| > c_2$.

A2: $\sup|f(s)| < \infty$.

Each element $(z_{ij}/\|z_{ij}\|)f(\|z_{ij}\|)$ of the sum in (5) is continuous at $z_{ij} = 0$ because $f(s)$ is a continuous function and $f(0) = 0$ by assumption A1. We will omit the easy proof of the fact that each $f_i((z_{ij})_{j\in\mathcal{N}_i})$ is Lipschitz continuous.

Examples of suitable interaction functions are $f(s) = \tanh(s)$ and

$$f(s) = \begin{cases} s, & \text{if } |s| \le 1\\ s/|s| & \text{if } |s| > 1.\end{cases} \tag{6}$$

In the latter case, feedback (5) reduces to

$$f_i((z_{ij})_{j\in\mathcal{N}_i}) = \begin{cases} z_{ij}, & \text{if } \|z_{ij}\| \le 1\\ z_{ij}/\|z_{ij}\|, & \text{if } \|z_{ij}\| > 1.\end{cases} \tag{7}$$

As we shall see in Section VI-A, feedback (5) globally asymptotically stabilizes the consensus subspace $\{z \in \mathbb{R}^{2n} : z_i = z_j,\ \forall i,j \in \mathbf{n}\}$ for any connected, undirected sensor graph $\mathcal{G}$. We do not claim originality of this result.

B. Attitude Synchronization

Now consider a collection of rotational integrators

$$\dot\psi_i = w_i,\ i \in \mathbf{n}, \tag{8}$$

where $\psi_i \in \mathbb{S}^1$. A feedback

$$w_i = g_i((\psi_{ij})_{j\in\mathcal{N}_i}, \eta) := \eta_i \sum_{j\in\mathcal{N}_i} b_{ij} g(\psi_{ij}), \tag{9}$$

where $\psi_{ij} := \psi_j - \psi_i$ is an *attitude synchronizer* if $\eta_i > 0$, $b_{ij} = b_{ji} > 0$, and $g : \mathbb{S}^1 \to \mathbb{R}$ is a continuously differentiable *interaction function* satisfying the following three assumptions [23]:

B1: $sg(s) > 0$ for all $s \in (-\pi,\pi)\backslash 0$, $g(0) = g(\pi) = 0$.

B2: $g(s)$ is an odd function: $g(-s) = -g(s)$ for $s \in (-\pi,\pi)$.

B3: $\dot g(s) > 0$, $\forall s \in (-\frac{\pi}{n-1}, \frac{\pi}{n-1})$ and $\dot g(s) < 0$, $\forall s \in (-\pi, -\frac{\pi}{n-1}) \cup (\frac{\pi}{n-1}, \pi)$.

A sample interaction function satisfying B1-B3 is shown in Figure 2.

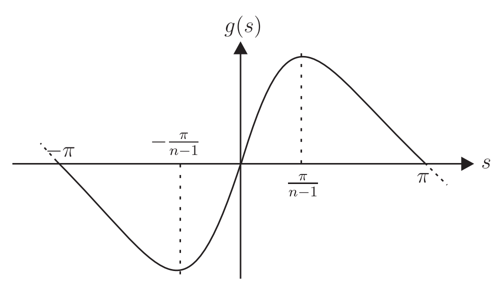

Fig. 2: Illustration of properties B1, B2 and B3.

The well-known Kuramoto model for attitude synchronization of angles in $\mathbb{S}^1$ with zero natural frequencies [24] corresponds to the choice $\eta_i = 1$ and $g(s) = \sin(s)$, and it satisfies properties B1-B2, but does not satisfy property B3. In [23], Mallada-Freeman-Tang showed that a feedback enjoying properties B1-B3 almost globally stabilizes the set $\{\psi \in \mathbb{T}^n : \psi_i = \psi_j,\ \forall i,j \in \mathbf{n}\}$ for almost all gains $b_{ij} = b_{ji} > 0$. The following result is a special case of Theorem 2 in [23].

Theorem 1 ([23])**.** *Consider system* (8) *with feedback* (9) *satisfying assumptions B1-B3, and assume that the sensor graph $\mathcal{G}$ is undirected and connected. There exists a set $N_b \subset (\mathbb{R}_{>0})^{|\mathcal{E}|}$ of Lebesgue measure zero such that for any collection of gains $(b_{ij})_{(i,j)\in\mathcal{E}} \in (\mathbb{R}_{>0})^{|\mathcal{E}|}\backslash N_b$, and for any $\eta_i > 0$, $i \in \mathbf{n}$, the set $\{\psi \in \mathbb{T}^n : \psi_i = \psi_j,\ \forall i,j \in \mathbf{n}\}$ is almost globally asymptotically stable.*

The above result follows from Theorem 2 in [23]. In particular, system (8) with feedback (9) satisfies the model in equations (14)-(16) in [23] by letting $\chi_i(s) = \eta_i s$, letting ζ be the identity function, and eliminating the integrator state γ_i.

The main difference here compared to the solution in [23, Theorem 2] is that, in [23], each system has an additional constant bias. The integrator state γ_i is used in [23] to compensate for this bias. In this paper, system (8) has zero bias, so the integrator state γ_i is not needed. Accounting for this small difference, the proof of Theorem 1 follows from minimal modifications to the proof of [23, Theorem 2].

C. Solution to the Formation Control Problem

In this section we present a solution to the formation control problem. Let $\bar{\alpha} > 0$ be a design parameter, and $d = (d^1_{i1})_{i\in\{2,\dots n\}} \in \mathsf{F}$ be a desired parallel formation. Define

$$\begin{aligned} &\alpha_1 := \bar{\alpha},\ \beta_1 := 0, \\ &\alpha_i := -d^1_{1i}\cdot e_1 + \bar{\alpha},\ \beta_i := -d^1_{1i}\cdot e_2, \quad i \in \mathbf{2:n}. \end{aligned} \tag{10}$$

Referring to Figure 3, attach the offset vector $\delta_i := \alpha_i R_i e_1 + \beta_i R_i e_2$ to the body frame of unicycle i, and let $\hat{x}_i := x_i + \delta_i$ be the endpoint of the offset vector in the coordinates of frame $\mathcal{I}$. Define further

$$\begin{aligned} &\hat{x}_{ij} := \hat{x}_j - \hat{x}_i, \\ &\hat{y}_i := (\hat{x}_{ij})_{j\in\mathcal{N}_i}, \quad \hat{y}^k_i := (\hat{x}^k_{ij})_{j\in\mathcal{N}_i}. \end{aligned}$$

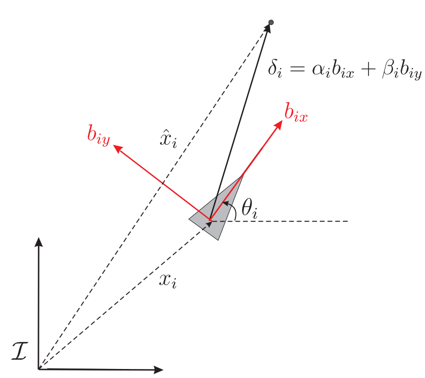

Fig. 3: Representation of the offset vector δ_i.

We now show that the formation control problem reduces to synchronizing the unicycles' heading angles and the endpoints $\hat{x}_i$. To this end, suppose that $\theta_{ij} = 0$ and $\hat{x}_{ij} = 0$ for all $i,j \in \mathbf{n}$. Then,

$$\begin{aligned} 0 = \hat{x}^i_{1i} &= [(x_i + \delta_i) - (x_1 + \delta_1)]^i \\ &= x^i_{1i} + (\delta_i - \delta_1)^i \\ &= x^i_{1i} - d^1_{1i} = x^1_{1i} - d^1_{1i}. \end{aligned}$$

The last identity follows from the fact that $R_i = R_1$. We conclude that $\theta_{ij} = 0$ and $\hat{x}_{ij} = 0$ for all $i,j \in \mathbf{n}$ implies $x^1_{1i} = d^1_{1i}$, so that the unicycles satisfy the parallel formation requirement. Vice versa, it is clear that if the unicycles form a parallel formation, then $\theta_{ij} = 0$ and $\hat{x}_{ij} = 0$ for all $i,j \in \mathbf{n}$.

We have thus shown that the formation control problem amounts to the simultaneous synchronization of the headings θ_i and the endpoints $\hat{x}_i$. We now present feedbacks that do just that. Let $f_i(\cdot)$ be a bounded integrator consensus controller as in (5), and $g_i(\cdot)$ be an attitude synchronizer as in (9). The feedbacks for unicycle i are defined as follows,

$$\begin{aligned} &u^\star_i(y^i_i, \varphi_i) = f_i(\hat{y}^i_i)\cdot e_1 + \beta_i\omega_i, \\ &\omega^\star_i(y^i_i, \varphi_i) = \frac{1}{\alpha_i}\left(f_i(\hat{y}^i_i)\cdot e_2 + k g_i(\varphi_i, \eta)\right),\ i \in \mathbf{n}, \end{aligned} \tag{11}$$

where $k > 0$ is a high-gain parameter, and $\eta = (\eta_1, \dots, \eta_n)$, $\eta_i := 1/\alpha_i$. Now the main result of this paper.

