ECE311 Control & Systems

Problem Set No. 2

Problem 2.28

a) To obtain the equilibrium point(s) for $\bar{u} = 1$, set $\frac{di}{dt} = 0$ and $\frac{dv}{dt} = 0$

$$\begin{aligned} 0 &= -\bar{i} + \bar{v} \\ 0 &= -\bar{i} + g(\bar{u} - \bar{v}), \end{aligned}$$

which yields

$$\begin{aligned} &\bar{v} = \bar{i} \\ &(1 - \bar{v})(1 - \bar{v} - 1)(1 - \bar{v} - 4) = \bar{i}, \end{aligned}$$

or

$$\begin{aligned} &\bar{v} = \bar{i} \\ &\bar{v}(\bar{v}^2 + 2\bar{v} - 2) = 0. \end{aligned}$$

Therefore,

$\bar{v} = \bar{i} = 0$
$\bar{v} = \bar{i} = -1 + \sqrt{3}$
$\bar{v} = \bar{i} = -1 - \sqrt{3}$

b) Define $x_1 = i = \bar{i} + \delta x_1$, $x_2 = v = \bar{v} + \delta x_2$, and $u = \bar{u} + \delta u$. On linearization of the system

$$\frac{dx_1}{dt} = -x_1 + x_2 \tag{1}$$

$$\frac{dx_2}{dt} = -x_1 + \underbrace{(u - x_2)(u - x_2 - 1)(u - x_2 - 4)}_{g(u - x_2)}, \tag{2}$$

about $\bar{u}$, $\bar{v}$, $\bar{i}$, one has

$$\begin{aligned} \delta \dot{x}_1 &= -\delta x_1 + \delta x_2 + 0 \delta u, && (3) \\ \delta \dot{x}_2 &= -\delta x_1 + \frac{\partial g(u - x_2)}{\partial x_2}\Big|_{\bar{u},\bar{v}} \delta x_1 + \frac{\partial g(u - x_2)}{\partial u}\Big|_{\bar{u},\bar{v}} \delta u. && (4) \end{aligned}$$

Note

$$\begin{aligned} \frac{\partial g(u - x_2)}{\partial x_2} &= -(u - x_2 - 1)(u - x_2 - 4) - (u - x_2)(u - x_2 - 4) - (u - x_2)(u - x_2 - 1), \\ \frac{\partial g(u - x_2)}{\partial u} &= (u - x_2 - 1)(u - x_2 - 4) + (u - x_2)(u - x_2 - 4) + (u - x_2)(u - x_2 - 1). \end{aligned}$$

For $\bar{u} = 1$ and $\bar{v} = \bar{i} = 0$ we have

$$\begin{aligned} \frac{\partial g(u - x_2)}{\partial x_2}\Big|_{\bar{u},\bar{v}} &= 3, \\ \frac{\partial g(u - x_2)}{\partial u}\Big|_{\bar{u},\bar{v}} &= -3. \end{aligned}$$

Therefore, equations (3)-(4) become

$$\begin{bmatrix} \delta \dot{x}_1 \\ \delta \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} \delta u.$$

c) For the equilibrium point $\bar{u} = 1$ and $\bar{v} = \bar{i} = -1 + \sqrt{3}$, one has

$$\begin{aligned} \frac{\partial g(u - x_2)}{\partial x_2}\Big|_{\bar{u},\bar{v}} &= -1.54, \\ \frac{\partial g(u - x_2)}{\partial u}\Big|_{\bar{u},\bar{v}} &= +1.54. \end{aligned}$$

Therefore, (3)-(4) yield

$$\begin{bmatrix} \delta \dot{x}_1 \\ \delta \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1.54 \end{bmatrix} + \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1.54 \end{bmatrix} \delta u.$$