Theorem 2 (Main Result)**.** *Consider the collection of n unicycles in* (1), (2) *with controller* (11)*, where the functions $f_i(\cdot), g_i(\cdot)$ are defined in* (5), (9) *and enjoy properties A1, A2 and B1-B3. Assume that sensor graph $\mathcal{G}$ is undirected and connected. For any parameters $a_{ij} = a_{ji} > 0$ in* (5) *and any parameters $b_{ij} = b_{ji} > 0$ in* (9) *satisfying $(b_{ij})_{(i,j)\in\mathcal{E}} \in (\mathbb{R}_{>0})^{|\mathcal{E}|}\backslash N_b$ as in Theorem 1, there exists $\alpha^\star > 0$ such that for any parallel formation $d = (d^1_{1i})_{i\in\mathbf{2:n}} \in \mathsf{F}$, choosing $\bar{\alpha} > \alpha^\star \max_{i\in\mathbf{2:n}}(-d^1_{1i}\cdot e_1)$ in* (10)*, the formation manifold $\Gamma(d)$ is almost semiglobally asymptotically stable with high-gain parameter k.*

The proof is presented in Section VI. Roughly speaking, the theorem states that letting the offset $\bar{\alpha}$ in (10) grow proportionally to the length of the formation (the quantity $\max_i(-d^1_{1i}\cdot e_1)$), and choosing k in (11) to be sufficiently large, the controller (11) ensures that almost all initial conditions in any given compact set are contained in the domain of attraction of the formation manifold $\Gamma(d)$. Another property of controller (11) is that $(u^\star_i, \omega^\star_i)\big|_{\Gamma(d)} = 0$ for all $i \in \mathbf{n}$, and therefore the unicycles come to a stop as $\Gamma(d)$ is approached, as required in the statement of the formation control problem in Section III. In the next section we further discuss the controller (11).

D. Discussion of the Control Solution

As we mentioned earlier, the philosophy behind controller (11) is to convert the formation control problem into a synchronization problem in which we make the offset vectors $\hat{x}_i$ and the heading angles θ_i converge to one another. This is illustrated in Figure 4, where the vectors $\hat{x}_i$, $i \in \mathbf{n}$ all meet at a common point at a distance $\bar{\alpha}$ in front of the formation, and all heading directions are aligned.

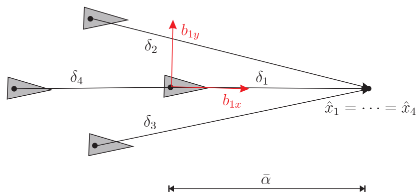

Fig. 4: Parallel formation where offset vectors $\hat{x}_i$ meet at a common point at a distance $\bar{\alpha}$ in front of the formation.

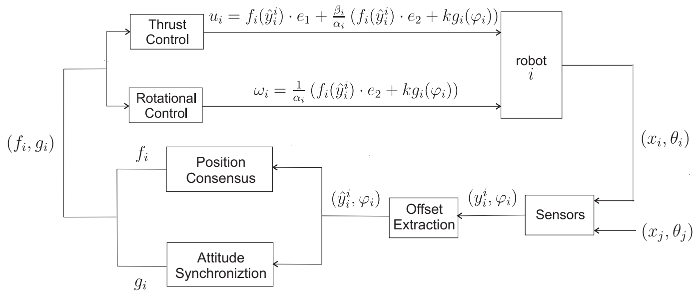

Fig. 5: Block diagram of the formation control system for robot i.

In (11), the terms containing $f_i(\hat{y}_i^i)$ aim to achieve consensus on the endpoints of the offset vectors $\hat{x}_i$, while the terms containing $g_i(\varphi_i, \eta)$ aim to achieve consensus on the unicycle angles. It will be shown in Section VI that the choice of (11) achieves both these, at times competing, objectives simultaneously by making use of gradient properties of the systems (4) and (8) with inputs (5) and (9) respectively.

The block diagram in Figure 5 summarizes the design of feedbacks $(u_i, \omega_i)_{i\in\mathbf{n}}$. From its sensors, unicycle i obtains the vector (y_i^i, φ_i) of its heading and displacement relative to its neighbours. These quantities can be measured locally in unicycle i's body frame using, for example, on-board cameras. The offset extraction block takes as input the vector (y_i^i, φ_i) and outputs $(\hat{y}_i^i, \varphi_i)$, where each component of $\hat{y}_i^i = (\hat{x}_{ij}^i)_{j\in\mathcal{N}_i}$ is computed as,

$$\hat{x}_{ij}^i = x_{ij}^i + \alpha_j R_j^i e_1 + \beta_j R_j^i e_2 - \begin{bmatrix}\alpha_i & \beta_i\end{bmatrix}^\top. \tag{12}$$

This computation requires that, in addition to (y_i^i, φ_i), unicycle i has access to the formation parameters $(\alpha_j, \beta_j)_{j\in\mathcal{N}_i}$ of its neighbours. These quantities must be stored in memory on-board unicycle i before deployment. Moreover, in order to compute $\hat{x}_{ij}^i$ in (12), unicycle i must be able to identify its neighbours so as to use, for each $j \in \mathcal{N}_i$, the appropriate bias constants (α_j, β_j). Such identification can be achieved, for instance, by means of visual markers. A consequence of using the constants (α_j, β_j) is that the unicycle feedbacks are not identical and the formation is not invariant to a relabelling of the agents. This is hardly surprising because, in our formulation of the formation control problem, we allow for general, non-symmetric formations.

An important property of the feedback in (11) is that it is local and distributed, since $u_i^\star$ and $\omega_i^\star$ depend on (y_i^i, φ_i). As a consequence of this feature, the asymptotic position and orientation of the formation with respect to the inertial frame depend only on the initial configuration of the unicycles.

E. Special cases: Line formations and full synchronization

As a by-product of the formation control solution, we present corresponding solutions for the special cases of parallel line formations and full synchronization.

A *parallel line formation* is a parallel formation satisfying $d_{1i}^1 \cdot e_1 = 0$ (and hence $\alpha_i = \bar{\alpha}$ for all $i \in \mathbf{2}\,{:}\,\mathbf{n}$). The set of all such formations will be denoted LF. Clearly, $\mathsf{LF} \subset \mathsf{F}$. In the case of *full synchronization*, the unicycles have the same position and orientation with respect to the inertial frame, i.e., $d_{1i}^1 = 0$ for all $i \in \mathbf{2}\,{:}\,\mathbf{n}$ (and therefore $\alpha_i = \bar{\alpha}$ and $\beta_i = 0$ for all $i \in \mathbf{n}$). Full synchronization, therefore, corresponds to the formation $0 \in \mathsf{F}$. Examples of a parallel line formation and full synchronization are illustrated in Figure 6.

According to Theorem 2, in both of these cases it suffices that $\bar{\alpha}$ satisfies the less strict condition $\bar{\alpha} > 0$. This is advantageous, as it will be discussed in Section V that large values of $\bar{\alpha}$ can slow down the rate of convergence of the unicycles to the formation. Arbitrarily choosing $\bar{\alpha} = 1$, the corresponding controller in (11) reduces to,

$$\begin{aligned} u_i^\star(y_i^i, \varphi_i) &= f_i(\hat{y}_i^i) \cdot e_1 + \beta_i \omega_i, \\ \omega_i^\star(y_i^i, \varphi_i) &= f_i(\hat{y}_i^i) \cdot e_2 + kg_i(\varphi_i, \eta), \;\; i \in \mathbf{n}, \end{aligned} \tag{13}$$

in which, $\hat{x}_{ij}^i = x_{ij}^i + R_j^i e_1 - e_1 + \beta_j R_j^i e_2 - \beta_i e_2$ and $\eta = (1, \ldots, 1)$. Since the values $\alpha_i = \bar{\alpha} = 1$ for all $i \in \mathbf{n}$ are equal, unicycle i only needs to store the quantities $(\beta_j)_{j\in\mathcal{N}_i}$ of its neighbours on-board. The next corollary is a specialization of Theorem 2 to parallel line formations.

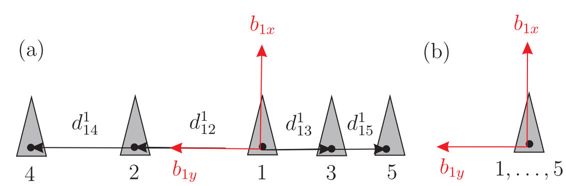

Fig. 6: (a) shows an example of a parallel line formation while (b) shows an example of full synchronization, a special case of a parallel line formation.

Corollary 1 (Parallel Line Formations)**.** *Consider the collection of n unicycles in* (1), (2) *with controller* (13), *where the functions $f_i(\cdot), g_i(\cdot)$ are defined as in* (5), (9) *and enjoy properties A1, A2 and B1-B3. Assume that sensor graph $\mathcal{G}$ is undirected and connected. For any parameters $a_{ij} = a_{ji} > 0$ in* (5), *any parameters $b_{ij} = b_{ji} > 0$ in* (9) *satisfying $(b_{ij})_{(i,j)\in\mathcal{E}} \in (\mathbb{R}_{>0})^{|\mathcal{E}|} \backslash N_b$ as in Theorem 1, and any parallel line formation $d \in \mathsf{LF}$, the formation manifold $\Gamma(d)$ is almost semiglobally asymptotically stable with high-gain parameter k.*

In the special case of full synchronization, $\beta_i = 0$ for all $i \in \mathbf{n}$, and the controller in (13) reduces to

$$\begin{aligned} u_i^\star(y_i^i, \varphi_i) &= f_i(\hat{y}_i^i) \cdot e_1, \\ \omega_i^\star(y_i^i, \varphi_i) &= f_i(\hat{y}_i^i) \cdot e_2 + kg_i(\varphi_i, \eta), \;\; i \in \mathbf{n}, \end{aligned} \tag{14}$$

in which, $\hat{x}_{ij}^i = x_{ij}^i + R_j^i e_1 - e_1$ and $\eta = (1, \ldots, 1)$. Since the α_i and β_i parameters are equal for all agents, unicycle i does not need to store any parameters of its neighbours on-board, the control inputs are identical for all unicycles, so that in this case the configuration is invariant to relabelling of agents. This is hardly surprising since the formation is symmetric in this case. The controller in (14) can be viewed as an extension of our previous result for unicycle rendezvous [25] (also closely related to our work on rendezvous of underactuated rigid bodies in three dimensions in [26]). In [25] the controller was given as

$$\begin{aligned} u_i^\star(y_i^i, \varphi_i) &= \|f_i(y_i^i)\| f_i(y_i^i) \cdot e_1, \\ \omega_i^\star(y_i^i, \varphi_i) &= -k f_i(y_i^i) \cdot e_2, \;\; i \in \mathbf{n}, \end{aligned} \tag{15}$$

in which $f_i(\cdot)$ is a linear single-integrator consensus controller, $f_i(y_i^i) = \sum_{j \in \mathcal{N}_i} a_{ij} x_{ij}^i$.

While the controller in (15) guarantees global rendezvous, in which only the unicycle positions are synchronized, the controller in (14) guarantees almost semiglobal full synchronization where both positions and angles of the unicycles are synchronized. The control inputs in (14) and (15) are similar in structure. The main difference is that the full synchronization controller in (14) has an additional term $k g_i(\varphi_i, \eta)$ responsible for aligning the unicycle heading angles, not required for rendezvous. In fact, for unicycle i, (14) depends on (y_i^i, φ_i) while (15) depends only on y_i^i.

V. Simulation Results

This section presents simulations for a group of five unicycles to illustrate our results and analyses the effect of the choice of the feedback gains on the system behaviour. The interaction function $f(s)$ for the bounded integrator consensus control is chosen as in (6) while the interaction function for the attitude synchronizer is chosen satisfying assumptions B1, B2 and B3 as in Figure 2. The sensing graph is cyclic with connections as shown in Figure 7 and the desired triangular formation is specified by $(\alpha_i)_{i \in \mathbf{1:5}} = (5, 15, 15, 25, 25)$ (corresponding to $\bar{\alpha} = 5$) and $(\beta_i)_{i \in \mathbf{1:5}} = (0, 5, -5, 10, -10)$ as illustrated in Figure 8. Let $a_{ij} = 30$ and $b_{ij} = \alpha_i + \alpha_j$ for all $j \in \mathcal{N}_i$ and $\eta_i = 1/\alpha_i$.

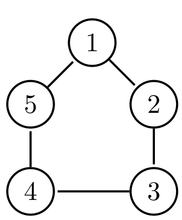

Fig. 7: Graph $\mathcal{G}$ under consideration in the simulation results.

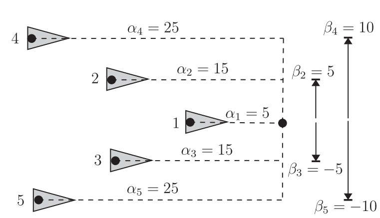

Fig. 8: Triangular formation specified by $(\alpha_i)_{i \in \mathbf{1:5}} = (5, 15, 15, 25, 25)$ and $(\beta_i)_{i \in \mathbf{1:5}} = (0, 5, -5, 10, -10)$.

We have chosen random initial unicycle positions on a 40m $\times$ 40m area with random initial angles. The corresponding plot of a simulation run is shown in Figure 9. We observe that the unicycles slow down rapidly as the formation is achieved, and seem to drift a finite distance.

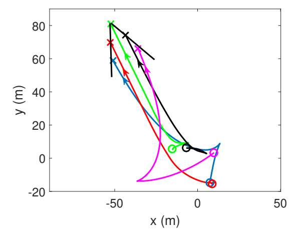

Fig. 9: Simulation for a triangle formation. Initial positions are indicated with $\circ$ and final positions are indicated with $\times$.

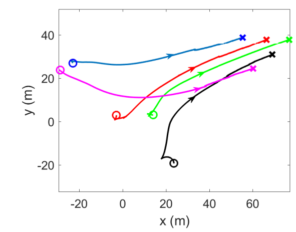

Fig. 10: Simulation result in the presence of disturbances and saturation.

TABLE I: Values of $\max_{i \in \mathbf{2:n}}(\|\hat{x}_{1i}\|)$ and $\max_{i \in \mathbf{2:n}}(|\theta_{1i}|)$ at the simulation termination time (400s).

case	$\max_{i \in \mathbf{2:n}}(\|\hat{x}_{1i}\|)$ (m)	$\max_{i \in \mathbf{2:n}}(\vert\theta_{1i}\vert)$ (rad)
(i)	0.2150	0.2001
(ii)	13.2168	0.2588
(iii)	7.1762	0.0998

Next, for the same formation considered in Figure 9, we study the effect of increasing the gains $\bar{\alpha}$ and k. This is done to illustrate potential drawbacks of the high-gain requirement in Theorem 2. In Figure 11 we plot $\|\hat{x}_{1i}\|$ versus time and θ_{1i} versus time for all $i \in \mathbf{2:5}$ and for three different scenarios: (i) nominal ($\bar{\alpha} = 5$, $k = 5$), (ii) high $\bar{\alpha}$ ($\bar{\alpha} = 55$, $k = 5$), (iii) high k ($\bar{\alpha} = 5$, $k = 50$). Then in Table I we list the values of $\max_{i \in \mathbf{2:n}}(\|\hat{x}_{1i}\|)$ and $\max_{i \in \mathbf{2:n}}(|\theta_{1i}|)$ at the simulation termination time (400s) for each case.

We observe that the effect of increasing $\bar{\alpha}$ (case (ii)), is a slow convergence of both $\hat{x}_{1i}$ and θ_{1i} quantities for $i \in \mathbf{2:n}$ compared to case (i), clearly negatively affecting the system response in all aspects. In case (iii), increasing the gain k causes the rotational terms in (11), $g_i(\varphi_i, \eta)$, to dominate the translational terms, $f_i(\hat{y}_i^i)$. We can see in the simulation results that while the rotational quantities θ_{1i} converge faster than in case 1, this has been at the expense of significantly slower convergence of the quantities $\hat{x}_{1i}$ for $i \in \mathbf{2:n}$, so much so that the overall performance of attaining formation degrades significantly. This illustrates the conflict between the translational and rotational terms, trying to accomplish the, often, competing objectives of driving $\hat{x}_{1i}$ and θ_{1i} to zero. For optimal performance, one is required to balance the strength of gains on the translational and rotational terms such as in case (i).

We have also performed a simulation to study the effectiveness of our control solution with sensor and input uncertainties by applying:

- an additive random noise with maximum magnitude of 0.25 m/s on the input u_i;
- an additive random noise with maximum magnitude of 0.25 rad/s on the input ω_i;
- an additive random noise with maximum magnitude of 0.25 rad on the quantity $g_i((\theta_{ij})_{j \in \mathcal{N}_i}, \eta)$ accounting for errors in measurements of relative headings;
- an additive random noise on the quantity $f_i(\hat{y}_i^i)$ account-

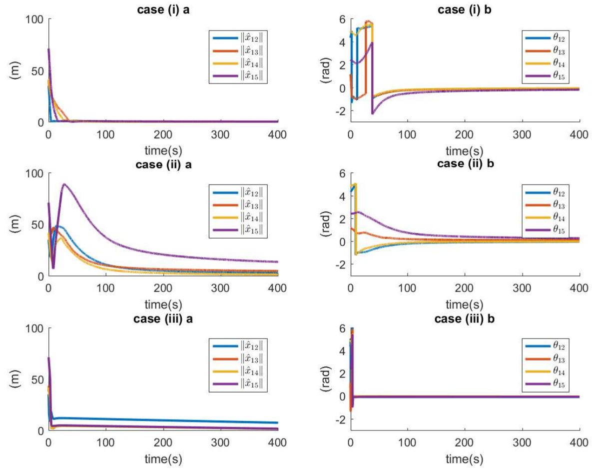

Fig. 11: Simulation results for 3 cases. (i) $\bar{\alpha} = 5$, $k = 5$; (ii) $\bar{\alpha} = 55$, $k = 5$; (iii) $\bar{\alpha} = 5$, $k = 50$. For each case, there are two plots: (a) $\|\hat{x}_{1i}\|$ versus time for $i \in \mathbf{2}\!:\!\mathbf{5}$; (b) θ_{1i} versus time for $i \in \mathbf{2}\!:\!\mathbf{5}$.

ing for errors in measurements of relative displacements of the vehicles. The direction of this vector has been rotated within 0.25 rad and the magnitude is scaled between 0.75 to 1.25 times the actual magnitude.