Finally, for the third equilibrium point $\bar{u} = 1$ and $\bar{v} = \bar{i} = -1 - \sqrt{3}$:

$$\begin{aligned} \frac{\partial g(u - x_2)}{\partial x_2}\Big|_{\bar{u},\bar{v}} &= -8.46, \\ \frac{\partial g(u - x_2)}{\partial u}\Big|_{\bar{u},\bar{v}} &= 8.46. \end{aligned}$$

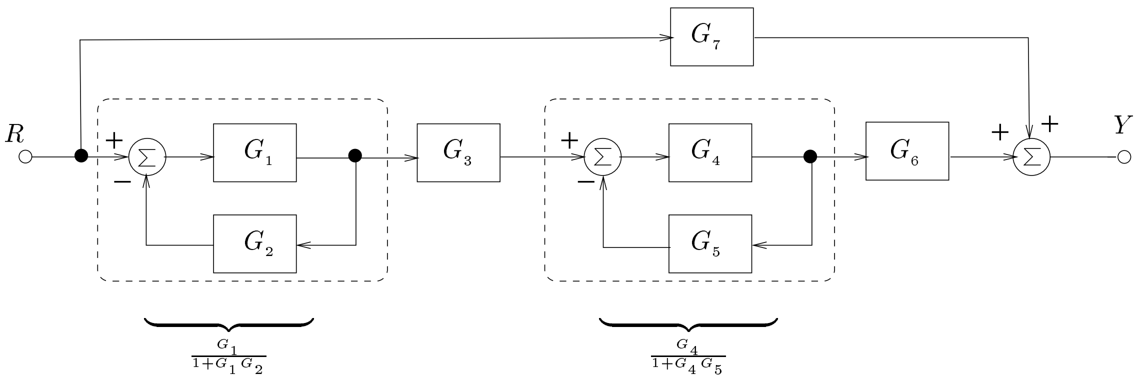
and

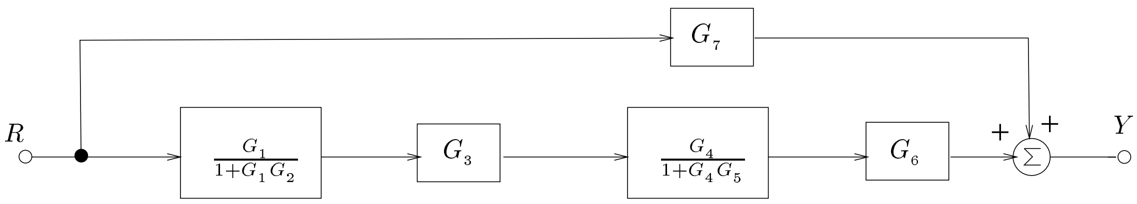
$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & -8.46 \end{bmatrix} + \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 8.46 \end{bmatrix} \delta u.$$

□

Problem 3.21

b)





$$\frac{Y}{R} = G_7 + \frac{G_1 G_3 G_4 G_6}{(1 + G_1 G_2)(1 + G_4 G_5)}$$

□

Problem 3.29

We perform a (stable) pole-zero cancellation to obtain a second-order characteristic equation. (Note that $s = -3$ is a stable pole of the system). For $z = 3$, the resultant closed-loop system is described by

$$Y(s) = \frac{K}{s^2 + ps + K} R(s),$$

for which the damping ratio and the natural frequency are

$$\zeta = \frac{1}{2}\frac{p}{\sqrt{K}}, \tag{5}$$

$$\omega_n = \sqrt{K}. \tag{6}$$

Next, we have to find the values for ζ and ω_n for which the second-order system above has the desired overshoot and settling time. To this end, recall that

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.1,$$

$$t_s = \frac{4.6}{\zeta\omega_n} = 1.5.$$

Solving the above two equations for ζ and ω, we get

$$\boxed{\zeta = 0.6, \qquad \omega_n = 5.11.}$$

Using the expressions in (5) and (6), we obtain the desired values of p and K as:

$$\boxed{p = 6.13, \qquad K = 26.11.}$$

□