Each unicycle samples its input 100 times per second. Moreover, the inputs u_i and ω_i are saturated by 5 m/s and $\pi/2$ rad/s respectively. The result of the simulation, presented in Figure 10, suggests that the proposed control strategy is robust with respect to perturbations and input saturations.

It is worth noticing that our control objective does not take into account collision avoidance. For the same triangular formation as before, we have run extensive simulations with random initial unicycle positions on a 60m $\times$ 60m area with random initial angles. Assuming unicycles have a 1m diameter, we have observed that there is a collision in roughly 68 percent of the 200 simulation trials run and that the colliding agents are precisely those sharing common α_i values. To avoid this, one possible solution would be to design a high level collision avoidance layer.

VI. Proof of Theorem 2

We divide the proof of Theorem 2 in several steps. We begin, in Section VI-A, by presenting preliminary results regarding the bounded integrator consensus controller in (5). Then in Section VI-B, we derive the closed-loop dynamics in $(\hat{x}_i, \theta_i)_{i\in\mathbf{n}}$ coordinates. In Sections VI-C and VI-D, we propose a Lyapunov function V for the closed-loop system, and carry out a Lyapunov analysis yielding the property $\dot{V} \leq 0$. In Section VI-E, we show that, for sufficiently large $\bar{\alpha} > 0$, the zero level set of $\dot{V}$ coincides with the formation manifold $\Gamma(d)$ on a neighbourhood of $\Gamma(d)$. This result will imply, via Lyapunov's direct method, asymptotic stability of $\Gamma(d)$. A further Lyapunov analysis is employed to show that $\Gamma(d)$ is in fact almost semiglobally asymptotically stable with high-gain parameter k. Each step of the proof will be presented in its own subsection.

A. Properties of Bounded Integrator Consensus Controller

Lemma 1. *Consider system* (4) *with feedback* (5)*. Assume that $f(s)$ satisfies assumptions A1 and A2 and the sensor graph $\mathcal{G}$ is undirected and connected. For any parameters $a_{ij} = a_{ji} > 0$, the consensus set $\{z \in \mathbb{R}^{2n} : (\forall i, j \in \mathbf{n})\ z_i = z_j\}$ is globally asymptotically stable.*

Proof. Consider system (4) with feedback (5). The feedback f_i in (5) for unicycle i points into the convex hull formed by its neighbours. By Corollary 3.9 in [27] the group of unicycles for system (4) achieves global consensus. □

Lemma 2. *If the sensor graph $\mathcal{G}$ is undirected and connected, then for any parameters $a_{ij} = a_{ji} > 0$, system* (4) *with feedback* (5)*, where $f(s)$ satisfies assumptions A1 and A2, is a gradient system, $\dot{z} = -\nabla V_t(z)$, with nonnegative storage function*

$$V_t(z) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} a_{ij}\int_0^{\|z_{ij}\|} f(s)\mathrm{d}s. \tag{16}$$

Moreover, $V^{-1}(0) = \{z \in \mathbb{R}^{2n} : (\forall i, j \in \mathbf{n})\ z_i = z_j\}$.

Proof. Assumption A1 implies that the function $z_{ij} \mapsto \int_0^{\|z_{ij}\|} f(s)\mathrm{d}s$ is nonnegative, and it attains its global minimum when $z_{ij} = 0$. Since $\mathcal{G}$ is connected, V_t attains a global minimum when $z_{ij} = 0$ for all $i, j \in \mathbf{n}$, and therefore V_t is positive definite. We now show the gradient property, i.e., $(\partial/\partial z_i)V_t = -f_i((z_{ij})_{j\in\mathcal{N}_i})^\top$. We have

$$\begin{aligned}
\frac{\partial V_t}{\partial z_i} &= \frac{1}{2}\sum_{j\in\mathcal{N}_i} a_{ij}\frac{\partial}{\partial\|z_{ij}\|}\left(\int_0^{\|z_{ij}\|} f(s)\mathrm{d}s\right)\frac{\partial\|z_{ij}\|}{\partial z_{ij}}\frac{\partial z_{ij}}{\partial z_i} \\
&\quad + \frac{1}{2}\sum_{j\in\mathcal{N}_i} a_{ji}\frac{\partial}{\partial\|z_{ji}\|}\left(\int_0^{\|z_{ji}\|} f(s)\mathrm{d}s\right)\frac{\partial\|z_{ji}\|}{\partial z_{ji}}\frac{\partial z_{ji}}{\partial z_i} \\
&= \frac{1}{2}\sum_{j\in\mathcal{N}_i} a_{ij} f(\|z_{ij}\|)\left(\frac{{z_{ij}}^\top}{\|z_{ij}\|}\right)(-1) \\
&\quad + \frac{1}{2}\sum_{j\in\mathcal{N}_i} a_{ji} f(\|z_{ji}\|)\left(\frac{{z_{ji}}^\top}{\|z_{ji}\|}\right)(1) \\
&= -\sum_{j\in\mathcal{N}_i} a_{ij}\frac{f(\|{z_{ij}}^\top\|)}{\|z_{ij}\|} z_{ij} = -f_i((z_{ij})_{j\in\mathcal{N}_i})^\top.
\end{aligned}$$

□

Lemma 3. *Assume $\mathcal{G}$ is undirected and connected, and consider system* (4) *with feedback* (5)*, where $f(s)$ satisfies assumptions A1 and A2. For any parameters $a_{ij} = a_{ji} > 0$, the following three properties hold:*

(i) $R_k^{-1} f_i((z_{ij})_{j\in\mathcal{N}_i}) = f_i((z_{ij}^k)_{j\in\mathcal{N}_i})$ *for all* $i, j, k \in \mathbf{n}$.
(ii) $\{z \in \mathbb{R}^{2n} : f_i((z_{ij})_{j\in\mathcal{N}_i}) = 0,\ \forall i \in \mathbf{n}\} = \{z \in \mathbb{R}^{2n} : z_i = z_j,\ \forall i, j \in \mathbf{n}\}$.
(iii) $\sum_i f_i(\cdot) = 0$.

Proof. To show (i), we use the fact that $\|z_{ij}^k\| = \|R_k^{-1} z_{ij}\| = \|z_{ij}\|$. Then,

$$R_k^{-1} f_i(\cdot) = \sum_{j\in\mathcal{N}_i} a_{ij} \frac{f(\|z_{ij}^k\|)}{\|z_{ij}^k\|} R_k^{-1} z_{ij} = f_i((z_{ij}^k)_{j\in\mathcal{N}_i}).$$

To show (ii), assume $f_i((z_{ij})_{j\in\mathcal{N}_i}) = 0$ for all $i \in \mathbf{n}$. Then system (4) is at a fixed point. By Lemma 1, the set $\{z \in \mathbb{R}^{2n} : z_i = z_j,\ \forall i, j \in \mathbf{n}\}$ is globally asymptotically stable, so it contains all fixed points. Therefore, $z_{ij} = 0$ for all $i, j \in \mathbf{n}$. Conversely, if $z_{ij} = 0$ for all $i, j \in \mathbf{n}$, then it follows by definition that $f_i = 0$ for all $i \in \mathbf{n}$. Finally, property (iii) follows by summing over the functions f_i in (5) and using the identities $a_{ij} = a_{ji}$, $z_{ij} = -z_{ji}$. □

B. System dynamics in $(\hat{x}_i, \theta_i)_{i\in\mathbf{n}}$ coordinates

To simplify the analysis, we consider new coordinates $(\hat{x}, \theta) = (\hat{x}_i, \theta_i)_{i\in\mathbf{n}}$ under the diffeomorphism $F : (x, \theta) \mapsto (\hat{x}, \theta)$ given by $F((x_i, \theta_i)_{i\in\mathbf{n}}) = (x_i + \delta(\theta_i), \theta_i)_{i\in\mathbf{n}}$. Computing the time derivative of $\hat{x}_i$ yields,

$$\begin{aligned}
\dot{\hat{x}}_i &= u_i R_i e_1 + R_i \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix} (\alpha_i e_1 + \beta_i e_2) \\
&= u_i R_i e_1 + \alpha_i \omega_i R_i e_2 - \beta_i \omega_i R_i e_1 \\
&= (u_i - \beta_i \omega_i) R_i e_1 + \alpha_i \omega_i R_i e_2,
\end{aligned}$$

from which we get

$$\begin{aligned}
\dot{\hat{x}}_i &= (u_i - \beta_i \omega_i) R_i e_1 + \alpha_i \omega_i R_i e_2 \\
\dot{\theta}_i &= \omega_i, \quad i \in \mathbf{n}.
\end{aligned} \tag{17}$$

Using Lemma 3(i) and the fact that the dot product is invariant to rotations, i.e, $R_i^{-1} f_i(\hat{y}_i) \cdot e_1 = f_i(\hat{y}_i) \cdot R_i e_1$, the feedbacks in (11) can be expressed as follows:

$$\begin{aligned}
u_i^\star(y_i^i, \varphi_i) &= f_i(\hat{y}_i) \cdot R_i e_1 + \beta_i \omega_i, \\
\omega_i^\star(y_i^i, \varphi_i) &= \frac{1}{\alpha_i} \left(f_i(\hat{y}_i) \cdot R_i e_2 + k g_i(\varphi_i, \eta) \right).
\end{aligned} \tag{18}$$

Substituting $u_i = u_i^\star(y_i^i, \varphi_i)$ and $\omega_i = \omega_i^\star(y_i^i, \varphi_i)$ from (18) into (17) and using the fact that $f_i(\hat{y}_i) = (f(\hat{y}_i) \cdot R_i e_1) R_i e_1 + (f(\hat{y}_i) \cdot R_i e_2) R_i e_2$ yields the closed-loop system in $(\hat{x}_i, \theta_i)_{i\in\mathbf{n}}$ coordinates,

$$\begin{aligned}
\dot{\hat{x}}_i &= f_i(\hat{y}_i) + k g_i(\varphi_i, \eta) R_i e_2 \\
\dot{\theta}_i &= \frac{1}{\alpha_i} \left(f_i(\hat{y}_i) \cdot R_i e_2 + k g_i(\varphi_i, \eta) \right), \quad i \in \mathbf{n}.
\end{aligned} \tag{19}$$

Notice that the control inputs in (18) are defined precisely in terms of $(\hat{x}, \theta)$ and so the equations of motion in (19) constitute a dynamical system. The parallel formation manifold Γ in (3) in $(\hat{x}, \theta)$ coordinates becomes,

$$\hat{\Gamma} := \left\{ (\hat{x}, \theta) \in \mathbb{R}^{2n} \times \mathbb{T}^n : \hat{x}_{1i} = 0,\ \theta_{1i} = 0,\ i \in \mathbf{n} \right\}. \tag{20}$$

C. Lyapunov analysis

From Lemma 2, system (4) is gradient with nonnegative storage function V_t. Inspired by [23], define a Lyapunov function $V_r(\theta)$ as,

$$V_r(\theta) := \frac{1}{2} \sum_{i=1}^n \sum_{j\in\mathcal{N}_i} b_{ij} \int_0^{\theta_{ij}} g(s) \mathrm{d}s. \tag{21}$$

Since $\mathcal{G}$ is connected, we have that $V_r \geq 0$ and $V_r^{-1}(0) = \{\theta \in \mathbb{T}^n : (\forall i, k \in \mathbf{n})\ \theta_i = \theta_j\}$. Next, combine $V_t(\hat{x})$ and $V_r(\theta)$ as follows:

$$V(\hat{x}, \theta) := V_t(\hat{x}) + k V_r(\theta).$$

Since V_t and V_r are nonnegative, V is nonnegative and $V^{-1}(0) = \hat{\Gamma}$.

Using (19), the time derivative of $V_t(\hat{x})$ is given by,

$$\begin{aligned}
\dot{V}_t &= \sum_{i=1}^n -f_i \cdot (f_i + k g_i R_i e_2) \\
&= \sum_{i=1}^n \left(-\|f_i\|^2 - (f_i \cdot R_i e_2) k g_i \right).
\end{aligned} \tag{22}$$

Since $g(\theta_{ij}) = -g(\theta_{ji})$, we have

$$\begin{aligned}
\frac{\partial V_r}{\partial \theta_i} &= \frac{1}{2} \sum_{j\in\mathcal{N}_i} b_{ij} \frac{\partial}{\partial \theta_{ij}} \left(\int_0^{\theta_{ij}} g(s) \mathrm{d}s \right) \frac{\partial \theta_{ij}}{\partial \theta_i} \\
&\quad + \frac{1}{2} \sum_{j\in\mathcal{N}_i} b_{ji} \frac{\partial}{\partial \theta_{ji}} \left(\int_0^{\theta_{ji}} g(s) \mathrm{d}s \right) \frac{\partial \theta_{ji}}{\partial \theta_i} \\
&= -\sum_{j\in\mathcal{N}_i} b_{ij} g(\theta_{ij}).
\end{aligned} \tag{23}$$

Using the above, identity (9), and the fact that $\eta_i = 1/\alpha_i$, we obtain

$$\begin{aligned}
\dot{V}_r &= -\sum_{i=1}^n \sum_{j\in\mathcal{N}_i} \frac{b_{ij}}{\alpha_i} g(\theta_{ij}) \left(f_i \cdot R_i e_2 + k g_i \right) \\
&= \sum_{i=1}^n -g_i \left(f_i \cdot R_i e_2 + k g_i \right) \\
&= \sum_{i=1}^n (-(f_i \cdot R_i e_2) g_i - k g_i^2).
\end{aligned} \tag{24}$$

Combining (22) and (24), we get

$$\begin{aligned}
\dot{V} &= \dot{V}_t + k \dot{V}_r \\
&= \sum_{i=1}^n \left(-\|f_i\|^2 - 2(f_i \cdot R_i e_2)(k g_i) - (k g_i)^2 \right) \\
&= \sum_{i=1}^n \left(-\|f_i \cdot R_i e_1\|^2 - \|f_i \cdot R_i e_2 + k g_i\|^2 \right) \leq 0.
\end{aligned} \tag{25}$$

D. Lyapunov analysis in relative coordinates

To further simplify the stability analysis, we perform another coordinate transformation with the intention of quotienting out the dynamics of unicycle 1. More precisely, consider the diffeomorphism

$$\begin{aligned}
&F : \mathbb{R}^{2n} \times \mathbb{T}^n \to \mathbb{R}^{2(n-1)} \times \mathbb{R}^2 \times \mathbb{T}^{(n-1)} \times \mathbb{S}^1, \\
&F(\hat{x}, \theta) = (\tilde{x}, \hat{x}_1^1, \tilde{\theta}, \theta_1),
\end{aligned}$$

where $\tilde{x} := (\hat{x}^1_{1i})_{i\in\mathbf{2:n}}$, $\tilde{\theta} := (\theta_{1i})_{i\in\mathbf{2:n}}$. Using Lemma 3(i) and the fact that $f_i(\hat{y}_i) \cdot R_i e_2 = f_i(\hat{y}^1_i) \cdot R^1_i e_2$, the dynamics in (19) can be written in new coordinates as,

$$\begin{aligned}
\dot{\hat{x}}^1_{1i} &= \left[f_i(\hat{y}^1_i) + kg_i(\varphi_i,\eta)R^1_i e_2 - f_1(\hat{y}^1_1) - kg_1(\varphi_1,\eta)e_2\right] \\
&\quad - \left(\frac{1}{\alpha_1}\left(f_1(\hat{y}^1_1)\cdot e_2 + kg_1(\varphi_1,\eta)\right)\right)^\times \hat{x}^1_{1i} \\
\dot{\theta}_{1i} &= \frac{1}{\alpha_i}\left(f_i(\hat{y}^1_i)\cdot R^1_i e_2 + kg_i(\varphi_i,\eta)\right) \\
&\quad - \frac{1}{\alpha_1}\left(f_1(\hat{y}^1_1)\cdot e_2 - kg_1(\varphi_1,\eta)\right) \\
\dot{\hat{x}}^1_1 &= f_1(\hat{y}^1_1) + kg_1(\varphi_1,\eta)e_2 - \omega_1^\times \hat{x}^1_1 \\
\dot{\theta}_1 &= \frac{1}{\alpha_1}\left(f_1(\hat{y}^1_1)\cdot e_2 - kg_1(\varphi_1,\eta)\right),
\end{aligned} \tag{26}$$

where $i \in \mathbf{2:n}$, and for $v \in \mathbb{R}$ we denote

$$v^\times := \begin{bmatrix} 0 & -v \\ v & 0 \end{bmatrix}.$$

We remark that $\hat{y}^1_i = (\hat{x}^1_{ij})_{j\in\mathcal{N}_i} = (\hat{x}^1_{1j} - \hat{x}^1_{1i})_{j\in\mathcal{N}_i}$, $\varphi_i = (\theta_{ij})_{j\in\mathcal{N}_i} = (\theta_{1j} - \theta_{1i})_{j\in\mathcal{N}_i}$ and R^1_i are functions of the relative quantities $(\tilde{x},\tilde{\theta})$, and do not depend on the absolute quantities $\hat{x}^1_1$ and θ_1. It follows that system (26) has a decoupled subsystem with state $(\tilde{x},\tilde{\theta}) \in \mathbb{R}^{2(n-1)} \times \mathbb{T}^{n-1}$. Moreover, $\Gamma(d)$ in new coordinates is given by

$$\left\{(\tilde{x},\hat{x}^1_1,\tilde{\theta},\theta_1) : \hat{x}^1_{1i} = 0,\ \theta_{1i} = 0,\ \forall\, i \in \mathbf{2:n}\right\}, \tag{27}$$

which is also independent of absolute quantities $\hat{x}^1_1$ and θ_1.

Based on these considerations, the variables $\hat{x}^1_1$ and θ_1 may be dropped, yielding a new dynamical system with state $(\tilde{x},\tilde{\theta}) \in \mathbb{R}^{2(n-1)} \times \mathbb{T}^{n-1}$. Proving almost semiglobal asymptotic stability of $\Gamma(d)$ for system (19) is equivalent to proving that the equilibrium

$$\tilde{\Gamma} := \left\{(\tilde{x},\tilde{\theta}) = (0,0) \in \mathbb{R}^{2(n-1)} \times \mathbb{T}^{(n-1)}\right\} \tag{28}$$

is almost semiglobally asymptotically stable for the $(\tilde{x},\tilde{\theta})$ subsystem.

We now return to the Lyapunov analysis of Section VI-C, expressing V in relative coordinates $(\tilde{x},\tilde{\theta})$. Using the fact that $\|\hat{x}_{ij}\| = \|R_1^{-1}\hat{x}_{ij}\| = \|R_1^{-1}(\hat{x}_{1j} - \hat{x}_{1i})\| = \|\hat{x}^1_{1j} - \hat{x}^1_{1i}\|$, and $\theta_{ij} = \theta_{1j} - \theta_{1i}$, we have

$$\begin{aligned}
\tilde{V}_t(\tilde{x},\tilde{\theta}) &:= V_t|_{(\hat{x},\theta)=F^{-1}(\tilde{x},\hat{x}^1_1,\tilde{\theta},\theta_1)} \\
&= \frac{1}{2}\sum_{i=1}^n \sum_{j\in\mathcal{N}_i} a_{ij}\int_0^{\|\hat{x}^1_{1j}-\hat{x}^1_{1i}\|} f(s)\mathrm{d}s \\
\tilde{V}_r(\tilde{x},\tilde{\theta}) &:= V_r|_{(\hat{x},\theta)=F^{-1}(\tilde{x},\hat{x}^1_1,\tilde{\theta},\theta_1)} \\
&= \frac{1}{2}\sum_{i=1}^n \sum_{j\in\mathcal{N}_i} b_{ij}\int_0^{\theta_{1j}-\theta_{1i}} g(s)\mathrm{d}s.
\end{aligned} \tag{29}$$

The identities in (29) imply that V can indeed be expressed in terms of relative quantities $(\tilde{x},\tilde{\theta})$, and in these coordinates it is given by $\tilde{V}(\tilde{x},\tilde{\theta}) := \tilde{V}_t(\tilde{x},\tilde{\theta}) + k\tilde{V}_r(\tilde{x},\tilde{\theta})$. Since $V^{-1}(0) = \Gamma$, it follows that $\tilde{V}^{-1}(0) = \tilde{\Gamma}$ and therefore $\tilde{V}$ is positive definite at $(\tilde{x},\tilde{\theta}) = (0,0)$. For any $c > 0$, the sublevel set $\tilde{V}_c = \{(\tilde{x},\tilde{\theta}) \in \mathbb{R}^{2(n-1)} \times \mathbb{T}^{(n-1)} : \tilde{V} \le c\}$ is closed since $\tilde{V}$ is continuous. Next we show that $\tilde{V}_c$ is bounded, and hence compact. In the set $\tilde{V}_c$,

$$a_{ij}\int_0^{\|\hat{x}^1_{ij}\|} f(s)\mathrm{d}s \le c,$$

for all $i \in \mathbf{n}$, $j \in \mathcal{N}_i$. If $\|\hat{x}^1_{ij}\| > c_2$ where c_2 is defined in A1, then this implies that $a_{ij}c_1(\|\hat{x}^1_{ij}\| - c_2) \le c$ where c_1 is defined in A1 and therefore $\|\hat{x}^1_{ij}\| \le (c/c_1 a_{ij}) + c_2$ is bounded. Since the undirected graph is connected, this proves boundedness of $(\tilde{x},\tilde{\theta})$. Moreover, using a standard result in [28, Proposition 8.16], the time derivative of $\tilde{V}$ satisfies,

$$\begin{aligned}
\dot{\tilde{V}} &= \dot{V}|_{(\hat{x},\theta)=F^{-1}(\tilde{x},\hat{x}^1_1,\tilde{\theta},\theta_1)} \\
&= \sum_{i=1}^n \left(-\|f_i(\hat{y}^1_i)\cdot R^1_i e_1\|^2 - \|f_i(\hat{y}^1_i)\cdot R^1_i e_2 + kg_i(\varphi_i,\eta)\|^2\right).
\end{aligned}$$

Once again, since $\hat{y}^1_i$, φ_i and R^1_i are functions of $(\tilde{x},\tilde{\theta})$, $\dot{\tilde{V}}(\tilde{x},\tilde{\theta})$ is independent of $\hat{x}^1_1$ and θ_1. In light of (25), $\dot{\tilde{V}} \le 0$, with equality if and only if $f_i(\hat{y}^1_i)\cdot R^1_i e_1 = 0$ and $f_i(\hat{y}^1_i)\cdot R^1_i e_2 = -kg_i(\varphi_i,\eta)$ for all $i \in \mathbf{n}$. Together, these conditions imply that on the set $E := \{(\tilde{x},\tilde{\theta}) : \dot{\tilde{V}}(\tilde{x},\tilde{\theta}) = 0\}$ it holds that

$$f_i(\hat{y}^1_i) = -kg_i(\varphi_i,\eta)R^1_i e_2,\ \forall i \in \mathbf{n}. \tag{30}$$

E. Local asymptotic stability of $\tilde{\Gamma}$

In this section we show that there exists $\epsilon > 0$ such that $E \cap \{(\tilde{x},\tilde{\theta}) : \|\tilde{\theta}\| \le \epsilon\} = \tilde{V}^{-1}(0) = \tilde{\Gamma}$, implying that $\dot{\tilde{V}}$ is negative definite, and $\tilde{\Gamma}$ is locally asymptotically stable by Lyapunov's direct method.

Let $(\tilde{x},\tilde{\theta}) \in E$ be arbitrary. By Lemma 3(iii), we have $0 = \sum_{i=1}^n f_i(\hat{y}^1_i) = R_1^{-1}\sum_{i=1}^n f_i(\hat{y}_i) = 0$. Using (30), we get $-\sum_{i=1}^n g_i(\varphi_i,\eta)R^1_i e_2 = 0$, and using (9), we get

$$-\sum_{i=1}^n \eta_i \sum_{j\in\mathcal{N}_i} b_{ij}g(\theta_{ij})R^1_i e_2 = 0.$$

We have $R^1_i e_2 = \begin{bmatrix} -\sin(\theta_{1i}) & \cos(\theta_{1i})\end{bmatrix}^\top$, so

$$-\sum_{i=1}^n \eta_i \sum_{j\in\mathcal{N}_i} b_{ij}g(\theta_{ij})\begin{bmatrix} -\sin(\theta_{1i}) \\ \cos(\theta_{1i})\end{bmatrix} = 0.$$

The first component of the above identity gives

$$\sum_{i=1}^n \eta_i \sum_{j\in\mathcal{N}_i} b_{ij}g(\theta_{ij})\sin(\theta_{1i}) = 0, \tag{31}$$

which depends solely on relative angles $\tilde{\theta}$. Expanding $g(s)$ and $\sin(s)$ about $s = 0$, we get

$$\begin{aligned}
g(s) &= \dot{g}(0)s + h_1(s)s \\
\sin(s) &= s + h_2(s)s,
\end{aligned}$$

where $\lim_{s\to 0} h_1(s) = 0$ and $\lim_{s\to 0} h_2(s) = 0$. Moreover, $\dot{g}(0) > 0$ by B3. Using the above identities in (31) we get

$$\begin{aligned}
\sum_{i=1}^n \eta_i \sum_{j\in\mathcal{N}_i} b_{ij}\,&[\dot{g}(0)\theta_{ij}\theta_{1i} + \dot{g}(0)h_2(\theta_{1i})\theta_{ij}\theta_{1i} \\
&+ h_1(\theta_{ij})\theta_{ij}\theta_{1i} + h_1(\theta_{ij})h_2(\theta_{1i})\theta_{ij}\theta_{1i}] = 0.
\end{aligned} \tag{32}$$

Dividing by $\dot{g}(0)$, we have

$$\sum_{i=1}^{n} \eta_i \sum_{j\in\mathcal{N}_i} b_{ij}\theta_{ij}\theta_{1i}\left[1 + h_2(\theta_{1i}) + \frac{h_1(\theta_{ij})}{\dot{g}(0)} + \frac{h_1(\theta_{ij})h_2(\theta_{1i})}{\dot{g}(0)}\right] = 0. \tag{33}$$

Ignoring, for now, higher order terms $h_1(\theta_{ij})$, $h_2(\theta_{1i})$ in (33) and substituting in $\eta_i = 1/\alpha_i$, we get

$$\sum_{i=1}^{n}(\theta_{1i}/\alpha_i)\sum_{j\in\mathcal{N}_i} b_{ij}[\theta_{1j} - \theta_{1i}] = 0. \tag{34}$$

Define a weighted Laplacian matrix L by $L(i,j) = -b_{ij}$ for $i \neq j$ and $L(i,i) = \sum_{j\in\mathcal{N}_i} b_{ij}$. Then L is a symmetric Laplacian for the connected graph $\mathcal{G}$, and therefore $\ker L = \operatorname{span}\{\mathbf{1}\}$. Next, let

$$\begin{aligned}\lambda_i(\bar{\alpha}, d) &:= \frac{\max_i \alpha_i}{\alpha_i} = \frac{\bar{\alpha} + \max_i(-d_{1i}^1 \cdot e_1)}{\bar{\alpha} - d_{1i}^1 \cdot e_1}, \\ \lambda &= (\lambda_1, \ldots, \lambda_n),\end{aligned} \tag{35}$$

and

$$D(\lambda) := \operatorname{diag}\left(\lambda_i\right)_{i\in\mathbf{n}}.$$

The identity in (34) can now be rewritten as

$$-\frac{1}{\max_i \alpha_i}\begin{bmatrix}0 & \tilde{\theta}^\top\end{bmatrix} D(\lambda(\bar{\alpha}, d))L\begin{bmatrix}0\\ \tilde{\theta}\end{bmatrix} = 0. \tag{36}$$

Letting

$$\tilde{L}(\lambda) := P^\top D(\lambda)LP,\ P = \begin{bmatrix}0_{1\times(n-1)}\\ I_{(n-1)}\end{bmatrix},$$

identity (36) implies

$$\tilde{\theta}^\top \tilde{L}(\lambda(\bar{\alpha}, d))\tilde{\theta} = 0. \tag{37}$$

Denoting by $M(\lambda) := (\tilde{L}(\lambda) + \tilde{L}(\lambda)^\top)/2$ the symmetric part of $\tilde{L}$, identity (37) becomes

$$\tilde{\theta}^\top M(\lambda(\bar{\alpha}, d))\tilde{\theta} = 0. \tag{38}$$

We will show that for large $\bar{\alpha} > 0$, $M(\lambda(\bar{\alpha}, d))$ is positive definite. Referring to the definition of λ_i in (35), note that

$$\lambda_i(\bar{\alpha}, d) \to 1 \text{ as } \frac{\bar{\alpha}}{\max_i(-d_{1i}^1 \cdot e_1)} \to \infty, \tag{39}$$

and $\lambda_i(\bar{\alpha}, d) = 1$ when $-d_{1i}^1 \cdot e_1 = 0$ for all $i \in \mathbf{n}$. In light of this observation, consider first the case in which $\lambda = \mathbf{1}$, so that $D(\lambda) = D(\mathbf{1}) = I_n$, the identity matrix. Then (36) reduces to

$$\begin{bmatrix}0 & \tilde{\theta}^\top\end{bmatrix} L\begin{bmatrix}0\\ \tilde{\theta}\end{bmatrix} \geq 0,$$

with equality if and only if $\begin{bmatrix}0 & \tilde{\theta}^\top\end{bmatrix} \in \operatorname{span}\{\mathbf{1}\}$ (since $\ker L = \operatorname{span}\{\mathbf{1}\}$), which can occur only if $\tilde{\theta} = 0$. Owing to the equivalence of (36) and (38), we have that $\tilde{\theta}^\top M(\mathbf{1})\tilde{\theta} \geq 0$, with equality holding if and only if $\tilde{\theta} = 0$, and thus $M(\mathbf{1})$ is positive definite and, since $M(\lambda)$ is symmetric, all its principal leading minors $m_i(\lambda)$, $i \in \mathbf{n}$, have the property that $m_i(\mathbf{1}) > 0$, $i \in \mathbf{n}$. Since the functions $m_i(\lambda)$ are continuous, there exists $\varepsilon > 0$ such that for all $\lambda \in \mathbb{R}^n$ such that $\|\lambda - \mathbf{1}\| < \varepsilon$, $m_i(\lambda) > 0$, $i \in \mathbf{n}$. From (39), we deduce that there exists $\alpha^\star > 0$ such that

$$\begin{aligned}\frac{\bar{\alpha}}{\max_i(-d_{1i}^1 \cdot e_1)} > \alpha^\star &\implies \|\lambda(\bar{\alpha}, d) - \mathbf{1}\| < \varepsilon \\ &\implies m_i(\lambda(\bar{\alpha}, d)) > 0\ i \in \mathbf{n}.\end{aligned}$$

We have thus established the existence of $\alpha^\star > 0$ such that, for all $\bar{\alpha} > \alpha^\star \max_i(-d_{1i}^1 \cdot e_1)$, the matrix $M(\lambda(\bar{\alpha}, d))$ is positive definite.

Now assuming that $\bar{\alpha}$ satisfies the above bound so that $M(\lambda(\bar{\alpha}, d))$ is positive definite, we return to identity (33) including higher-order terms, and rewrite it as

$$\tilde{\theta}^\top M(\lambda(\bar{\alpha}, d))\tilde{\theta} + r(\tilde{\theta}) = 0, \tag{40}$$

where $M(\cdot)$ is as before and

$$r(\tilde{\theta}) = \sum_{i=1}^{n}\eta_i \sum_{j\in\mathcal{N}_i} b_{ij}\theta_{ij}\theta_{1i}\left[h_2(\theta_{1i}) + \frac{h_1(\theta_{ij})}{\dot{g}(0)} + \frac{h_1(\theta_{ij})h_2(\theta_{1i})}{\dot{g}(0)}\right].$$

We will show, using similar arguments to [29, Proof of Theorem 6.1], that there exists $\epsilon > 0$ such that in an ϵ-neighborhood of $\tilde{\theta} = 0$, identity (40) holds only if $\tilde{\theta} = 0$.

Condition (40) holds only if $\|\tilde{\theta}^\top M(\cdot)\tilde{\theta}\| = \|r(\tilde{\theta})\|$. Suppose for a moment that

$$\lim_{\tilde{\theta}\to 0}\frac{\|r(\tilde{\theta})\|}{\|\tilde{\theta}^\top M(\cdot)\tilde{\theta}\|} = 0. \tag{41}$$

Then for sufficiently small $\tilde{\theta}$, $\|r(\tilde{\theta})\| \leq \|\tilde{\theta}^\top M(\cdot)\tilde{\theta}\|/2$, and the unique solution to (40) is $\tilde{\theta} = 0$, as desired. To show that (41) holds, express $\tilde{\theta}$ as $\tilde{\theta} = \|\tilde{\theta}\|\phi$ where $\phi = (\phi_{1i})_{i\in\mathbf{2}:\mathbf{n}} \in \mathbb{S}^{n-1}$ is a unit vector. Correspondingly, $\theta_{1i} = \|\tilde{\theta}\|\phi_{1i}$ for all $i \in \mathbf{2}:\mathbf{n}$. One can then write,

$$\begin{aligned}\tilde{\theta}^\top M(\cdot)\tilde{\theta} &= \|\tilde{\theta}\|^2\phi^\top M(\cdot)\phi, \\ r(\tilde{\theta}) &= \|\tilde{\theta}\|^2\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} b_{ij}\eta_i\phi_{ij}\phi_{1i}\left[h_2(\theta_{1i}) + \frac{h_1(\theta_{ij})}{\dot{g}(0)} + \frac{h_1(\theta_{ij})h_2(\theta_{1i})}{\dot{g}(0)}\right] = \|\tilde{\theta}\|^2 h(\tilde{\theta}, \phi),\end{aligned}$$

where $h(\cdot,\cdot)$ has the property that $\lim_{\tilde{\theta}\to 0} h(\tilde{\theta}, \phi) = 0$. Then,

$$\lim_{\tilde{\theta}\to 0}\frac{\|r(\tilde{\theta})\|}{\|\tilde{\theta}^\top M(\cdot)\tilde{\theta}\|} = \lim_{\tilde{\theta}\to 0}\frac{\|h(\tilde{\theta}, \phi)\|}{\phi^\top M(\cdot)\phi} = 0,$$

since $\lim_{\tilde{\theta}\to 0} h(\tilde{\theta}, \phi) = 0$ and $\min_{\phi\in\mathbb{S}^{n-1}}(\phi^\top M(\cdot)\phi) > 0$ because $M(\cdot)$ positive definite and ϕ is a unit vector.

To summarize, there exists $\epsilon > 0$ such that if $\|\tilde{\theta}\| \leq \epsilon$, then (33) is zero only if $\tilde{\theta} = 0$, implying that $g_i = 0$ for all $i \in \mathbf{n}$. By (30) this implies that $f_i = 0$ for all $i \in \mathbf{n}$ and therefore, by Lemma 3(ii), $\hat{x}_i = \hat{x}_j$ for all $i, j \in \mathbf{n}$. It follows that $E \cap (\mathbb{R}^{2(n-1)} \times \{\tilde{\theta} : \|\tilde{\theta}\| \leq \epsilon\}) = \tilde{\Gamma}$.

To summarize our findings so far, we have shown that $\tilde{V}$ is positive definite, $\tilde{V}^{-1}(0) = \{(\tilde{x}, \tilde{\theta}) = (0, 0)\} = \tilde{\Gamma}$, and $\dot{\tilde{V}} = 0$ is negative definite on $(\mathbb{R}^{2(n-1)} \times \{\tilde{\theta} : \|\tilde{\theta}\| \leq \epsilon\})$, a neighbourhood of $\tilde{\Gamma}$. By Lyapunov's stability theorem, the equilibrium $\tilde{\Gamma}$ is locally asymptotically stable for the $(\tilde{x}, \tilde{\theta})$ subsystem.

F. Almost semiglobal asymptotic stability of $\tilde{\Gamma}$

Having established that for $\bar{\alpha} > 0$ sufficiently large, the equilibrium $\tilde{\Gamma}$ is asymptotically stable for the $(\tilde{x}, \tilde{\theta})$ subsystem, we now prove that $\tilde{\Gamma}$ is almost semiglobally asymptotically stable with high-gain parameter k. The idea is to show that, for sufficiently large k, for almost all initial conditions in any given compact set the solutions of the $(\tilde{x}, \tilde{\theta})$ subsystem enter in finite time and remain inside the set $(\mathbb{R}^{2(n-1)} \times \{\tilde{\theta} : \|\tilde{\theta}\| \leq \epsilon\})$ on which $\dot{\tilde{V}}$ is negative definite, which implies that they converge to $\tilde{\Gamma}$.

Rewrite the dynamics of the $\tilde{\theta}$ subsystem in (26) as,

$$\dot{\tilde{\theta}} = kF(\tilde{\theta}) + \Delta(\tilde{x}, \tilde{\theta}), \tag{42}$$

where

$$F_i(\tilde{\theta}) := \left(\frac{1}{\alpha_i} g_i(\varphi_i, \eta) - \frac{1}{\alpha_1} g_1(\varphi_1, \eta) \right)$$
$$\Delta_i(\tilde{x}, \tilde{\theta}) := \frac{1}{\alpha_i} f_i(\hat{y}_i^1) \cdot R_i^1 e_2 - \frac{1}{\alpha_1} f_1(\hat{y}_1^1) \cdot e_2.$$

After the time scaling $\tau = kt$, system (42) reads as

$$\tilde{\theta}' = F(\tilde{\theta}) + \frac{1}{k}\Delta(\tilde{x}, \tilde{\theta}), \tag{43}$$

where prime denotes differentiation with respect to τ. In what follows, we denote by $\Sigma(0)$ the nominal system $\tilde{\theta}' = F(\tilde{\theta})$, and by $\Sigma(k)$ the perturbed system (43).

The vector field F coincides with the attitude synchronization dynamics of the collection of rotational integrators in (8) with feedback (9), expressed relative to integrator 1. Therefore, by Theorem 1, the equilibrium $\tilde{\theta} = 0$ is almost globally asymptotically stable for $\Sigma(0)$. Let $D(0)$ be the domain of attraction of $\tilde{\theta} = 0$ for $\Sigma(0)$, a set of full-measure.

The term $(1/k)\Delta$ acts as a perturbation in (43). Since, by assumption A2, the functions $f_i(\hat{y}_i^1) \cdot R_i^1 e_2$ and $f_1(\hat{y}_1^1) \cdot e_2$ are uniformly bounded, the map Δ is uniformly bounded, i.e., there exists $\bar{\Delta} > 0$ such that $\sup \|\Delta\| < \bar{\Delta}$. The uniform bound on the perturbation $(1/k)\Delta$ tends to zero as $k \to \infty$.

Since $\tilde{\theta} = 0$ is asymptotically stable for $\Sigma(0)$, there exists $r > 0$ and a C^1 positive definite Lyapunov function $W : B_r(0) \to \mathbb{R}$ whose derivative along $\Sigma(0)$, $L_F W : B_r(0) \to \mathbb{R}$, is negative definite. We may assume, without loss of generality, that $r \leq \epsilon$. Let $c > 0$ be such that the sublevel set $W_c := \{\tilde{\theta} : W(\tilde{\theta}) < c\}$ is contained in $B_r(0) \subset B_\epsilon(0)$, and let $\epsilon' > 0$ be such that $B_{\epsilon'}(0) \subset W_c \subset B_\epsilon(0)$. Since $L_F W\big|_{\partial W_c} < 0$, W_c is positively invariant for $\Sigma(0)$. Moreover, letting

$$k_0 = \frac{\max_{\partial W_c} \|\partial W / \partial \tilde{\theta}\|}{\min_{\partial W_c} |L_F W|} \bar{\Delta},$$

we have that for all $k > k_0$ it holds that $L_{F+(1/k)\Delta} W\big|_{\partial W_c} < 0$, and thus W_c is positively invariant for the perturbed system $\Sigma(k)$ in (43).

Let $\bar{\theta} \in D(0)$ be arbitrary. Then the solution of $\Sigma(0)$ through $\bar{\theta}$ converges to 0, and let $T > 0$ be the first time when such solution enters $B_{\epsilon'}(0)$. By continuity of solutions with respect to initial conditions and bounded perturbations [30, Theorem 3.4], there exists $\mu > 0$ and $\bar{k} \geq k_0$ such that for all $k > \bar{k}$, all solutions of $\Sigma(k)$ through initial conditions in $B_\mu(\bar{\theta})$ are contained in W_c at time T. Since W_c is positively invariant for $\Sigma(k)$ and contained in $B_\epsilon(0)$, all solutions of $\Sigma(k)$ through $B_\mu(\bar{\theta})$ enter in finite time and remain inside the set $B_\epsilon(0)$, for all $k > \bar{k}$.

Let $K \subset D(0)$ be an arbitrary compact set. The arguments above yield an open cover of K by balls $B_\mu(\bar{\theta})$ and associated gains $\bar{k}$, where μ and $\bar{k}$ depend on $\bar{\theta}$. Taking a finite subcover, we obtain points $\{\bar{\theta}_i, i \in \mathbf{k}\} \subset \mathbb{T}^{n-1}$, associated balls $B_{\mu_i}(\bar{\theta}_i)$, and gains $\bar{k}_i$, $i \in \mathbf{k}$. Letting $k^\star = \max_{i \in \mathbf{k}} \bar{k}_i$, for all $k > k^\star$ all solutions of $\Sigma(k)$ through points in K enter and remain inside $B_\epsilon(0)$.

Returning to the $(\tilde{x}, \tilde{\theta})$ dynamics, the $\tilde{x}$ subsystem has no finite escape times because $\tilde{V}$ is proper and nonincreasing along solutions. Then, from the results just obtained we have that, for any compact subset K of $D(0)$ (a set of full-measure in $\mathbb{T}^{n-1}$), there exists $k^\star > 0$ such that, for all $k > k^\star$, all solutions of the $(\tilde{x}, \tilde{\theta})$ subsystem through initial conditions in $\mathbb{R}^{2(n-1)} \times K$ enter in finite time and remain inside the closed set $\mathbb{R}^{2(n-1)} \times \{\tilde{\theta} : \|\tilde{\theta}\| \leq \epsilon\}$. For any $c > 0$, the set $\tilde{V}_c \cap (\mathbb{R}^{2(n-1)} \times \{\tilde{\theta} : \|\tilde{\theta}\| \leq \epsilon\})$ is compact because the sublevel set $\tilde{V}_c$ is compact. Since $\dot{\tilde{V}}$ is negative definite on this set, all solutions through $\mathbb{R}^{2(n-1)} \times K$ converge to $\tilde{\Gamma}$. This proves that $\tilde{\Gamma}$ is almost semiglobally asymptotically stable.

VII. Conclusion

We have presented a class of local and distributed control laws to drive a group of kinematic unicycles to a desired parallel formation in which all unicycles have a common heading direction and achieve a desired predefined spacing between their positions. While converging to the formation, the unicycles come to a stop. In our setup, the sensor graph is assumed to be undirected and connected. As a special case, we solved the full synchronization problem where both the unicycle heading directions and positions coincide. The formations in this paper are invariant under rigid translations and rotations. We presented simulation results that validate the results in this paper. Some drawbacks of the approach are the requirement of a high-gain parameter and the assumption of fixed, undirected graphs. In future work, one could consider extending the current approach to directed, time-dependent or state-dependent graphs and relax the high gain parameter requirement.

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Ashton Roza received the Bachelor of Applied Science (BASc) degree in electrical engineering in 2010 from the University of Waterloo, Canada and the Master of Applied Science (MASc) from the University of Toronto, Canada, in 2012. He is currently a PhD student in the Edward S. Rogers Sr. Department of Electrical and Computer Engineering, University of Toronto, Canada. His research focuses on the application of nonlinear control and geometric methods to coordination in networks of autonomous robots.

Manfredi Maggiore was born in Genoa, Italy. He received the "Laurea" degree in Electronic Engineering in 1996 from the University of Genoa and the PhD degree in Electrical Engineering from the Ohio State University, USA, in 2000. Since 2000 he has been with the Edward S. Rogers Sr. Department of Electrical and Computer Engineering, University of Toronto, Canada, where he is currently Professor. He has been a visiting Professor at the University of Bologna (2007-2008), and the Laboratoire des Signaux et Systèmes, Ecole CentraleSupélec (2015-2016). His research focuses on mathematical nonlinear control, and relies on methods from dynamical systems theory and differential geometry.

Luca Scardovi (M '06) received the Laurea degree and Ph.D. degree in Electronic and Computer Engineering from the University of Genoa, Italy, in 2001 and 2005 respectively. He held research associate positions at the Department of Electrical Engineering and Computer Science at the University of Liège, Belgium (2005-2007) and at the Department of Mechanical and Aerospace Engineering at Princeton University (2007-2009). From 2009 to 2011 he was an Assistant Professor at the Technische Universität München, Munich, Germany. Since 2011 he has been with the Edward S. Rogers Sr. Department of Electrical and Computer Engineering, University of Toronto, Canada, where he is currently an Associate Professor. In 2018 he was a Visiting Scientist at the Biocore Institute at INRIA Sophia Antipolis, France, and a Visiting Professor at the University of Cagliari, Italy. His research interests focus on dynamical systems with special emphasis in the analysis and control of emergent dynamics in networked dynamical systems. He is an Associate Editor for the journals *Systems & Control Letters* and *IEEE Control Systems Letters*.